ON f_q-DERIVATIONS OF BF-ALGEBRAS

KIMBERLY V. ALBINO[1,*], MICHELLE T. PANGANDUYON[2], MARY JOY R. LATAYADA[1]

[1]Department of Mathematics, Caraga State University, Ampayon, Butuan City, Philippines

[2]College of Arts and Sciences, Surigao del Norte State University, Surigao City, Philippines

[*]Corresponding author: kimalbino122801@gmail.com

Received Apr. 8, 2026

Abstract. This paper introduces the concept of f_q-derivations in BF-algebras and investigates their fundamental properties. Various types of f_q-derivations, including inside, outside, and (ℓ, r) and (r, ℓ) derivations, are defined and analyzed within the framework of associative BF-algebras. A characterization of associativity is established through the identity $c * (a * b) = (b * c) * a$, which provides an equivalent condition to the standard associative law. Several results concerning the behavior of f_q-derivations at the zero element are obtained, including explicit expressions for $d_q^f(0)$ under different derivation types. It is shown that, under associativity and the condition $0*a = a$, the mapping d_q^f satisfies both inside and outside properties, and hence becomes an f_q-derivation. Special cases, such as d_0^f, are also examined. Furthermore, additional results are established for BF-algebras that also satisfy the axioms of a B-algebra, particularly in relation to regular f_q-derivations. These findings extend derivation theory to BF-algebras and provide a foundation for further studies on generalized derivations in non-classical algebraic structures.

2026 Mathematics Subject Classification. 06F35.

Key words and phrases. BF algebras; derivation; f_q derivation; algebra.

1. Introduction

The study of non-classical algebraic structures such as BCI- and BCK-algebras has significantly contributed to the development of abstract algebra, particularly in understanding logical and algebraic systems. These structures were later generalized into B-algebras, introduced by Neggers and Kim [10], which provide a broader framework for analyzing binary operations and their properties. Extending this line of generalization, BF-algebras were introduced by Walendziak [12], characterized by the identity $0 * (a * b) = b * a$, offering a richer structure for algebraic investigation.

Derivations have long been recognized as important tools in algebra for studying the structural behavior of algebraic systems. In non-classical systems such as BCK-, BCI-, and B-algebras, derivations have been extensively explored, leading to various generalizations including f-derivations. A further extension is

DOI: 10.28924/APJM/13-98

the f_q-derivation, which incorporates both an endomorphism and a fixed element into the mapping. Recent studies have developed the concept of f_q-derivations in several algebraic systems. In B-algebras, f_q-derivations were studied by Muangkarn et al. (2021) [11]. This concept was later extended to BN-algebras by Gemawati et al. (2023) [7], and further investigated in BP-algebras by Gemawati et al. (2021) [8]. These works revealed important structural properties such as regularity, symmetry, and the behavior of derivations under different algebraic conditions.

However, despite these developments, the concept of f_q-derivations has not been fully explored within the framework of BF-algebras. This gap motivates the present study, which aims to introduce and investigate f_q-derivations in BF-algebras. In particular, this work establishes new results concerning associativity conditions, structural identities, and the behavior of derivations, especially at the zero element. Furthermore, it examines the conditions under which f_q-derivations satisfy inside, outside, and full derivation properties in associative BF-algebras.

2. Preliminaries

In this section, we recall some fundamental definitions and results related to B-algebras, BF-algebras, and endomorphisms, which are essential for the development of the present study. These concepts provide the necessary algebraic framework for investigating the properties of f_q-derivations in BF-algebras.

Definition 1. [10] A B-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(B1) $a * a = 0$,

(B2) $a * 0 = a$,

(B3) $(a * b) * c = a * (c * (0 * b))$

for all $a, b, c \in X$.

Example 1. [10] Let $X := \{0, 1, 2\}$ be a set with the following operation table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(X, *, 0)$ is a B-algebra.

Definition 2. [12] A BF-algebra is a non-empty set C with a binary operation $*$ and a constant 0 such that for all $a, b \in C$:

(B1) $a * a = 0$,

(B2) $a * 0 = a$,

(BF) $0 * (a * b) = b * a$.

Example 2. [12] Let $A = [0, +\infty) = \{x \in \mathbb{R} : x \geq 0\}$ and define a binary operation $*$ on A by

$$x * y = |x - y| \quad \text{for all } x, y \in A.$$

Then $(A, *, 0)$ is a BF-algebra.

Example 3. Let $C = \{0, a, b, c\}$ be a nonempty set with binary operation $*$ defined in the following Cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then C is a BF-algebra.

Note 1. Let C be a BF-algebra. If C satisfies B3, that is

$$(x * y) * z = x * (z * (0 * y))$$

for all $x, y, z \in C$, then C is also a B-algebra.

Definition 3. [6] Let C be a BF-algebra. A mapping $f : C \to C$ is called an endomorphism of C if for all $a, b \in C$,

$$f(a * b) = f(a) * f(b).$$

Example 4. Consider the BF-algebra defined in the Example 3. Let $f : C \to C$ be defined by

$$f(0) = 0, \quad f(a) = a, \quad f(b) = 0, \quad f(c) = a.$$

Hence, the map f is an endomorphism of the given BF-algebra.

Associativity is a fundamental property in algebraic structures such as groups, rings, and semigroups, given by $(a * b) * c = a * (b * c)$. It ensures that the grouping of elements does not affect the result. In BF-algebras, the operation $*$ is not necessarily associative, especially in the study of (r, l)-f_q and (l, r)-f_q derivations. To address this, BF-algebras satisfying the associative law, called associative BF-algebras, are considered.

Proposition 1. Let C be a BF-algebra. Then C is associative if and only if

$$c * (a * b) = (b * c) * a$$

for all $a, b, c \in C$.

Proof:

Suppose C is associative. Then

$$(a * b) * c = a * (b * c).$$

Applying the BF-axiom $0 * x = x$, we obtain

$$0 * ((a * b) * c) = 0 * (a * (b * c)).$$

Hence

$$c * (a * b) = (b * c) * a.$$

Conversely, suppose

$$c * (a * b) = (b * c) * a$$

for all $a, b, c \in C$. Applying $0*$ to both sides gives

$$0 * (c * (a * b)) = 0 * ((b * c) * a).$$

Thus

$$(a * b) * c = a * (b * c).$$

Therefore, C is associative.

Example 5. Consider the BF-algebra C in Example 3. It follows that, C is an associative BF-algebra.

Lemma 1. Let (C) be an associative BF-algebra. Then $0 * x = x$ for all $x \in C$.

Proof:

By axiom $(\mathrm{B1})$, $0 * 0 = 0$. Using Proposition 2, we have

$$(0 * x) * 0 = x * (0 * 0) = x.$$

On the other hand, by axiom $(\mathrm{B2})$,

$$(0 * x) * 0 = 0 * x.$$

Hence $0 * x = x$.

Proposition 2. Let $(C, *, 0)$ be an associative BF-algebra. Then, for all $a, b \in C$,

$$a * b = b * a.$$

Proof:

Let $a, b \in C$. By the BF-axiom,

$$0 * (a * b) = b * a.$$

Since $0 * a = a$ for all $a \in C$, we have

$$0 * (a * b) = a * b.$$

Thus,

$$a * b = b * a.$$

3. Main Results

This section presents the main results on f_q-derivations in BF-algebras. Various types of f_q-derivations, including inside, outside, and (ℓ, r) and (r, ℓ) derivations, are introduced and analyzed. Their fundamental properties in associative BF-algebras are established, together with results for BF-algebras that also satisfy the axioms of a B-algebra.

3.1. f_q-Derivations of BF-Algebras.

The relevant definitions are provided below to serve as a foundation for the study of their properties.

Definition 4. Let C be a BF-algebra, let f be an endomorphism of C, and let $q \in C$. The self-map $d_q^f : C \to C$ is defined by

$$\forall a \in C, \quad d_q^f(a) = f(a) * q.$$

Definition 5. Let C be a BF-algebra and let f be an endomorphism of C. A self-map $d_q^f : C \to C$ is called:

(1) an outside f_q-derivation of C if

$$\forall a, b \in C, \quad d_q^f(a * b) = f(a) * d_q^f(b);$$

(2) an inside f_q-derivation of C if

$$\forall a, b \in C, \quad d_q^f(a * b) = d_q^f(a) * f(b);$$

(3) an f_q-derivation of C if it is both an outside and inside f_q-derivation.

Example 6. Let $C = \{0, a, b, c\}$ be a BF-algebra in Example 3 and consider the endomorphism f defined in Example 4. Then the self-map d_q^f is both an inside and outside f_q-derivation of C. Hence, d_q^f is an f_q-derivation of C.

Definition 6. Let C be a BF-algebra and let f be an endomorphism of C. A self-map $d_q^f : C \to C$ is called:

(1) a left-right (ℓ, r) f_q-derivation (l, r) of C if

$$\forall a, b \in C, \quad d_q^f(a * b) = (d_q^f(a) * f(b)) \wedge (f(a) * d_q^f(b));$$

(2) a right-left (r, ℓ) f_q-derivation of C if

$$\forall a, b \in C, \quad d_q^f(a * b) = (f(a) * d_q^f(b)) \wedge (d_q^f(a) * f(b)).$$

Example 7. Let $C = \{0, a, b, c\}$ be a BF-algebra in Example 3 and consider the endomorphism f defined in Example 4. Then the self-map d_q^f is both an (ℓ, r) and (r, ℓ) f_q derivation of C.

Definition 7. The f_q-derivation d_q^f is called regular if

$$d_q^f(0) = 0.$$

3.2. **Properties of f_q-Derivations of Associative BF-Algebra.** By incorporating the associativity condition, several identities and structural behaviors of f_q-derivations are established and simplified. These results provide deeper insight into the interaction between the algebraic operation and the endomorphic mapping, leading to useful characterizations and applications in the study of BF-algebra structures.

Theorem 1. Let C be an associative BF-algebra, and let d_q^f be an (r, ℓ)- or (ℓ, r)-f_q-derivation on C. Then for any $a \in C$,

$$d_q^f(0) = \begin{cases} d_q^f(a) * f(a), & \text{if } d_q^f \text{ is an } (r, \ell)\text{-}f_q\text{-derivation,} \\ f(a) * d_q^f(a), & \text{if } d_q^f \text{ is an } (\ell, r)\text{-}f_q\text{-derivation.} \end{cases}$$

Proof:

Let C be an associative BF-algebra.

Case 1. If d_q^f be an $(r, l) f_q$ derivation of C, then for any $a \in C$,

$$\begin{aligned} d_q^f(0) &= d_q^f(a * a) \\ &= (f(a) * d_q^f(a)) \wedge (d_q^f(a) * f(a)) \\ &= (d_q^f(a) * f(a)) * \left[(d_q^f(a) * f(a)) * (f(a) * d_q^f(a))\right] \end{aligned}$$

Using B2 axiom,

$$(d_q^f(a) * f(a)) * (f(a) * d_q^f(a)) = [(d_q^f(a) * f(a)) * (f(a) * d_q^f(a))] * 0$$

Let $a = 0$, $b = d_q^f(a) * f(a)$, and $c = f(a) * d_q^f(a)$. By associativity,

$$\begin{aligned} [(d_q^f(a) * f(a)) * (f(a) * d_q^f(a))] * 0 &= (f(a) * d_q^f(a)) * [0 * (d_q^f(a) * f(a))] \\ &= (f(a) * d_q^f(a)) * (f(a) * d_q^f(a)) \end{aligned}$$

$$= 0$$

So,

$$\begin{aligned}(d_q^f(a) * f(a)) * \left[(d_q^f(a) * f(a)) * (f(a) * d_q^f(a))\right] &= (d_q^f(a) * f(a)) * 0 \\ &= (d_q^f(a) * f(a)).\end{aligned}$$

Therefore,

$$d_q^f(0) = d_q^f(a) * f(a).$$

Case 2. The proof is done similarly.

Theorem 2. Let C be an associative BF-algebra and let f be an endomorphism of C. Then, the following holds:

(i) d_0^f is an f_0-derivation; and

(ii) for any $q \in C$, d_q^f is an f_q-derivation of C.

Proof:

Let C be an associative BF-algebra and let f be an endomorphism of C.

For any $a, b \in C$,

$$d_q^f(a * b) = f(a * b) * q = (f(a) * f(b)) * q.$$

If $q = 0$, then

$$d_0^f(a * b) = f(a) * f(b).$$

Using (BF) and associativity,

$$f(a) * f(b) = (f(a) * 0) * f(b) = d_0^f(a) * f(b),$$

so d_0^f is an inside f_0-derivation. By Lemma 1, then

$$f(a) * f(b) = (f(b) * 0) * f(a) = d_0^f(b) * f(a) = f(a) * d_0^f(b),$$

so d_0^f is also an outside f_0-derivation.

For arbitrary q, associativity gives

$$(f(a) * f(b)) * q = f(b) * (f(a) * q) = f(b) * d_q^f(a),$$

and

$$(f(a) * f(b)) * q = f(a) * (f(b) * q) = f(a) * d_q^f(b).$$

Thus, d_q^f is both inside and outside, hence an f_q-derivation.

Example 8. Consider the BF-algebra in Example 3 and the endomorphism map in Example 4. Define $d_0^f(x) = f(x) * 0$ for all $x \in C$.

Let $x = a$ and $y = b$. From the Cayley table,

$$a * b = c.$$

Then

$$d_0^f(a * b) = f(c) * 0 = a * 0 = a.$$

Also,

$$d_0^f(a) * f(b) = (a * 0) * 0 = a * 0 = a.$$

Thus,

$$d_0^f(a * b) = d_0^f(a) * f(b),$$

so d_0^f is an inside f_0-derivation.

Now choose $x = b$ and $y = a$. From the table,

$$b * a = c.$$

Then

$$d_0^f(b * a) = f(c) * 0 = a * 0 = a.$$

On the other hand,

$$f(b) * d_0^f(a) = 0 * (f(a) * 0) = 0 * (a * 0) = 0 * a.$$

Using in the Cayley Table,

$$f(b) * d_0^f(a) = 0 * a = a.$$

Hence,

$$d_0^f(b * a) = f(b) * d_0^f(a).$$

This shows that the outside property depends essentially when the identity $0 * a = a$ is satisfied. And since C is an associative BF-algebra, $0 * a = a$, $\forall\ a \in C$. Therefore, d_0^f is always an f_0-derivation. Consequently, under this condition, d_q^f is an f_q-derivation for all $q \in C$.

Theorem 3. Let C be an associative BF-algebra and f be an endomorphism of C. If d_q^f is an inside derivation of C, then $d_q^f(0) = q$.

Proof:

Let C be an associative BF-algebra and f be an endomorphism of C.Then,

$$\begin{aligned} d_q^f(0) &= d_q^f(a) * f(a) \\ &= (f(a) * q) * f(a) \\ &= q * (f(a) * f(a)) \end{aligned}$$

$$= q.$$

Remark 1. The result shows that for inside f_q-derivations, the value of $d_q^f(0)$ is uniquely determined by the parameter q.

Theorem 4. Let C be an associative BF-algebra, and let d_q^f be an (ℓ, r)- or (r, ℓ)-f_q-derivation on C. If $d_q^f(a) = 0$ and $d_q^f(b) = 0$ for some $a, b \in C$, then

$$d_q^f(a * b) = \begin{cases} f(a) \wedge f(b), & \text{if } d_q^f \text{ is an } (r, \ell)\text{-}f_q\text{-derivation,} \\ f(b) \wedge f(a), & \text{if } d_q^f \text{ is an } (\ell, r)\text{-}f_q\text{-derivation.} \end{cases}$$

Proof:
Case 1. Let C be an associative BF-algebra and let d_q^f be an (r, l)-f_q derivation on C. Then, for some $a, b \in C$, let

$$d_q^f(a) = 0 \quad \text{and} \quad d_q^f(b) = 0.$$

$$\begin{aligned} d_q^f(a * b) &= (f(a) * d_q^f(b)) \wedge (d_q^f(a) * f(b)) \\ &= (f(a) * 0) \wedge (0 * f(b)) = f(a) \wedge (0 * f(b)) \end{aligned}$$

By Theorem 1,

$$(0 * f(b)) * 0 = f(b) * (0 * 0) = f(b).$$

Hence,

$$d_q^f(a * b) = f(a) \wedge f(b).$$

Case 2. The proof is done similarly.

Corollary 1. Let (C) be an associative BF-algebra, and let d_q^f be an f_q-derivation on C. If $d_q^f(a) = d_q^f(b) = 0$, then $d_q^f(a * b)$ is determined entirely by $f(a)$ and $f(b)$. More explicitly,

$$d_q^f(a * b) = \begin{cases} f(a) \wedge f(b), & \text{in the } (r, \ell)\text{-case,} \\ f(b) \wedge f(a), & \text{in the } (\ell, r)\text{-case.} \end{cases}$$

This corollary shows that the value of the derivation at $a * b$ depends only on the images of a and b under f whenever both are annihilated by d_q^f.

Example 9. Let $C = \{0, a, b, c\}$ be a BF-algebra with the operation given in Example 3. Define an endomorphism $f : C \to C$ by

$$f(0) = 0, \quad f(a) = a, \quad f(b) = a, \quad f(c) = 0,$$

Let $q = a$, then

$$d_q^f(x) = f(x) * q.$$

It follows that,

$$d_q^f(a) = a * a = 0, \qquad d_q^f(b) = a * a = 0.$$

Thus, $d_q^f(a) = 0$ and $d_q^f(b) = 0$.

i. (r, ℓ)-case:

$$d_q^f(a * b) = (f(a) * d_q^f(b)) \wedge (d_q^f(a) * f(b)).$$

Substituting,

$$= (a * 0) \wedge (0 * a) = a * (0 * 0) = a.$$

Hence,

$$d_q^f(a * b) = f(a) \wedge f(b).$$

ii. (ℓ, r)-case:

$$d_q^f(a * b) = (d_q^f(a) * f(b)) \wedge (f(a) * d_q^f(b)).$$

Substituting,

$$= (0 * a) \wedge (a * 0) = a * (0 * 0) = a.$$

Hence,

$$d_q^f(a * b) = f(b) \wedge f(a).$$

3.3. **Properties of f_q-Derivations of BF-Algebra that Is Also a B-Algebra.** It is known that not every BF-algebra is a B-algebra. As introduced by Walendziak et al. [12], BF-algebras are a generalization of B-algebras. This motivates the study of those BF-algebras that also satisfy the defining identity of B-algebras.

Theorem 5. Let C be a BF-algebra which is also a B algebra. If d_q^f is regular, then the following holds,

(i) if d_q^f be an (r, ℓ)-f_q-derivation of C, then $d_q^f(0) = f(a) * d_q^f(a)$.

(ii) if d_q^f be an (ℓ, r)-f_q-derivation of C, then $d_q^f(0) = d_q^f(a) * f(a)$.

$Proof$:

i. Let C be a BF-algebra, and let d_q^f be an (r, ℓ)-f_q-derivation on C. Assume further that C is also a B-algebra and d_q^f is regular. Then, $d_q^f(0) = 0$ and for any $a \in C$,

$$\begin{aligned} d_q^f(0) &= d_q^f(a * a) \\ &= (f(a) * d_q^f(a)) \wedge (d_q^f(a) * f(a)) \\ &= (d_q^f(a) * f(a)) * ((d_q^f(a) * f(a)) * (f(a) * d_q^f(a))) \end{aligned}$$

Since C is also a B algebra, applying $a * (b * c) = (a * (0 * c)) * b$ with $a = (d_q^f(a) * f(a))$, $b = (d_q^f(a) * f(a))$, and $c = (f(a) * d_q^f(a))$, we obtain

$$d_q^f(0) = ((d_q^f(a) * f(a)) * (0 * (f(a) * d_q^f(a)))) * (d_q^f(a) * f(a))$$

$$= ((d_q^f(a) * f(a)) * (d_q^f(a) * f(a))) * (d_q^f(a) * f(a))$$
$$= 0 * (d_q^f(a) * f(a))$$
$$= f(a) * d_q^f(a)$$

ii. The proof is done similarly.

4. Conclusion

This study introduced the concept of f_q-derivations in BF-algebras and investigated their fundamental properties within this framework. The notions of inside, outside, and (ℓ, r) and (r, ℓ) f_q-derivations were defined, providing a systematic approach to analyzing derivation behavior in BF-algebras.

A significant result of this study is the characterization of associativity in BF-algebras through the identity

$$c * (a * b) = (b * c) * a,$$

which is equivalent to the usual associative law and serves as a useful tool in simplifying computations involving derivations.

Moreover, several identities involving the zero element were established. In particular, explicit expressions for $d_q^f(0)$ were obtained under different derivation types, showing that its value depends on both the endomorphism and the algebraic structure. It was also shown that, under associativity and the condition $0 * a = a$, the mapping d_q^f satisfies both inside and outside properties, and hence becomes an f_q-derivation.

Special cases, including d_0^f, were examined and shown to satisfy derivation properties under suitable conditions. Furthermore, additional results were obtained for BF-algebras that also satisfy the axioms of a B-algebra, particularly in relation to regular f_q-derivations.

Overall, this work extends the theory of derivations to BF-algebras and provides foundational results that contribute to a deeper understanding of generalized derivations in non-classical algebraic systems. These results may serve as a basis for further investigations in algebraic structures and their applications.

Authors' Contributions. Kimberly V. Albino developed the main theoretical framework and carried out the mathematical analysis and proofs. Michelle T. Panganduyon, as an adviser, contributed to the mathematical analysis of the theorems and proofs, validated the results, and provided critical revisions to improve the manuscript. Mary Joy R. Latayada, also as an adviser, guided the organization of the study and contributed to identifying the depth and sufficiency of the paper, ensuring its clarity and completeness. All authors reviewed and approved the final version of the manuscript.

Acknowledgement. The authors would like to express their deepest gratitude to the Department of Science and Technology (DOST), particularly under the Strand N program, for the financial assistance and continuous support throughout the duration of the research. This support greatly contributed to the successful completion of the research by enabling access to academic resources, research opportunities, and professional development.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] D. Al-Kadi, f_q-Derivations of G-Algebra, Int. J. Math. Math. Sci. 2016 (2016), 9276096. https://doi.org/10.1155/2016/9276096.

[2] D. Al-Kadi, On (f,g)-Derivations of G-Algebras, Pure Math. Sci. 6 (2017), 95–104. https://doi.org/10.12988/pms.2017.7911.

[3] N. Al-Shehrie, Derivations of B-Algebras, J. King Abdulaziz Univ. Sci. 22 (2010), 71–83. https://doi.org/10.4197/Sci.22-1.5.

[4] L.K. Ardekani, B. Davvaz, On Generalized Derivations of BCI-Algebras and Their Properties, J. Math. 2014 (2014), 207161. https://doi.org/10.1155/2014/207161.

[5] G.T. Dejen, F.G. Tegegne, Derivations of BF-Algebras, Int. J. Innov. Sci. Res. Technol. 5 (2020), 329–333.

[6] J. Endam, J. Vilela, Homomorphism of BF-Algebras, Math. Slovaca 64 (2014), 13–20. https://doi.org/10.2478/s12175-013-0182-6.

[7] S. Gemawati, Mashadi, Musraini, E. Fitria, F_q-Derivation of BP-Algebras, J. Indones. Math. Soc. 29 (2023), 235–244. https://doi.org/10.22342/jims.29.2.1448.235-244.

[8] S. Gemawati, A. Sirait, M. M, E. Fitria, F_q-Derivations of BN_1-Algebras, Int. J. Math. Trends Technol. 67 (2021), 1–13. https://doi.org/10.14445/22315373/IJMTT-V67I11P501.

[9] E.D. Jayanti, Suatu Fq-Derivasi Pada BE-Aljabar, JOSTECH J. Sci. Technol. 4 (2024), 39–48. https://doi.org/10.15548/jostech.v4i1.8376.

[10] J. Neggers, H.S. Kim, On B-Algebras, Mat. Vesnik 54 (2002), 21–29.

[11] P. Muangkarn, C. Suanoom, P. Pengyim, A. Iampan, f_q-Derivations of b-Algebras, J. Math. Comput. Sci. 11 (2021), 2047–2057. https://doi.org/10.28919/jmcs/5472.

[12] A. Walendziak, On BF-Algebras, Math. Slovaca 57 (2007), 119–128. https://doi.org/10.2478/s12175-007-0003-x.