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NEW ESTIMATES OF LYAPUNOV-TYPE FUNCTIONS AND STABILITY ANALYSIS FOR A CLASS OF GENERALIZED FRACTIONAL DERIVATIVE WITH NON-SINGULAR KERNEL AND APPLICATIONS

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Abstract. In this paper, we propose an estimation of a class of generalized weighted fractional derivatives with non-singular kernel of Hattaf in the Caputo sense, applied to certain functions useful in the construction of Lyapunov-type candidate functions. We derive inequalities associated with the Caputo-Fabrizio fractional derivative in the Caputo sense, as well as with the Atangana-Baleanu fractional derivative in the Caputo sense. Moreover, we also establish an extension of the direct Lyapunov method to analyze the asymptotic stability of fractional differential equations with the generalized weighted fractional derivative of Hattaf in the sense of Caputo. Furthermore, we apply these results to prove the asymptotic stability of some examples of systems of differential equations governed by the generalized fractional derivative of Hattaf. In addition, for each example we carry out numerical simulations to illustrate our theoretical results.

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1. Introduction

The concept of fractional-order derivatives was defined in the 19th century by Riemann and Liouville. The aim was to generalize the notion of derivatives or integrals to non-integer and integer orders. This generalization led to several approaches to fractional differentiation, including the Riemann-Liouville (1832's) [15], Grünwald-Letnikov (1867's) [21], Hadamard (1892's) [21] and Caputo fractional derivatives (1967's) [9]. Applications of these fractional derivatives can be found in various fields such as ecology [32], hydrology, economics [18], automatic [8], control theory, rheology [4,23] and bioengineering [9,10,17,29].

In recent years, to more accurately model complex phenomena, researchers have introduced non

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local fractional differential operators with non-singular kernels, such as the Caputo-Fabrizio fractional derivative (2015's) [11] and Atangana-Baleanu fractional derivative (2016's) [3]. The concept of singular and non-singular weighted fractional derivatives was proposed with the aim of better accounting for memory effect in the dynamics of systems with memory [1]. Notable examples include the weighted fractional derivative of Caputo and Riemann-Liouville [1], the weighted fractional derivative of Caputo-Fabrizio [2], and the weighted fractional derivative of Atangana-Baleanu. In 2020, Hattaf [25] proposed a new definition of the generalized weighted fractional derivative in the sense of Riemann-Liouville and Caputo, with a non-singular kernel based on the Mittag-Leffler function. This fractional-order differential operator generalizes a broad class of classical non-local fractional derivatives with non-singular kernels. The theory and applications of these differential operators are discussed in [1,22]. Since the emergence of fractional-order derivatives, considerable attention has been devoted to the analysis of differential systems governed by weighted fractional derivatives. In [25,28], the author investigates the properties of Hattaf's generalized fractional derivative and establishes stability results using the direct Lyapunov method for nonlinear fractional differential equations associated with this operator. However, the extension of the direct Lyapunov method to fractional differential systems remains at an early stage and calls for further investigation, primarily due to the computational challenges involved in evaluating fractional derivatives of some Lyapunov candidate functions, particularly with Hattaf's fractional derivative. The fractional derivative presented in [25] encompasses a broad class of non-local fractional differential operators with non-singular kernels, thereby offering a unified framework for modeling memory dependent phenomena. Since the classical Leibniz rule is not satisfied by these operators, specialized analytical techniques are required to enable the application of the direct Lyapunov method. Accordingly, several inequalities related to these fractional differential operators have been developed in both the continuous and discrete settings [13,39,40]. This work has been motivated by more recent developments in the field [14,16].

The main objective of this work is to first establish inequalities with a class of the generalized fractional derivative of Hattaf in the sense of Caputo and to propose an extension of the direct method of Lyapunov for the analysis of the asymptotic stability of systems of generalized fractional differential equations of Hattaf in the sense of Caputo of order $\theta \in (0;1)$ and parameter $\beta \in (0;1]$. Then in a second step, to carry out direct applications of these results in the stability analysis of fractional differential systems.

The remainder of this paper is organized as follows. In section 2, we recall some basic notions useful for the rest of this work. Then, in section 3, we establish estimates of a class of generalized weighted fractional derivatives of Hattaf in the Caputo sense, applied to power-type functions as well as to an integral Lyapunov-type function. These inequalities extend to a large class of non-local differential operators with non-singular kernels, such as those of Caputo-Fabrizio and Atangana-Baleanu. In addition, we propose an extension result of the direct Lyapunov method for the analysis of the asymptotic stability of generalized nonlinear differential systems. Section 4 is devoted to the application of these results to some examples, along with numerical simulations. Finally, we end with a conclusion in section 5.

2. Preliminaries on the fractional calculus

In this section, we recall some notions that will be used in the next sections.

Throughout this paper, $\mathscr{C}([p;\infty))$ denotes the space of continuous functions defined on $[p;\infty)$, $\mathscr{C}^n([p;\infty))$ stands the class of all real valued functions defined on $[p;\infty)$ which have continuous n-th order derivative and $\mathbb{N}^*$ be the set of non-zero natural integers. Furthermore, we denote by $\mathbb{R}$ the set of real numbers; $\mathbb{R}_+ := \{z \in \mathbb{R}, z \geq 0\}$; $\mathbb{R}_{++} := \{z \in \mathbb{R}, z > 0\}$; $\mathbb{R}^n_+ := \{(z_1, z_2, \ldots, z_n), z_i \in \mathbb{R}_+, i = 1, 2, \ldots, n\}$ and $\mathbb{R}^n_{++} := \{(z_1, z_2, \ldots, z_n), z_i \in \mathbb{R}_{++}, i = 1, 2, \ldots, n\}$.

Let us recall the definition of the Mittag-Leffler function.

Definition 2.1. ([30,32]) *Let $\theta,\ \nu \in \mathbb{R}_{++}$. Then, the two-parameter Mittag-Leffler function $E_{\theta,\nu}(\cdot)$ is defined by the series expansion:*

$$E_{\theta,\nu}(q) = \sum_{n=0}^{\infty} \frac{q^n}{\Gamma(\nu + \theta n)}, \quad q \in \mathbb{R}$$

where $\Gamma(\cdot)$ is Gamma function.

As a particular case, when $\nu = 1$, we obtain the Mittag-Leffler function in one parameter:

$$E_{\theta,1}(q) = \sum_{n=0}^{\infty} \frac{q^n}{\Gamma(1 + \theta n)} =: E_\theta(q), \quad q \in \mathbb{R}.$$

Remark 2.1. ([12]) *If $\theta \in (0;1]$ and $\lambda < 0$, then $E_\theta(\lambda t^\theta) \xrightarrow[t\to\infty]{} 0$.*

We recall below the definitions of the Caputo-Fabrizio, Atangana-Baleanu, and Hattaf fractional derivatives with non-singular kernels, in the sense of Caputo.

Definition 2.2. ([11]) *Let $\theta \in [0;1)$ and $h \in \mathscr{C}^1([p;\infty))$. The Caputo-Fabrizio fractional derivative of order θ of h in Caputo sense is the function ${}^{CF}\mathcal{D}^\theta_p[h]$ defined on $[p;\infty)$ by:*

$$ {}^{CF}\mathcal{D}^\theta_p[h](t) := \frac{H(\theta)}{1-\theta} \int_p^t exp\left[-\frac{\theta}{1-\theta}(t-\zeta)\right] h'(\zeta) d\zeta \tag{1}$$

where h' is the first derivative of h and $H(\cdot)$ is a positive normalization function such that: $H(0) = H(1) = 1$.

Definition 2.3. ([3]) *Let $\theta \in [0;1)$ and $h \in \mathscr{C}^1([p;\infty))$. The Atangana-Baleanu fractional derivative of order θ of h in Caputo sense is the function ${}^{ABC}\mathcal{D}^\theta_p[h]$ defined on $[p;\infty)$ by:*

$$ {}^{ABC}\mathcal{D}^\theta_p[h](t) := \frac{H(\theta)}{1-\theta} \int_p^t E_\theta\left[-\frac{\theta}{1-\theta}(t-\zeta)^\theta\right] h'(\zeta) d\zeta. \tag{2}$$

Definition 2.4. ([25]) *Let $\theta \in [0;1)$, $\beta, \gamma \in \mathbb{R}_{++}$ and $h \in \mathscr{C}^1([p;\infty))$. The new generalized Hattaf fractional derivative of order θ of h in Caputo sense with respect to the weight function w is the function ${}^C\mathcal{D}_{p,w}^{\theta,\beta,\gamma}[h]$ defined on $[p;\infty)$ by:*

$$ {}^C\mathcal{D}_{p,w}^{\theta,\beta,\gamma}[h](t) := \frac{H(\theta)}{1-\theta}\frac{1}{w(t)}\int_p^t E_\beta\left[-\frac{\theta}{1-\theta}(t-\zeta)^\gamma\right](hw)'(\zeta)d\zeta \tag{3} $$

where $w \in \mathscr{C}^1([p;\infty))$, $w > 0$ on $[p;\infty)$.

Remark 2.2. *Note that formula (3) constitutes a generalization of a large class of fractional differential operators. If w is the identity function on $\mathbb{R}_+$, we have:*

(i) *If $\beta = \gamma \in [0;1]$, then (3) extends the generalized fractional derivative in Caputo sense [36];*
(ii) *If $\beta = \gamma = \theta$, then (3) extends the fractional derivative of Atangana-Baleanu [3];*
(iii) *If $\beta = \gamma = 1$, then (3) extends the fractional derivative of Caputo-Fabrizio [11].*

Definition 2.5. ([25]) *Let $\theta \in [0;1)$, $\beta \in \mathbb{R}_{++}$ and $h \in \mathscr{C}([p;\infty))$. The generalized fractional integral of order θ of h with respect to the weight function w associated with ${}^C\mathcal{D}_{p,w}^{\theta,\beta,\beta}[h]$ is the function $\mathcal{I}_{p,w}^{\theta,\beta,\beta}[h]$ defined on $[p;\infty)$ by:*

$$ \mathcal{I}_{p,w}^{\theta,\beta,\beta}[h](t) := \frac{1-\theta}{H(\theta)}h(t) + \frac{\theta}{H(\theta)}\frac{1}{\Gamma(\beta)}\frac{1}{w(t)}\int_p^t (t-\zeta)^{\beta-1}w(\zeta)h(\zeta)d\zeta $$

where $w \in \mathscr{C}([p;\infty))$, $w > 0$ on $[p;\infty)$.

Furthermore, we have:

$$ \mathcal{I}_{p,w}^{\theta,\beta,\beta}\left[{}^C\mathcal{D}_{p,w}^{\theta,\beta,\beta}[h]\right](t) = h(t) - \frac{w(p)h(p)}{w(t)}. \tag{4} $$

Lemma 2.1. ([25]) *The Laplace transform of $w{}^C\mathcal{D}_{0,w}^{\theta,\beta,\beta}[h]$ is given by:*

$$ \mathscr{L}\left\{w(t){}^C\mathcal{D}_{0,w}^{\theta,\beta,\beta}[h](t)\right\}(s) = \frac{H(\theta)}{1-\theta}\frac{s^\beta\mathscr{L}\{w(t)h(t)\}(s) - s^{\beta-1}w(0)h(0)}{s^\beta + \frac{\theta}{1-\theta}} $$

where $t \geq 0$, $\mathscr{L}\{\cdot\}$ stands the Laplace transform and s is the variable in the Laplace domain.

Definition 2.6. ([34]) *A continuous function $\Upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ function, if:*

(i) *Υ is strictly increasing;*
(ii) *$\Upsilon(0) = 0$;*
(iii) *$\Upsilon(t) > 0$, $t \in \mathbb{R}_{++}$.*

3. Useful inequalities and stability analysis

The following lemma establishes inequalities associated with the generalized fractional derivative with power-type functions. These types of inequalities are useful in the stability analysis of generalized fractional differential systems in the sense of Caputo by the direct Lyapunov method.

Lemma 3.1. *Let* $\theta \in (0;1)$, $\beta \in (0;1]$, $\gamma \in \mathbb{R}_{++}$, $f \in \mathscr{C}^1(\mathbb{R}_+)$ *and* $t_0 \in \mathbb{R}_+$.

For all $t \in \mathbb{R}_+$ *such that* $t \geq t_0$, *we have:*

(i) ${}^C\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f^{2m}\right](t) \leq 2f^m(t)\,{}^C\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f^m\right](t)$, $m \in \mathbb{N}^*$.

(ii) ${}^C\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f^{2^m}\right](t) \leq 2^m f^{2^m-1}(t)\,{}^C\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f\right](t)$, $m \in \mathbb{N}^*$.

(iii) ${}^C\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f^{\frac{2m}{n}}\right](t) \leq \frac{2m}{2m-n} f(t)\,{}^C\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f^{\frac{2m}{n}-1}\right](t)$, $m, n \in \mathbb{N}^*$ *with* $m \geq n$ *and* n *odd.*

(iv) ${}^C\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f^{\frac{2m}{n}}\right](t) \leq \frac{2m}{n} f^{\frac{2m}{n}-1}(t)\,{}^C\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f\right](t)$, $m, n \in \mathbb{N}^*$ *with* $2m \geq n$ *and* n *odd.*

Proof.

(i) It is enough to prove that:

$${}^C\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f^{2m}\right](t) - 2f^m(t)\,{}^C\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f^m\right](t) \leq 0.$$

For this, we pose: for all $t \in \mathbb{R}_+$ such that $t \geq t_0$

$$\Psi_1(t) = {}^C\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f^{2m}\right](t) - 2f^m(t)\,{}^C\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f^m\right](t).$$

Using the Definition 2.4, we have

$$\begin{aligned}
\Psi_1(t) &= \frac{H(\theta)}{1-\theta}\int_{t_0}^{t} E_\beta\left[-\frac{\theta}{1-\theta}(t-\zeta)^\gamma\right] 2mf'(\zeta)f^{2m-1}(\zeta)d\zeta \\
&- \frac{2f^m(t)H(\theta)}{1-\theta}\int_{t_0}^{t} E_\beta\left[-\frac{\theta}{1-\theta}(t-\zeta)^\gamma\right] mf'(\zeta)f^{m-1}(\zeta)d\zeta \\
&= \frac{H(\theta)}{1-\theta}\int_{t_0}^{t} E_\beta\left[-\frac{\theta}{1-\theta}(t-\zeta)^\gamma\right] 2mf'(\zeta)f^{m-1}(\zeta)\left[f^m(\zeta)-f^m(t)\right]d\zeta.
\end{aligned}$$

By setting $g_1(\zeta) = f^m(\zeta) - f^m(t)$, with $g_1'(\zeta) = mf'(\zeta)f^{m-1}(\zeta)$. It follows that:

$$\Psi_1(t) = \frac{H(\theta)}{1-\theta}\int_{t_0}^{t} E_\beta\left[-\frac{\theta}{1-\theta}(t-\zeta)^\gamma\right] 2g_1'(\zeta)g_1(\zeta)d\zeta. \tag{5}$$

By integration by parts (5), we obtain

$$\begin{aligned}
\Psi_1(t) &= \left[\frac{H(\theta)}{1-\theta}g_1^2(\zeta)E_\beta\left[-\frac{\theta(t-\zeta)^\gamma}{1-\theta}\right]\right]_{t_0}^{t} \\
&-\frac{\gamma\theta H(\theta)}{\beta(1-\theta)^2}\int_{t_0}^{t}(t-\zeta)^{\gamma-1}E_{\beta,\beta}\left[-\frac{\theta(t-\zeta)^\gamma}{1-\theta}\right]g_1^2(\zeta)d\zeta \\
&= -\frac{H(\theta)}{1-\theta}g_1^2(t_0)E_\beta\left[-\frac{\theta(t-t_0)^\gamma}{1-\theta}\right] \\
&-\frac{\gamma\theta H(\theta)}{\beta(1-\theta)^2}\int_{t_0}^{t}(t-\zeta)^{\gamma-1}E_{\beta,\beta}\left[-\frac{\theta(t-\zeta)^\gamma}{1-\theta}\right]g_1^2(\zeta)d\zeta \leq 0.
\end{aligned}$$

(ii) We prove that ${}^C\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f^{2^m}\right](t) - 2^m f^{2^m-1}(t)\,{}^C\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f\right](t) \leq 0$.

For this, we pose for all $t \in \mathbb{R}_+$ such that $t \geq t_0$

$$\Psi_2(t) = {}^C\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f^{2^m}\right](t) - 2^m f^{2^m-1}(t)\,{}^C\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f\right](t).$$

By using the Definition 2.4, we have

$$\begin{aligned}\Psi_2(t) &= \frac{H(\theta)}{1-\theta}\int_{t_0}^{t} E_\beta\left(-\frac{\theta}{1-\theta}(t-\zeta)^\gamma\right) 2^m f'(\zeta) f^{2^m-1}(\zeta)d\zeta \\ &- 2^m f^{2^m-1}(t)\frac{H(\theta)}{1-\theta}\int_{t_0}^{t} E_\beta\left(-\frac{\theta}{1-\theta}(t-\zeta)^\gamma\right) f'(\zeta)d\zeta \\ &= \frac{H(\theta)}{1-\theta}\int_{t_0}^{t} E_\beta\left(-\frac{\theta}{1-\theta}(t-\zeta)^\gamma\right)\left[2^m f'(\zeta) f^{2^m-1}(\zeta) - 2^m f^{2^m-1}(t) f'(\zeta)\right] d\zeta.\end{aligned}$$

We pose for all $\zeta \in [t_0, t]$, $g_2(\zeta) = f^{2^m}(\zeta) - 2^m f^{2^m-1}(t) f(\zeta) + (2^m - 1) f^{2^m}(t)$. As a result,

$$\Psi_2(t) = \frac{H(\theta)}{1-\theta}\int_{t_0}^{t} E_\beta\left(-\frac{\theta}{1-\theta}(t-\zeta)^\gamma\right) g_2'(\zeta)d\zeta. \tag{6}$$

By integration by parts (6), we obtain

$$\begin{aligned}\Psi_2(t) &= \left[\frac{g_2(\zeta)H(\theta)}{1-\theta}E_\beta\left(-\frac{\theta(t-\zeta)^\gamma}{1-\theta}\right)\right]_{t_0}^{t} \\ &\quad -\frac{\gamma\theta H(\theta)}{\beta(1-\theta)^2}\int_{t_0}^{t}(t-\zeta)^{\gamma-1}E_{\beta,\beta}\left(-\frac{\theta(t-\zeta)^\gamma}{1-\theta}\right) g_2(\zeta)d\zeta \\ &= -\frac{g_2(t_0)H(\theta)}{1-\theta}E_\beta\left(-\frac{\theta(t-t_0)^\gamma}{1-\theta}\right) \\ &\quad -\frac{\gamma\theta H(\theta)}{\beta(1-\theta)^2}\int_{t_0}^{t}(t-\zeta)^{\gamma-1}E_{\beta,\beta}\left(-\frac{\theta(t-\zeta)^\gamma}{1-\theta}\right) g_2(\zeta)d\zeta.\end{aligned}$$

We still have to prove that for all $\zeta \in [t_0; t]$, $g_2(\zeta) \geq 0$. By applying Young's inequality ([14]), we find

$$f(\zeta)f^{2^m-1}(t) \leq |f(\zeta)||f^{2^m-1}(t)| \leq \frac{2^m-1}{2^m}f^{2^m}(t) + \frac{1}{2^m}f^{2^m}(\zeta).$$

Consequently

$$\begin{aligned}g_2(\zeta) &= f^{2^m}(\zeta) - 2^m f^{2^m-1}(t) f(\zeta) + (2^m-1) f^{2^m}(t) \\ &\geq f^{2^m}(\zeta) - 2^m\left(\frac{2^m-1}{2^m}f^{2^m}(t) + \frac{1}{2^m}f^{2^m}(\zeta)\right) + (2^m-1) f^{2^m}(t) = 0.\end{aligned}$$

Therefore, $\Psi_2(t) \leq 0$, hence ${}^{C}\mathcal{D}_{t_0,1}^{\theta,\beta,\gamma}\left[f^{2^m}\right](t) \leq 2^m f^{2^m-1}(t)\,{}^{C}\mathcal{D}_{t_0,1}^{\theta,\beta,\gamma}[f](t)$.

The inequalities (iii) and (iv) can be proved similarly to (ii).

□

Remark 3.1. *We further observe that:*

(1) *If $m = 1$ in inequalities (i)-(ii) or $m = n$ in inequalities (iii)-(iv) of Lemma 3.1, we obtain the estimate with the quadratic Lyapunov function established in [27]: for all $t \in \mathbb{R}_+$ such that $t \geq t_0$*

$$ {}^{C}\mathcal{D}_{t_0,1}^{\theta,\beta,\gamma}\left[f^2\right](t) \leq 2f(t)\,{}^{C}\mathcal{D}_{t_0,1}^{\theta,\beta,\gamma}[f](t) \tag{7}$$

where $\theta \in (0;1)$, $\beta \in (0;1]$, $\gamma \in \mathbb{R}_{++}$ and $f \in \mathscr{C}^1(\mathbb{R}_+)$.

(2) *If $n = 1$, we obtain the following inequalities: for all $t \in \mathbb{R}_+$ such that $t \geq t_0$*

(8)
$$^{C}\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f^{2m}\right](t) \leq \frac{2m}{2m-1} f(t)^{C}\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f^{2m-1}\right](t)$$

(9)
$$^{C}\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[f^{2m}\right](t) \leq 2m f^{2m-1}(t)^{C}\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}[f](t)$$

where $m \in \mathbb{N}^*$, $\theta \in (0;1)$, $\beta \in (0;1]$, $\gamma \in \mathbb{R}_{++}$ and $f \in \mathscr{C}^1(\mathbb{R}_+)$.

We immediately deduce the following corollaries.

Corollary 3.1. *Let $\theta \in (0;1)$, $f \in \mathscr{C}^1(\mathbb{R}_+)$ and $t_0 \in \mathbb{R}_+$. For all $t \in \mathbb{R}_+$ such that $t \geq t_0$, we have:*

(i) $^{ABC}\mathcal{D}^{\theta}_{t_0}\left[f^{2m}\right](t) \leq 2f^m(t)^{ABC}\mathcal{D}^{\theta}_{t_0}\left[f^m\right](t)$, $m \in \mathbb{N}^*$.
(ii) $^{ABC}\mathcal{D}^{\theta}_{t_0}\left[f^{2^m}\right](t) \leq 2^m f^{2^m-1}(t)^{ABC}\mathcal{D}^{\theta}_{t_0}\left[f\right](t)$, $m \in \mathbb{N}^*$.
(iii) $^{ABC}\mathcal{D}^{\theta}_{t_0}\left[f^{\frac{2m}{n}}\right](t) \leq \frac{2m}{2m-n} f(t)^{ABC}\mathcal{D}^{\theta}_{t_0}\left[f^{\frac{2m}{n}-1}\right](t)$, *$m, n \in \mathbb{N}^*$ with $m \geq n$ and n odd.*
(iv) $^{ABC}\mathcal{D}^{\theta}_{t_0}\left[f^{\frac{2m}{n}}\right](t) \leq \frac{2m}{n} f^{\frac{2m}{n}-1}(t)^{ABC}\mathcal{D}^{\theta}_{t_0}\left[f\right](t)$, *$m, n \in \mathbb{N}^*$ with $2m \geq n$ and n odd.*

Proof. This is a direct consequence of the Lemma 3.1. Indeed, by setting $\beta = \gamma = \theta$, we obtain the result. □

Corollary 3.2. *Let $\theta \in (0;1)$, $f \in \mathscr{C}^1(\mathbb{R}_+)$ and $t_0 \in \mathbb{R}_+$. For all $t \in \mathbb{R}_+$ such that $t \geq t_0$, we have:*

(i) $^{CF}\mathcal{D}^{\theta}_{t_0}\left[f^{2m}\right](t) \leq 2f^m(t)^{CF}\mathcal{D}^{\theta}_{t_0}\left[f^m\right](t)$, $m \in \mathbb{N}^*$.
(ii) $^{CF}\mathcal{D}^{\theta}_{t_0}\left[f^{2^m}\right](t) \leq 2^m f^{2^m-1}(t)^{CF}\mathcal{D}^{\theta}_{t_0}\left[f\right](t)$, $m \in \mathbb{N}^*$.
(iii) $^{CF}\mathcal{D}^{\theta}_{t_0}\left[f^{\frac{2m}{n}}\right](t) \leq \frac{2m}{2m-n} f(t)^{CF}\mathcal{D}^{\theta}_{t_0}\left[f^{\frac{2m}{n}-1}\right](t)$, *$m, n \in \mathbb{N}^*$ with $m \geq n$ and n odd.*
(iv) $^{CF}\mathcal{D}^{\theta}_{t_0}\left[f^{\frac{2m}{n}}\right](t) \leq \frac{2m}{n} f^{\frac{2m}{n}-1}(t)^{CF}\mathcal{D}^{\theta}_{t_0}\left[f\right](t)$, *$m, n \in \mathbb{N}^*$ with $2m \geq n$ and n odd.*

Proof. This is a direct consequence of the Lemma 3.1. Indeed, by taking $\beta = \gamma = 1$, we obtain the result. □

Remark 3.2. *Corollaries 3.1 and 3.2 extend an original estimate for quadratic-type functions, which is useful for the stability analysis of nonlinear fractional order differential systems with Caputo-Fabrizio and Atangana-Baleanu fractional derivatives in the Caputo sense ([33,35]).*

Many works in the literature use an integral Lyapunov-type function as the main tool to study the stability of equilibrium points ([7,19,24,38], etc.). More recently, estimates of this type of function have been proposed in the Caputo, Caputo-Fabrizio and Atangana-Baleanu-Caputo fractional differential operators ([5,6]). In the following Lemma, we establish a new estimation of the generalized Hattaf fractional derivative in the Caputo sense applied to an integral Lyapunov-type function.

Lemma 3.2. *Let $z \in \mathscr{C}^1(\mathbb{R}_+)$ be a positive function, $z^* \in \mathbb{R}_{++}$ and $t_0 \in \mathbb{R}_+$. For all $t \in \mathbb{R}_+$ such that $t \geq t_0$*

(10)
$$^{C}\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}\left[\mathscr{W}_u(z)\right](t) \leq \left(1 - \frac{u(z^*)}{u(z(t))}\right)^{C}\mathcal{D}^{\theta,\beta,\gamma}_{t_0,1}[z](t), \quad \theta \in (0;1), \beta \in (0;1], \gamma \in \mathbb{R}_{++}$$

with $\mathscr{W}_u(z) = z - z^* - \int_{z^*}^{z} \frac{u(z^*)}{u(\zeta)} d\zeta$*; and* $u : \mathbb{R}_+ \to \mathbb{R}_+$ *a differentiable and strictly increasing function on* $\mathbb{R}_+$*.*

Proof. By using the Definition 2.4, we have

$$ {}^{C}\mathcal{D}_{t_0,1}^{\theta,\beta,\gamma} \left[\mathscr{W}_u(z)\right](t) = \frac{H(\theta)}{1-\theta} \int_{t_0}^{t} E_\beta \left(-\frac{\theta}{1-\theta}(t-\zeta)^\gamma\right) \left(1 - \frac{u(z^*)}{u(z(\zeta))}\right) z'(\zeta) d\zeta \tag{11} $$

and

$$ {}^{C}\mathcal{D}_{t_0,1}^{\theta,\beta,\gamma}[z](t) = \frac{H(\theta)}{1-\theta} \int_{t_0}^{t} E_\beta \left(-\frac{\theta}{1-\theta}(t-\zeta)^\gamma\right) z'(\zeta) d\zeta. \tag{12} $$

Since u is a function taking values in $\mathbb{R}_+$, expression (10) can be written as follows

$$ u(z(t))^{C}\mathcal{D}_{t_0,1}^{\theta,\beta,\gamma} \left[\mathscr{W}_u(z)\right](t) - u(z(t))^{C}\mathcal{D}_{t_0,1}^{\theta,\beta,\gamma}[z](t) + u(z^*)^{C}\mathcal{D}_{t_0,1}^{\theta,\beta,\gamma}[z](t) \leq 0. $$

Let $G(t) = u(z(t))^{C}\mathcal{D}_{t_0,1}^{\theta,\beta,\gamma} \left[\mathscr{W}_u(z)\right](t) - u(z(t))^{C}\mathcal{D}_{t_0,1}^{\theta,\beta,\gamma}[z](t) + u(z^*)^{C}\mathcal{D}_{t_0,1}^{\theta,\beta,\gamma}[z](t)$.
From (11)-(12), we arrive at

$$ G(t) = \frac{H(\theta)}{1-\theta} \int_{t_0}^{t} u(z^*) \left(1 - \frac{u(z(t))}{u(z(\zeta))}\right) E_\beta \left[-\frac{\theta}{1-\theta}(t-\zeta)^\gamma\right] z'(\zeta) d\zeta. $$

Using the integration by parts technique, we obtain

$$ \begin{aligned} G(t) &= -\left[\frac{H(\theta)u(z^*)E_\beta\left[-\frac{\theta(t-t_0)^\gamma}{1-\theta}\right]}{1-\theta} \left(z(t_0) - z(t) - \int_{z(t)}^{z(t_0)} \frac{u(z(t))}{u(\tau)} d\tau\right)\right] \\ &- \frac{\gamma\theta H(\theta)}{\beta(1-\theta)^2} \int_{t_0}^{t} (t-\zeta)^{\gamma-1} E_{\beta,\beta}\left(-\frac{\theta(t-\zeta)^\gamma}{1-\theta}\right) \mathscr{W}_u(z(\zeta)) d\zeta \\ &+ \lim_{\zeta \to t} \left[\frac{H(\theta)u(z^*)E_\beta\left(-\frac{\theta}{1-\theta}(t-\zeta)^\gamma\right)}{1-\theta} \left(z(\zeta) - z(t) - \int_{z(t)}^{z(\zeta)} \frac{u(z(t))}{u(\tau)} d\tau\right)\right]. \end{aligned} $$

Since $\lim_{\zeta \to t} \left[\frac{H(\theta)u(z^*)E_\beta\left(-\frac{\theta}{1-\theta}(t-\zeta)^\gamma\right)}{1-\theta} \left(z(\zeta) - z(t) - \int_{z(t)}^{z(\zeta)} \frac{u(z(t))}{u(\tau)} d\tau\right)\right] = 0$. We get

$$ \begin{aligned} G(t) &= -\left[\frac{H(\theta)u(z^*)E_\beta\left(-\frac{\theta(t-t_0)^\gamma}{1-\theta}\right)}{1-\theta} \left(z(t_0) - z(t) - \int_{z(t)}^{z(t_0)} \frac{u(z(t))}{u(\tau)} d\tau\right)\right] \\ &- \frac{\gamma\theta H(\theta)}{\beta(1-\theta)^2} \int_{t_0}^{t} (t-\zeta)^{\gamma-1} E_{\beta,\beta}\left(-\frac{\theta(t-\zeta)^\gamma}{1-\theta}\right) \mathscr{W}_u(z(\zeta)) d\zeta \leq 0. \end{aligned} $$

Therefore,

$$ {}^{C}\mathcal{D}_{t_0,1}^{\theta,\beta,\gamma} \left[\mathscr{W}_u(z)\right](t) - \left(\frac{u(z(t)) - u(z^*)}{u(z(t))}\right) {}^{C}\mathcal{D}_{t_0,1}^{\theta,\beta,\gamma}[z](t) \leq 0. $$

This completes the proof. □

Remark 3.3. *We note that our estimation with the generalized fractional derivative established in the Lemma 3.2 extends the inequalities established in recent work ([6,27,35]):*

(1) *If u is the identity function on $\mathbb{R}_+$, then (10) becomes*

$$ {}^{C}\mathcal{D}_{t_0,1}^{\theta,\beta,\gamma}\left[z - z^* - z^*\ln\left(\frac{z}{z^*}\right)\right](t) \le \left(1 - \frac{z^*}{z(t)}\right){}^{C}\mathcal{D}_{t_0,1}^{\theta,\beta,\gamma}[z](t), \tag{13}$$

where $\theta \in (0;1)$, $\beta \in (0;1]$ and $\gamma \in \mathbb{R}_{++}$. It is important to note that inequality (13) corresponds to the estimate of Lyapunov function established in [27] with the generalized fractional derivative in the Caputo sense.

(2) *For $\theta = \beta = \gamma$, the inequality (10) corresponds to the estimation of the Lyapunov function associated with the Atangana-Baleanu fractional derivation in the sense of Caputo introduced in [6]. Moreover, when u is the identity function on $\mathbb{R}_+$, we obtain the inequality established in [35] with the fractional derivation of Atangana-Baleanu.*

(3) *When $\beta = \gamma = 1$, in this case inequality (10) extends the estimation of the Lyapunov function with the Caputo-Fabrizio fractional derivation [6]. In particular, if u is the identity function on $\mathbb{R}_+$, then the inequality (10) corresponds to the estimate established with the Caputo-Fabrizio fractional derivative of the Lyapunov type function (see [6]).*

Using the Lyapunov's direct method and positive definite functions, we can obtain an extension of the asymptotic stability of fractional systems with the generalized weighted Hattaf fractional derivative in Caputo sense.

Theorem 3.1. *Let zero be an equilibrium point of the following non-autonomous generalized fractional system:*

$$\begin{cases} {}^{C}\mathcal{D}_{0,w}^{\theta,\beta,\beta}[y](t) = \mathscr{F}(t, y(t)), \\ y(0) = y_0. \end{cases}$$

With $\mathscr{F} : [0,\infty) \times D \to \mathbb{R}^n$ is a $\mathscr{C}^1$ function with $D \subset \mathbb{R}^n$ is a domain that contains the origin $y = 0$. Assume that there exists a continuously differentiable function $L(t, y) : [0,\infty) \times D \to \mathbb{R}$ and $\alpha_i (i = 1, 2, 3)$ three continuous positive definite functions on D such that

(i) $\alpha_1(y) \le L(t, y) \le \alpha_2(y)$,

(ii) ${}^{C}\mathcal{D}_{0,w}^{\theta,\beta,\beta}[L(\cdot, y(\cdot))](t) \le -\alpha_3(y(t))$,

for all $t \ge 0$, $\theta \in (0;1)$, $\beta \in (0;1]$, $w \in \mathscr{C}^1([0;\infty))$, $w > 0$ and $w' \ge 0$ on $[0;\infty)$. Then zero is asymptotically stable.

Proof. Let $B_r = \{y \in \mathbb{R}^n : \|y\| \le r\} \subset D$ for some $r > 0$. Since α_1, α_2 and α_3 are continuous and positive definite functions [20,34], then there exist three functions of class $\mathcal{K}$, β_1, β_2 and β_3 defined on $[0, r]$ such that $\alpha_1(y) \ge \beta_1(\|y\|)$; $\alpha_2(y) \le \beta_2(\|y\|)$ and $\alpha_3(y) \ge \beta_3(\|y\|)$. According to assumptions (i) and (ii) of Theorem 3.1, we have:

$$\beta_1(\|y\|) \le L(t, y) \le \beta_2(\|y\|) \tag{14}$$

and

$$ {}^{C}\mathcal{D}_{0,w}^{\theta,\beta,\beta}\left[L(\cdot,y(\cdot))\right](t) \leq -\beta_3(\|y\|). \tag{15}$$

We show that $y=0$ is Lyapunov stable.
β_3 being a class $\mathcal{K}$ function, from inequality (15), we obtain that ${}^{C}\mathcal{D}_{0,w}^{\theta,\beta,\beta}\left[L(\cdot,y(\cdot))\right](t) \leq 0$.
Consequently, $L(t,y) \leq L(0,y_0)$. Let $\epsilon > 0$ such that $\epsilon \leq r$ and $\eta = \beta_2^{-1} \circ \beta_1(\epsilon)$. If $\|y_0\| \leq \eta$, then: $L(0,y_0) \leq \beta_2(\|y_0\|) \leq \beta_2(\eta)$. Therefore $\beta_1(\|y\|) \leq \beta_2(\eta) \Rightarrow \|y\| \leq \beta_1^{-1} \circ \beta_2(\eta) = \epsilon$. We conclude that $y=0$ is stable.
Let us now show that $\lim_{t\to\infty} y(t) = 0$.
By combining relations (14) and (15) we arrive at

$$ {}^{C}\mathcal{D}_{0,w}^{\theta,\beta,\beta}\left[L(\cdot,y(\cdot))\right](t) \leq -\beta_3(\|y\|) \leq -\beta_3 \circ \beta_2^{-1}(L(t,y)).$$

Note that $\beta_3 \circ \beta_2^{-1}$ is a class $\mathcal{K}$ function defined on $[0,r]$. Using the comparison principle ([28], Lemma 1.), we obtain that $L(t,y)$ is bounded by the positive solution of the following fractional differential system

$$\begin{cases} {}^{C}\mathcal{D}_{0,w}^{\theta,\beta,\beta}[k](t) = -\beta_3 \circ \beta_2^{-1}(k(t)) \\ k(0) = L(0,y_0) \geq 0. \end{cases} \tag{16}$$

From the first equation of the system (16), we obtain that for all $t \in [0;\infty)$,

$$ {}^{C}\mathcal{D}_{0,w}^{\theta,\beta,\beta}[k](t) \leq 0. \tag{17}$$

By applying the generalized fractional integral operator to the inequality (17) and taking into account relation (4), we have:

$$k(t) - \frac{w(0)k(0)}{w(t)} \leq 0.$$

As for all $t \geq 0$, $\frac{w(0)}{w(t)} \leq 1$, it follows that $k(t) \leq k(0)$.
If $k(0) = 0$ then $k(t) = 0$ for all $t \in [0,\infty)$.
If $k(0) \neq 0$ then $\lim_{t\to\infty} k(t) = 0$. Indeed, suppose there exists $\epsilon > 0$ such that $k(t) \geq \epsilon$ for all $t \in [0,\infty)$.
From the first equation of system (16), we have

$$\begin{aligned} {}^{C}\mathcal{D}_{0,w}^{\theta,\beta,\beta}[k](t) &= -\beta_3 \circ \beta_2^{-1}(k(t)) \\ &\leq -\beta_3 \circ \beta_2^{-1}(\epsilon) \\ &= \frac{-\beta_3 \circ \beta_2^{-1}(\epsilon)}{k(0)} k(0). \end{aligned}$$

Given that $k(t) \leq k(0)$, it follows that

$$ {}^{C}\mathcal{D}_{0,w}^{\theta,\beta,\beta}[k](t) \leq \frac{-\beta_3 \circ \beta_2^{-1}(\epsilon)}{k(0)} k(t). \tag{18}$$

It follows from ([26], Corollary 1.), and under the additional assumption that $\beta \in (0,1]$, we obtain that:

$$k(t) \leq k(0)E_{\beta}\left(\rho t^{\beta}\right),$$

where $\rho = \frac{-\beta_3 \circ \beta_2^{-1}(\epsilon)\theta}{k(0)H(\theta)+\beta_3\circ\beta_2^{-1}(\epsilon)(1-\theta)} < 0$. According to Remark 2.1, $E_{\beta}(\rho t^{\beta}) \xrightarrow[t\to\infty]{} 0$, for all $\beta \in (0,1]$, which contradicts the hypothesis. As for all $t \in [0,\infty)$, $L(t,y(t)) \leq k(t)$, $L(t,y(t))$ converges to zero when $t \to \infty$. Therefore, $\|y(t)\| \leq \beta_1^{-1}(L(t,y(t)))$. We conclude that the origin is asymptotically stable. □

4. Illustrative examples

In this section we present two examples of use of the results established in section 3 then in each example we carry out some numerical simulations to illustrate our results. We use the numerical method developed in [28] to obtain the numerical solutions.

4.1. **Empirical example.**

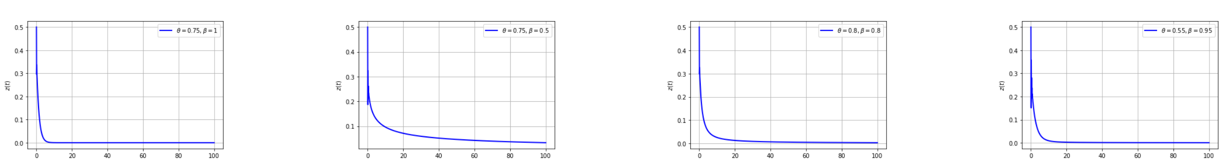
We consider the following nonlinear generalized fractional differential equation in the sense of Caputo:

$$\begin{cases} {}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[z](t) = -z^5(t) - z(t) \\ z(0) = 0.5 \end{cases} \tag{19}$$

where $z(t) \in \mathbb{R}$, ${}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}$ denotes the generalized fractional derivative of Hattaf in the sense of Caputo with $\theta \in (0,1)$ and $\beta \in (0,1]$. We will show that $z = 0$ is asymptotically stable. For this, we use Theorem 3.1. We consider the following Lyapunov candidate function $L(t,z(t)) = z^{\frac{6}{5}}(t)$. It is easy to see that $u_1(z(t)) \leq L(t,z(t)) \leq u_2(z(t))$ where $u_1(z(t)) = \frac{1}{2}z^{\frac{6}{5}}(t)$ and $u_2(z(t)) = 2z^{\frac{6}{5}}(t)$. Furthermore, by differentiating the Lyapunov function in time with the generalized fractional derivative and then using inequality (iv) of Lemma 3.1, we obtain:

$${}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[L(\cdot,z(\cdot))](t) \leq \frac{6}{5}z^{\frac{1}{5}}(t)\,{}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[z](t) = -\frac{6}{5}z^{\frac{26}{5}}(t) - \frac{6}{5}z^{\frac{6}{5}}(t).$$

By setting $u_3(z(t)) = \frac{6}{5}z^{\frac{26}{5}}(t) + \frac{6}{5}z^{\frac{6}{5}}(t)$, we obtain that ${}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[L(\cdot,z(\cdot))](t) \leq -u_3(z(t))$. According to Theorem 3.1, the fractional system (19) is asymptotically stable.



(a) (b) (c) (d)

Figure 1. Numerical simulation of the asymptotic behavior of the solution z of the fractional model (19) for certain values of the order of derivation θ and the parameter β.

4.2. **Application in ecology.** Motivated by the work presented in [31], we propose the generalized fractional-order model with memory effects of the transmission dynamics of foliar diseases in maize plants. Thus by replacing the classical derivatives in the ordinary differential system with the generalized fractional derivatives of Hattaf of the Caputo type, we obtain the following generalized model:

$$\begin{cases} {}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[S](t) = \Lambda_1 - f(S,Y) - \mu_1 S, \\ {}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[I](t) = f(S,Y) - (\delta + \mu_1) I, \\ {}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[X](t) = \Lambda_2 - g(X,I) - \mu_2 X, \\ {}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[Y](t) = g(X,I) - \mu_2 Y, \end{cases} \tag{20}$$

with the following initial conditions:

$$S(0) = S_0 > 0, \quad I(0) = I_0 \geq 0, \quad X(0) = X_0 \geq 0, \quad Y(0) = Y_0 \geq 0. \tag{21}$$

Where $\theta \in (0,1)$, $\beta \in (0,1]$; ${}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}$ denotes the generalized fractional derivative with $w(t) = 1$; and $f(S,Y) = \beta_1 SY$, $g(X,I) = \beta_2 XI$. In Table 1, we provide a description of all the states of the fractional system and a description of the biological parameters of the model.

TABLE 1. Notations and descriptions of the different classes and the biological parameters of the model.

Notations	Descriptions
S	class of susceptible of maize plants
I	class of infected of maize plants
X	class of vectors susceptible to foliar disease
Y	class of vectors infected to foliar disease
Λ_1	recruitment rate of susceptible maize plants
Λ_2	recruitment rate of susceptible vectors
μ_1	natural death rate of maize plants
μ_2	natural death rate of vectors
β_1	infection rate of susceptible maize by vector
β_2	infection rate of susceptible vector by maize
δ	mortality rate from infected state

In the following, we denote: $\frac{\partial f}{\partial S} = \partial_S f$, $\frac{\partial f}{\partial Y} = \partial_Y f$, $\frac{\partial g}{\partial X} = \partial_X g$ and $\frac{\partial g}{\partial I} = \partial_I g$.

Proposition 4.1. *Let $\epsilon > 0$, there exists $t_0 > 0$ such that for every solution $(S, I, X, Y) \in \mathbb{R}_+^4$ of the fractional system (20), we have:*

$$0 \leq S(t) + I(t) \leq \frac{\Lambda_1}{\mu_1} + \epsilon, \ 0 \leq X(t) + Y(t) \leq \frac{\Lambda_2}{\mu_2} + \epsilon, \ t \geq t_0. \tag{22}$$

Proof. By setting $N_1(t) = S(t) + I(t)$, we get

$$^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[N_1](t) = {}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[S](t) + {}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[I](t).$$

Consequently, by adding the first two equations of the generalized fractional system (20), we obtain

$$\begin{aligned} {}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[N_1](t) &= \Lambda_1 - \delta I(t) - \mu_1 I(t) - \mu_1 S(t) \\ &\leq \Lambda_1 - \min\{\delta + \mu_1; \mu_1\}(I(t) + S(t)) \\ &= \Lambda_1 - \mu_1 N_1(t). \end{aligned} \tag{23}$$

By applying the Laplace transform to both sides of the inequality (23), we obtain:

$$\begin{aligned} \frac{H(\theta)}{1-\theta}\frac{s^{\beta}\mathscr{L}\{N_1(t)\}(s) - s^{\beta-1}N_1(0)}{s^{\beta}+\eta} &\leq \frac{\Lambda_1}{s} - \mu_1\mathscr{L}\{N_1(t)\}(s) \\ \frac{s^{\beta}\mathscr{L}\{N_1(t)\}(s) - s^{\beta-1}N_1(0)}{s^{\beta}+\eta} &\leq \frac{(1-\theta)}{H(\theta)}\frac{\Lambda_1}{s} - \frac{\mu_1(1-\theta)}{H(\theta)}\mathscr{L}\{N_1(t)\}(s) \\ \left[\frac{s^{\beta}}{s^{\beta}+\eta} + \frac{\mu_1(1-\theta)}{H(\theta)}\right]\mathscr{L}\{N_1(t)\}(s) &\leq \frac{(1-\theta)}{H(\theta)}\frac{\Lambda_1}{s} + \frac{s^{\beta-1}N_1(0)}{s^{\beta}+\eta} \end{aligned}$$

where $\eta = \frac{\theta}{1-\theta}$ and $N_1(0) = S_0 + I_0$. Consequently,

$$\begin{aligned} \mathscr{L}\{N_1(t)\}(s) &\leq \frac{(1-\theta)\Lambda_1}{H(\theta)+\mu_1(1-\theta)} \times \frac{s^{\beta-1} + \eta s^{-1}}{s^{\beta} + \frac{\mu_1\eta(1-\theta)}{H(\theta)+\mu_1(1-\theta)}} + \frac{H(\theta)N_1(0)}{H(\theta)+\mu_1(1-\theta)}\frac{s^{\beta-1}}{s^{\beta} + \frac{\mu_1\eta(1-\theta)}{H(\theta)+\mu_1(1-\theta)}} \\ &= \frac{(1-\theta)\Lambda_1}{H(\theta)+\mu_1(1-\theta)}\left[\frac{s^{\beta-1}}{s^{\beta} + \frac{\mu_1\eta(1-\theta)}{H(\theta)+\mu_1(1-\theta)}} + \frac{\eta s^{-1}}{s^{\beta} + \frac{\mu_1\eta(1-\theta)}{H(\theta)+\mu_1(1-\theta)}}\right] \\ &\quad + \frac{H(\theta)N_1(0)}{H(\theta)+\mu_1(1-\theta)}\frac{s^{\beta-1}}{s^{\beta} + \frac{\mu_1\eta(1-\theta)}{H(\theta)+\mu_1(1-\theta)}}. \end{aligned} \tag{24}$$

By now applying the inverse Laplace transform to both sides of the inequality (24), we obtain

$$\begin{aligned} N_1(t) &\leq \frac{(1-\theta)\Lambda_1}{H(\theta)+\mu_1(1-\theta)}\left[\mathscr{L}^{-1}\left\{\frac{s^{\beta-1}}{s^{\beta} + \frac{\mu_1\eta(1-\theta)}{H(\theta)+\mu_1(1-\theta)}}\right\} + \mathscr{L}^{-1}\left\{\frac{\eta s^{-1}}{s^{\beta} + \frac{\mu_1\eta(1-\theta)}{H(\theta)+\mu_1(1-\theta)}}\right\}\right] \\ &\quad + \frac{H(\theta)N_1(0)}{H(\theta)+\mu_1(1-\theta)}\mathscr{L}^{-1}\left\{\frac{s^{\beta-1}}{s^{\beta} + \frac{\mu_1\eta(1-\theta)}{H(\theta)+\mu_1(1-\theta)}}\right\} \\ &= \frac{(1-\theta)\Lambda_1}{H(\theta)+\mu_1(1-\theta)}\left[E_{\beta}\left(-\frac{\mu_1(1-\theta)\eta t^{\beta}}{H(\theta)+\mu_1(1-\theta)}\right) + \eta t^{\beta}E_{\beta,\beta+1}\left(-\frac{\mu_1(1-\theta)\eta t^{\beta}}{H(\theta)+\mu_1(1-\theta)}\right)\right] \\ &\quad + \frac{H(\theta)N_1(0)}{H(\theta)+\mu_1(1-\theta)}E_{\beta}\left(-\frac{\mu_1(1-\theta)\eta}{H(\theta)+\mu_1(1-\theta)}t^{\beta}\right), \end{aligned}$$

where $\mathscr{L}^{-1}\{\cdot\}$ denotes the inverse Laplace transform. Thus,

$$\begin{aligned} N_1(t) &\leq \left[E_{\beta}\left(-\frac{\mu_1(1-\theta)\eta}{H(\theta)+\mu_1(1-\theta)}t^{\beta}\right) + \frac{\mu_1(1-\theta)\eta t^{\beta}}{H(\theta)+\mu_1(1-\theta)}E_{\beta,\beta+1}\left(-\frac{\mu_1(1-\theta)\eta}{H(\theta)+\mu_1(1-\theta)}t^{\beta}\right)\right] \\ &\quad \times \max\left\{\frac{(1-\theta)\Lambda_1}{H(\theta)+\mu_1(1-\theta)}; \frac{\Lambda_1}{\mu_1}\right\} + \frac{H(\theta)N_1(0)}{H(\theta)+\mu_1(1-\theta)}E_{\beta}\left(-\frac{\mu_1(1-\theta)\eta}{H(\theta)+\mu_1(1-\theta)}t^{\beta}\right) \end{aligned}$$

$$\begin{aligned} &= \max\left\{\frac{(1-\theta)\Lambda_1}{H(\theta)+\mu_1(1-\theta)};\frac{\Lambda_1}{\mu_1}\right\} + \frac{H(\theta)N_1(0)}{H(\theta)+\mu_1(1-\theta)}E_\beta\left(-\frac{\mu_1(1-\theta)\eta}{H(\theta)+\mu_1(1-\theta)}t^\beta\right) \\ &= \frac{\Lambda_1}{\mu_1} + \frac{H(\theta)N_1(0)}{H(\theta)+\mu_1(1-\theta)}E_\beta\left(-\frac{\mu_1(1-\theta)}{H(\theta)+\mu_1(1-\theta)}t^\beta\right) =: P(t). \end{aligned}$$

Consequently, if $t \to \infty$, $P(t) \to \frac{\Lambda_1}{\mu_1}$. Thus, for all $\epsilon > 0$, there exists $t_0 > 0$ for which $S(t)+I(t) \leq \frac{\Lambda_1}{\mu_1}+\epsilon$, $t \geq t_0$.

Similarly, we set $N_2(t) = X(t) + Y(t)$. Then, we obtain

$$N_2(t) \leq \frac{\Lambda_2}{\mu_2} + \frac{H(\theta)N_2(0)}{H(\theta)+\mu_2(1-\theta)}E_\beta\left(-\frac{\mu_2(1-\theta)}{H(\theta)+\mu_2(1-\theta)}t^\beta\right) =: Q(t)$$

where $N_2(0) = X_0 + Y_0$. Consequently, if $t \to \infty$, $Q(t) \to \frac{\Lambda_2}{\mu_2}$. Thus, for all $\epsilon > 0$, there exists $t_0 > 0$ such that $X(t) + Y(t) \leq \frac{\Lambda_2}{\mu_2} + \epsilon$, $t \geq t_0$. □

Remark 4.1. *The generalized fractional model (20) admits a disease-free equilibrium point* $\phi_0^* = (S_0^*, 0, X_0^*, 0)^T$ *with* $S_0^* = \frac{\Lambda_1}{\mu_1}$ *and* $X_0^* = \frac{\Lambda_2}{\mu_2}$.

Remark 4.2. *By using the Van Den Driessche and Watmough method [37], we obtain that the basic reproduction number of the generalized fractional model (20) given by:*

$$\mathcal{R}_0 = \sqrt{\frac{\partial_Y f(S_0^*,0)\partial_I g(X_0^*,0)}{\mu_2(\delta+\mu_1)}} = \sqrt{\frac{\beta_1\beta_2\Lambda_1\Lambda_2}{\mu_1\mu_2^2(\delta+\mu_1)}}. \tag{25}$$

Proposition 4.2. *If* $\mathcal{R}_0 > 1$, *then the generalized fractional model (20) admits a unique endemic equilibrium point* $\phi^* = (S^*, I^*, X^*, Y^*)^T$ *with*

$$S^* = \frac{\Lambda_1\mu_2(\beta_2 I^*+\mu_2)}{\beta_1\beta_2\Lambda_2 I^*+\mu_1\mu_2(\beta_2 I^*+\mu_2)} \qquad X^* = \frac{\Lambda_2}{\mu_2+\beta_2 I^*}$$
$$I^* = \frac{\mu_1\mu_2^2}{\beta_1\beta_2\Lambda_2+\mu_1\mu_2\beta_2}\left(\mathcal{R}_0^2-1\right) \qquad Y^* = \frac{\beta_2\Lambda_2 I^*}{\mu_2^2+\mu_2\beta_2 I^*}$$

where $\mathcal{R}_0$ *is given in (25).*

Proof. Let $\phi^* = (S^*, I^*, X^*, Y^*)^T$ be a positive equilibrium point of the system (20). The search for the equilibrium points of the generalized fractional model (20) is similar to the classical model. This amounts to solving the following system of equations:

$$\begin{cases} \Lambda_1 - \beta_1 S^*Y^* - \mu_1 S^* = 0, \\ \beta_1 S^*Y^* - (\delta+\mu_1)I^* = 0, \\ \Lambda_2 - \beta_2 X^*I^* - \mu_2 X^* = 0, \\ \beta_2 X^*I^* - \mu_2 Y^* = 0. \end{cases} \tag{26}$$

According to system (26), we obtain:

$$X^* = \frac{\Lambda_2}{\mu_2+\beta_2 I^*},$$

$$Y^* = \frac{\beta_2 X^* I^*}{\mu_2} = \frac{\beta_2 \Lambda_2 I^*}{\mu_2^2 + \mu_2 \beta_2 I^*},$$

$$S^* = \frac{\Lambda_1}{\beta_1 Y^* + \mu_1} = \frac{\Lambda_1 \mu_2 (\beta_2 I^* + \mu_2)}{\beta_1 \beta_2 \Lambda_2 I^* + \mu_1 \mu_2 (\beta_2 I^* + \mu_2)},$$

and

$$-(\delta + \mu_1)(\beta_1 \beta_2 \Lambda_2 + \mu_1 \mu_2 \beta_2) I^{*2} + (\Lambda_1 \Lambda_2 \beta_1 \beta_2 - \mu_1 \mu_2 (\delta + \mu_1)) I^* = 0. \tag{27}$$

It follows from equation (27) that: $I^* = 0$ or $I^* = \frac{\mu_1 \mu_2^2}{\beta_1 \beta_2 \Lambda_2 + \mu_1 \mu_2 \beta_2} \left(\mathcal{R}_0^2 - 1\right)$.
If $\mathcal{R}_0 \leq 1$, $I^* = \frac{\mu_1 \mu_2^2}{\beta_1 \beta_2 \Lambda_2 + \mu_1 \mu_2 \beta_2} \left(\mathcal{R}_0^2 - 1\right) \leq 0$, therefore $I^* = 0$ and in this case we obtain the disease-free equilibrium point $\phi_0^* = (S_0^*, 0, X_0^*, 0)^T$.
However, if $\mathcal{R}_0 > 1$, $I^* = \frac{\mu_1 \mu_2^2}{\beta_1 \beta_2 \Lambda_2 + \mu_1 \mu_2 \beta_2} \left(\mathcal{R}_0^2 - 1\right) > 0$, then we get the endemic equilibrium point $\phi^* = (S^*, I^*, X^*, Y^*)^T$. This completes the proof. □

Proposition 4.3. *When $\mathcal{R}_0 > 1$, the endemic equilibrium point ϕ^* of the generalized fractional model (20) is asymptotically stable.*

Proof. In order to use Theorem 3.1, we consider the candidate Lyapunov function:

$$\begin{aligned} \mathscr{W}(\phi)(t) &= p\left[S - S^* - \int_{S^*}^{S} \frac{f(S^*, I^*)}{f(\tau, I^*)} d\tau\right] + p\left[I - I^* - \int_{I^*}^{I} \frac{I^*}{v} dv\right] \\ &+ q\left[X - X^* - \int_{X^*}^{X} \frac{g(X^*, Y^*)}{g(\tau, Y^*)} d\tau\right] + q\left[Y - Y^* - \int_{Y^*}^{Y} \frac{Y^*}{v} dv\right] \end{aligned}$$

where $\phi = (S, I, X, Y)^T \in \mathbb{R}_{++}^4$, $p = g(X^*, I^*)$ and $q = f(S^*, Y^*)$.
It is easy to see that $\mathscr{W}(\phi^*) = 0$ and $\mathscr{W}(\phi)(t) \neq 0$ for $\phi \neq \phi^*$. We have for all $(S, I, X, Y) \in \mathbb{R}_{++}^4$, the Lyapunov function $\mathscr{W}$ satisfies $U_1(S, I, X, Y) \leq \mathscr{W}(S, I, X, Y) \leq U_2(S, I, X, Y)$, where $U_1(S, I, X, Y) = \frac{1}{2}\mathscr{W}(S, I, X, Y)$ and $U_2(S, I, X, Y) = 2\mathscr{W}(S, I, X, Y)$. U_1 and U_2 are continuous and positive definite on $\mathbb{R}_{++}^4$. Indeed, $U_1(\phi^*) = 0$, $U_1(S, I, X, Y) > 0$ for $(S, I, X, Y) \neq (S^*, I^*, X^*, Y^*)$; and $U_2(\phi^*) = 0$, $U_2(S, I, X, Y) > 0$ for $(S, I, X, Y) \neq (S^*, I^*, X^*, Y^*)$.
By differentiating $\mathscr{W}$ in time in the sense of generalized fractional derivative and taking account the linearity of ${}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}$ and using Lemma 3.2, we have:

$$\begin{aligned} {}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[\mathscr{W}(\phi)](t) &= p\,{}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}\left[S - S^* - \int_{S^*}^{S} \frac{f(S^*, Y^*)}{f(\tau, Y^*)} d\tau\right](t) + p\,{}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}\left[I - I^* - \int_{I^*}^{I} \frac{I^*}{v} dv\right](t) \\ &+ q\,{}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}\left[X - X^* - \int_{X^*}^{X} \frac{g(X^*, I^*)}{g(\tau, I^*)} d\tau\right](t) \\ &+ q\,{}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}\left[Y - Y^* - \int_{Y^*}^{Y} \frac{Y^*}{v} dv\right](t) \\ &\leq p\left(1 - \frac{f(S^*, Y^*)}{f(S, Y^*)}\right) {}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[S](t) + p\left(1 - \frac{I^*}{I(t)}\right) {}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[I](t) \\ &+ q\left(1 - \frac{g(X^*, I^*)}{g(X, I^*)}\right) {}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[X](t) + q\left(1 - \frac{Y^*}{Y(t)}\right) {}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[Y](t) \end{aligned}$$

$$
\begin{aligned}
&= p\left(1-\frac{f(S^*,Y^*)}{f(S,Y^*)}\right)[\Lambda_1 - f(S,Y) - \mu_1 S] \\
&+ q\left(1-\frac{g(X^*,I^*)}{g(X,I^*)}\right)[\Lambda_2 - g(X,I) - \mu_2 X] + q\left(1-\frac{Y^*}{Y(t)}\right)[g(X,I) - \mu_2 Y] \\
&+ p\left(1-\frac{I^*}{I(t)}\right)[f(S,Y) - (\delta+\mu_1)I].
\end{aligned}
$$

Using the following conditions for the endemic equilibrium:

$$
\begin{cases}
\Lambda_1 = f(S^*,Y^*) + \mu_1 S^* \\
f(S^*,Y^*) = (\delta-\mu_1)I^* \\
\Lambda_2 = g(X^*,I^*) + \mu_2 X^* \\
g(X^*,I^*) = \mu_2 Y^*
\end{cases}
$$

we obtain

$$
\begin{aligned}
{}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[\mathscr{W}(\phi)](t) &\leq g(X^*,I^*)\left(1-\frac{f(S^*,Y^*)}{f(S,Y^*)}\right)[f(S^*,Y^*) + \mu_1 S^* - f(S,Y) - \mu_1 S] \\
&+ g(X^*,I^*)\left(1-\frac{I^*}{I(t)}\right)\left[f(S,Y) - \frac{f(S^*,Y^*)I}{I^*}\right] \\
&+ f(S^*,Y^*)\left(1-\frac{g(X^*,I^*)}{g(X,I^*)}\right)[g(X^*,I^*) + \mu_2 X^* - g(X,I) - \mu_2 X] \\
&+ f(S^*,Y^*)\left(1-\frac{Y^*}{Y(t)}\right)\left[g(X,I) - \frac{g(X^*,I^*)Y}{Y^*}\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
{}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[\mathscr{W}(\phi)](t) &\leq g(X^*,I^*)f(S^*,Y^*)\left[2 - \frac{I}{I^*} + \frac{f(S,Y)}{f(S,Y^*)} - \frac{f(S^*,Y^*)}{f(S,Y^*)} - \frac{f(S,Y)I^*}{f(S^*,Y^*)I}\right] \\
&- \mu_2 f(S^*,Y^*)(X-X^*)\left(1-\frac{g(X^*,I^*)}{g(X,I^*)}\right) \\
&+ f(S^*,Y^*)g(X^*,I^*)\left[2 - \frac{Y}{Y^*} + \frac{g(X,I)}{g(X,I^*)} - \frac{g(X^*,I^*)}{g(X,I^*)} - \frac{g(X,I)Y^*}{g(X^*,I^*)Y}\right] \\
&- \mu_1 g(X^*,I^*)(S-S^*)\left(1-\frac{f(S^*,Y^*)}{f(S,Y^*)}\right) \\
&= -\mu_1 g(X^*,I^*)(S-S^*)\left(1-\frac{f(S^*,Y^*)}{f(S,Y^*)}\right) \\
&- \mu_2 f(S^*,Y^*)(X-X^*)\left(1-\frac{g(X^*,I^*)}{g(X,I^*)}\right) \\
&+ f(S^*,Y^*)g(X^*,I^*) \times P(S,I,X,Y)
\end{aligned}
$$

where $P(S,I,X,Y) = \left[4 - \frac{I}{I^*} - \frac{Y}{Y^*} + \frac{f(S,Y)}{f(S,Y^*)} - \frac{f(S^*,Y^*)}{f(S,Y^*)} - \frac{f(S,Y)I^*}{f(S^*,Y^*)I} + \frac{g(X,I)}{g(X,I^*)} - \frac{g(X^*,I^*)}{g(X,I^*)} - \frac{g(X,I)Y^*}{g(X^*,I^*)Y}\right]$.
Furthermore, by adding and the subtracting $2 + \ln\frac{f(S,Y)}{f(S,Y^*)} + \ln\frac{f(S^*,Y^*)}{f(S,Y^*)} + \ln\frac{I}{I^*} + \ln\frac{g(X,I)}{g(X,I^*)} + \ln\frac{g(X^*,I^*)}{g(X,I^*)} +$

$\ln \frac{Y}{Y^*}$ from the expression of $P(S,I,X,Y)$, it follows that

$$\begin{aligned}
{}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[\mathscr{W}(\phi)](t) &\leq -\mu_1(S-S^*)p\left(1-\frac{f(S^*,Y^*)}{f(S,Y^*)}\right) - \mu_2 q(X-X^*)\left(1-\frac{g(X^*,I^*)}{g(X,I^*)}\right) \\
&+ pq\left(1-\frac{f(S^*,Y^*)}{f(S,Y^*)}+\ln\frac{f(S^*,Y^*)}{f(S,Y^*)}\right) + pq\left(1-\frac{f(S,Y)I^*}{f(S^*,Y^*)I}+\ln\frac{I^*f(S,Y)}{If(S^*,Y^*)}\right) \\
&+ pq\left(\frac{f(S,Y)}{f(S,Y^*)}-1-\ln\frac{f(S,Y)}{f(S,Y^*)}\right) + pq\left(1-\frac{g(X^*,I^*)}{g(X,I^*)}+\ln\frac{g(X^*,I^*)}{g(X,I^*)}\right) \\
&+ pq\left(1-\frac{Y^*g(X,I)}{Yg(X^*,I^*)}+\ln\frac{Y^*g(X,I)}{Yg(X^*,I^*)}\right) + pq\left(\frac{g(X,I)}{g(X,I^*)}-1-\ln\frac{g(X,I)}{g(X,I^*)}\right) \\
&+ pq\left(1-\frac{I}{I^*}+\ln\frac{I}{I^*}\right) + pq\left(1-\frac{Y}{Y^*}+\ln\frac{Y}{Y^*}\right). \qquad (28)
\end{aligned}$$

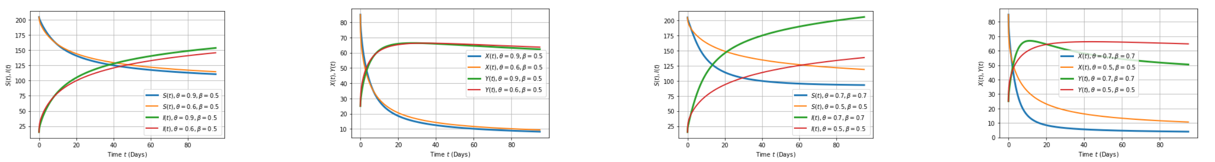
Let us note that for all $(S,I,X,Y) \in \mathbb{R}^4_{++}$, $\partial_S f(S,Y) > 0$ and $\partial_X g(X,I) > 0$. It follows that: if $S \leq S^*$, then $1-\frac{f(S^*,Y^*)}{f(S,Y^*)} \leq 0$ and if $S \geq S^*$, then $1-\frac{f(S^*,Y^*)}{f(S,Y^*)} \geq 0$. This implies that $(S-S^*)\left(1-\frac{f(S^*,Y^*)}{f(S,Y^*)}\right) \geq 0$. Similarly, we have: if $X \leq X^*$, then $1-\frac{g(X^*,I^*)}{g(X,I^*)} \leq 0$, and if $X \geq X^*$, then $1-\frac{g(X^*,I^*)}{g(X,I^*)} \geq 0$. Therefore, $(X-X^*)\left(1-\frac{g(X^*,I^*)}{g(X,I^*)}\right) \geq 0$. Moreover, considering L the function defined on $\mathbb{R}_{++}$ by $L(s) = s-1-\ln(s)$, we obtain $L\left(\frac{f(S,Y)}{f(S,Y^*)}\right) = L\left(\frac{Y}{Y^*}\right)$ and $L\left(\frac{g(X,I)}{g(X,I^*)}\right) = L\left(\frac{I}{I^*}\right)$.

Let us set

$$\begin{aligned}
U_3(S,I,X,Y) &= \mu_1(S-S^*)p\left(1-\frac{f(S^*,Y^*)}{f(S,Y^*)}\right) + \mu_2(X-X^*)q\left(1-\frac{g(X^*,I^*)}{g(X,I^*)}\right) \\
&+ pq\left(\frac{f(S^*,Y^*)}{f(S,Y^*)}-1-\ln\frac{f(S^*,Y^*)}{f(S,Y^*)}\right) + pq\left(\frac{f(S,Y)I^*}{f(S^*,Y^*)I}-1-\ln\frac{I^*f(S,Y)}{If(S^*,Y^*)}\right) \\
&+ pq\left(1-\frac{f(S,Y)}{f(S,Y^*)}+\ln\frac{f(S,Y)}{f(S,Y^*)}\right) + pq\left(\frac{g(X^*,I^*)}{g(X,I^*)}-1-\ln\frac{g(X^*,I^*)}{g(X,I^*)}\right) \\
&+ pq\left(\frac{Y^*g(X,I)}{Yg(X^*,I^*)}-1-\ln\frac{Y^*g(X,I)}{Yg(X^*,I^*)}\right) + pq\left(1-\frac{g(X,I)}{g(X,I^*)}+\ln\frac{g(X,I)}{g(X,I^*)}\right) \\
&+ pq\left(\frac{I}{I^*}-1-\ln\frac{I}{I^*}\right) + pq\left(\frac{Y}{Y^*}-1-\ln\frac{Y}{Y^*}\right) \\
&= \mu_1(S-S^*)p\left(1-\frac{f(S^*,Y^*)}{f(S,Y^*)}\right) + \mu_2(X-X^*)q\left(1-\frac{g(X^*,I^*)}{g(X,I^*)}\right) \\
&+ pq\left[L\left(\frac{f(S^*,Y^*)}{f(S,Y^*)}\right) + L\left(\frac{I}{I^*}\right) + L\left(\frac{f(S,Y)I^*}{f(S^*,Y^*)I}\right) - L\left(\frac{f(S,Y)}{f(S,Y^*)}\right)\right] \\
&+ pq\left[L\left(\frac{Y^*g(X,I)}{Yg(X^*,I^*)}\right) - L\left(\frac{g(X,I)}{g(X,I^*)}\right) + L\left(\frac{Y}{Y^*}\right) + L\left(\frac{g(X^*,I^*)}{g(X,I^*)}\right)\right] \\
&= \mu_1(S-S^*)p\left(1-\frac{f(S^*,Y^*)}{f(S,Y^*)}\right) + \mu_2(X-X^*)q\left(1-\frac{g(X^*,I^*)}{g(X,I^*)}\right) \\
&+ pq\left[L\left(\frac{f(S^*,Y^*)}{f(S,Y^*)}\right) + L\left(\frac{f(S,Y)I^*}{f(S^*,Y^*)I}\right) + L\left(\frac{g(X^*,I^*)}{g(X,I^*)}\right) + L\left(\frac{Y^*g(X,I)}{Yg(X^*,I^*)}\right)\right].
\end{aligned}$$

We observe that U_3 is continuous positive definite function. From relation (28), it follows that ${}^{C}\mathcal{D}_{0,1}^{\theta,\beta,\beta}[\mathscr{W}(\phi)](t) \leq -U_3(S,I,X,Y)$. Therefore, according to Theorem 3.1, the endemic equilibrium point ϕ^* is asymptotically stable when $\mathcal{R}_0 > 1$. $\square$

For numerical simulations, we select the normalized function H given by: $H(\theta) = 1 - \theta + \frac{\theta}{\Gamma(\theta)}$, $\theta \in (0; 1)$. The values of the model parameters used for the numerical simulations from the literature [31] are: $\Lambda_1 = 10\text{day}^{-1}$, $\Lambda_2 = 5.32\text{day}^{-1}$, $\mu_1 = 0.0103\text{day}^{-1}$, $\mu_2 = 0.1236\text{day}^{-1}$, $\beta_1 = 0.024\text{day}^{-1}$, $\beta_2 = 0.0078\text{day}^{-1}$ and $\delta = 0.0206\text{day}^{-1}$.



(a) (b) (c) (d)

FIGURE 2. Numerical simulation of the asymptotic behavior of the fractional model (20) for certain values of the order of derivation θ and the parameter β.

5. Conclusion

In this paper, we established new inequalities using power-type and integral-type functions associated with the generalized fractional derivative with non-singular kernel of Hattaf in the sense of Caputo. We also proposed an extension of the direct Lyapunov method to analyze the asymptotic stability of nonlinear differential systems involving the generalized weighted fractional derivative of Hattaf of order $\theta \in (0; 1)$ and the parameter $\beta \in (0; 1]$. Two examples were presented to illustrate the application of these results to study of the asymptotic stability of generalized fractional systems of Hattaf in the sense of Caputo. Numerical simulations were carried out to illustrate our results. This work is part of a framework of the application of generalized fractional derivatives to the analysis of complex systems.

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