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SOME OPERATIONS IN HUB-INTEGRITY OF GRAPHS

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Abstract. The hub-integrity of a connected graph $G = (V(G), E(G))$ is denoted as $HI(G)$ and defined by $HI(G) = \min\{|S| + m(G-S), S \; is \; a \; hub \; set \; of \; G\}$, where $m(G-S)$ is the order of a maximum component of $G-S$. In this paper we discuss hub-integrity of square graph of path as well as we give some results connecting the hub-integrity of binary graph operations.

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Key words and phrases. integrity, hub-integrity, square graph, binary operation.

1. Introduction

We begin with simple, finite, connected and undirected graph G with vertex set $V(G)$ and edge set $E(G)$. For graph theoretic terminology, we refer to Harary [9]. In the remaining portion of this section we will give brief summary of definitions and information related to the present work.

The vulnerability of network have been studied in various contexts including road transportation system, information security, structural engineering and communication network. A graph structure is vulnerable if ąőany small damage produces large consequencesąŕ. In a communication network, the vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations (junctions) or communication links (connections). In the theory of graphs, the vulnerability implies a lack of resistance(weakness) of graph network arising from deletion of vertices or edges or both. Communication networks must be so designed that they do not easily get disrupted under external attack and even if they get disturbed then they should be easily reconstructible. Many graph theoretic parameters have been introduced to describe the vulnerability of communication networks including binding number, rate of disruption, toughness, neighbor-connectivity, integrity, mean integrity, edge-connectivity and tenacity. In the analysis of the vulnerable communication network two quantities are playing vital role, namely (i) the number of elements that are not functioning (ii) the size of the largest remaining (survived) sub network within which mutual communication can still occur. In adverse relationship it is desirable that an opponentąŕs network would be such that the above referred two quantities can be made simultaneously small. Here the first parameter provides an information about nodes which can be targeted for more disruption while the later gives the impact of damage after disruption. To estimate these quantities Barefoot et al. [2] have introduced the concept of integrity, which is defined as follows.

Definition 1.1. [2] *The integrity of a graph G is denoted by $I(G)$ and defined by $I(G) = \min\{|S| + m(G-S) : S \subseteq V(G)\}$, where $m(G-S)$ denotes the order of a maximum component of $G-S$.*

Definition 1.2. [2] *A subset S of $V(G)$ is said to be an I-set, if $I(G) = |S| + m(G-S)$.*

The parameters of integrity and edge-integrity were introduced by Barefoot, Entringer and Swart in [2] and were studied more extensively by the same authors in [3]. Computational aspects of these parameters were studied in [4, 5]. Some general results on the interrelations between integrity and other graph parameters are investigated by Goddard and Swart [7] while Mamut and Vumar [11] have determined the integrity of middle graph of some graphs. It is also observed that bigger the integrity of network, more reliable functionality of the network after any disruption caused by nonfunctional devices (elements). The connectivity is useful to identify local weaknesses in some respect while integrity gives brief account of vulnerability of the graph network.

Suppose that $H \subseteq V(G)$ and let $x, y \in V(G)$. An H-path between x and y is a path where all intermediate vertices are from H. (This includes the degenerate cases where the path consists of the single edge xy or a single vertex x if $x = y$, call such an H-path trivial). A set $H \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) - H$, there is an H-path in G between x and y. The smallest size of a hub set in G is called the hub number of G, and is denoted by $h(G)$ [8, 12].
The concept of hub-integrity was introduced by Sultan et al. [10].

Definition 1.3. [10] *The hub-integrity of a graph G denoted by $HI(G)$ is defined by, $HI(G) = \min\{|S| + m(G-S), S\ is\ a\ hub\ set\ of\ G\}$, where $m(G-S)$ is the order of a maximum component of $G-S$.*

Definition 1.4. [9] *For a simple connected graph G the square of G denoted by G^2, is defined as the graph with the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 in G .*

Many results on the integrity of graphs in the context of union, join, composition and product of two graphs have been reported by Goddard and Swart (1988) [6]. The present work is intended to investigate the hub-integrity of a square graph of P_p, composition (lexicographic product), (Cartesian)product and join of two graphs. We need the following to prove main results.

Theorem 1.5. [1] *For any graphs G and H, $I(G+H) = min\{I(G)+|V(H)|, I(H)+|V(G)|\}$.*

Theorem 1.6. [2] *The integrity of*

- **(a):** *the complete graph K_p is p,*
- **(b):** *the star $K_{1,n}$ is 2,*
- **(c):** *the path P_p is $\lceil 2\sqrt{p+1} \rceil - 2$,*
- **(d):** *the cycle C_p is $\lceil 2\sqrt{p} \rceil - 1$.*

Proposition 1.7. [10]

(1) *The hub-integrity of the complete graph K_p is p.*
(2) *For any path P_p with $p \geq 3$, $HI(P_p) = p-1$.*
(3) *For any cycle C_p,*

$$HI(C_p) = \begin{cases} p-1, & \text{if } p = 4,5; \\ p-2, & \text{if } p \geq 6. \end{cases}$$

2. Main Results

Theorem 2.1.

$$HI(P_p^2) = \begin{cases} 2 & \text{if } p = 2, \\ 3 & \text{if } p = 3,4, \\ \frac{2p}{3} & \text{if } p \geq 5 \text{ and } p \equiv 0 (mod\ 3), \\ \frac{2(p-1)}{3}+1 & \text{if } p \geq 5 \text{ and } p \equiv 1 (mod\ 3), \\ \frac{2p+2}{3} & \text{if } p \geq 5 \text{ and } p \equiv 2 (mod\ 3). \end{cases}$$

Proof. Let $V(P_p) = \{v_1, v_2, ..., v_p\}$. Then, $|V(P_p^2)| = p$ and $|E(P_p^2)| = 2p-3$. We consider the following three cases:
Case 1: $p = 2$. P_2^2 is isomorphic to complete graph K_2. By Proposition 1.7, $HI(P_2^2) = 2$.
Case 2: $p = 3,4$. For $p = 3$, P_3^2 is isomorphic to complete graph K_3. By Proposition 1.7, $HI(P_3^2) = 3$. For $p = 4$, consider $S = \{v_2, v_3\}$ which is a hub set for P_4^2 and $m(P_4^2 - S) = 1$. Therefore, $|S| + m(P_4^2 - S) = 3$. For $S = \{v_1, v_3\}$, $\{v_2, v_4\}$, $\{v_1, v_2\}$ or $\{v_3, v_4\}$, $m(P_4^2 - S) = 2$, then we get $|S| + m(P_4^2 - S) = 4$. If $S = \{v_i\}$, $i = 1,2,3,4$, then

$m(P_4^2 - S) = 3$, so $|S| + m(P_4^2 - S) = 4$. Hence $HI(P_4^2) = 3$.
Case 3: $p \geq 5$. We consider a hub set S of $V(P_p^2)$ as below:

- If $p \equiv 0(mod\ 3)$, then $p = 3k$ for some integer $k \geq 2$. Consider

$$S = \{v_{3+3i}, v_{4+3i}/0 \leq i \leq k-2\}\ and\ |S| = 2(k-1).$$

  We have, $|S| = \frac{2p}{3} - 2$.
- If $p \equiv 1(mod\ 3)$ then $p = 3k+1$ for some integer $k \geq 2$. Consider

$$S = \{v_{3+3i}, v_{4+3i}/0 \leq i \leq k-2\} \cup \{v_{p-1}\}\ and\ |S| = 2k-1.$$

  We have, $|S| = \frac{2(p-1)}{3} - 1$.
- If $p \equiv 2(mod\ 3)$ then $p = 3k-1$ for some integer $k \geq 2$. Consider

$$S = \{v_{3+3i}, v_{4+3i}/0 \leq i \leq k-2\}\ and\ |S| = 2k-2.$$

  We have, $|S| = \frac{2p+2}{3} - 2$.

In all the above cases S is a hub set for P_p^2 and $m(P_p^2 - S) = 2$.
Now, we discuss the minimality of $|S| + m(P_p^2 - S)$. If we consider any hub set S_1 of P_p^2 such that, $|S_1| \leq |S|$, then due to the construction of P_p^2 (i.e., to convert $P_p^2 - S_1$ into disconnected graph, we must include at least two consecutive vertices in S_1), it generates large value of $m(P_p^2 - S_1)$ such that,
$|S| + m(P_p^2 - S) < |S_1| + m(P_p^2 - S_1)$. (1)
Let S_2 be any hub set of P_p^2 such that $m(P_p^2 - S_2) = 1$. Then
$|S| + m(P_p^2 - S) \leq |S_2| + m(P_p^2 - S_2)$, **(2)**
From (1) and (2) we have,
$|S| + m(P_p^2 - S) = min\{|X| + m(G - X) : X\ is\ a\ hub\ set\}$
$= HI(P_p^2)$.
Hence,

$$HI(P_p^2) = \begin{cases} 2 & \text{if } p = 2, \\ 3 & \text{if } p = 3, 4, \\ \frac{2p}{3} & \text{if } p \geq 5 \text{ and } p \equiv 0(mod\ 3), \\ \frac{2(p-1)}{3} + 1 & \text{if } p \geq 5 \text{ and } p \equiv 1(mod\ 3), \\ \frac{2p+2}{3} & \text{if } p \geq 5 \text{ and } p \equiv 2(mod\ 3). \end{cases}$$

□

Definition 2.2. [9] *The composition $G[H]$ of two graphs G and H has its vertex set $V(G) \times V(H)$, with (u_1, u_2) adjacent to (v_1, v_2) if either u_1 is adjacent to v_1 in G or $u_1 = v_1$ and u_2 is adjacent to v_2 in H.*

Theorem 2.3.

$$HI(K_2[P_p]) = p + \lceil 2\sqrt{p+1} \rceil - 1.$$

Proof. Consider K_2 with vertices u_1, u_2 and P_p with $v_1, v_2, ..., v_p$. Let G be the graph $K_2[P_p]$. Then,

$$V(G) = \{(u_i, v_j)/1 \leq i \leq 2, 1 \leq j \leq p\}$$

and

$E(G) = \{(u_1, v_j)(u_2, v_k)/1 \leq j \leq p, 1 \leq k \leq p\} \cup \{(u_1, v_j)(u_1, v_{j+1}), (u_2, v_j)(u_2, v_{j+1}/1 \leq j \leq p-1\}$.

For the sake of convenience, we denote the vertices $(u_1, v_j) = w_{1j}, 1 \leq j \leq p$ and

$$(u_2, v_j) = w_{2j}, 1 \leq j \leq p.$$

Then graph of $K_2[P_6]$ is shown in Figure 1 for better understanding of the notation and arrangement of vertices. Moreover, $K_{p,p}$ is a subgraph of G and $HI(K_{p,p)} = p + 1$, so $HI(G) > p + 1$.
Consider $S_1 = \{w_{2j}/1 \leq j \leq p\}$, $|S_1| = p$. Then, S_1 is a hub set of G and $G - S_1 = P_p$, so $m(G - S_1) = p$.

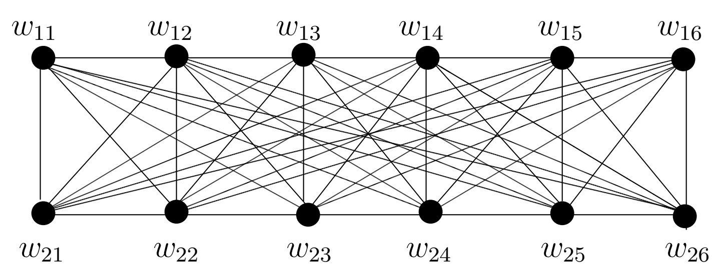

Figure 1

Let $S_2 = \{w_{1k} = (u_1, v_k)/v_k \in I - set\ of\ P_p\}$. Take $V_1 = \{v_k/v_k \in I - set\ of\ P_p\}$ so $|S_2| = |V_1|$.
Consider $S = S_1 \cup S_2$. Then, S is also a hub set of G (as $S_1 \subset S$). We have,

$$|S| = |S_1| + |S_2| = |S_1| + |V_1|\ and\ G - S = P_p - V_1,\ so\ m(G - S) = m(P_p - V_1).$$

By Theorem 1.6, we have

$$\begin{aligned}|S| + m(G - S) &= |S_1| + |V_1| + m(P_p - V_1)\\ &= |S_1| + I(P_p).\\ &= p + \lceil 2\sqrt{p+1} \rceil - 2 > p + 1.\end{aligned}$$

Hence,

$$|S| + m(G - S) = p + \lceil 2\sqrt{p+1} \rceil - 2 > p + 1. \ (3)$$

Now we discuss the minimality of $|S| + m(G - S)$. If S_3 is any hub set of G which is not containing S_1 or S_2 as a proper subset and $|S_3| = k < 2p$. Then, due to construction of $G(w_{1i}\ is\ adjacent\ to\ w_{2k}\ for\ 1 \leq i, k \leq p)$,

$$|S_3| + m(G - S_3) = k + 2p - k = 2p > |S| + m(G - S). \ (4)$$

Let S_5 be another hub set of G such that $S_5 = S_4 \cup S_2$, where $S_4 \subset S_1$ with $|S_4| < p$. In G, w_{1i} is adjacent to w_{2k} for $1 \leq i, k \leq p$. Therefore,

$$m(G - S_5) = |S_2| + p - |S_4|.$$

Hence,

$$\begin{aligned} |S_5| + m(G - S_5) &= |S_2| + |S_4| + |S_2| + p - |S_4| \\ &= 2|S_2| + p. \\ &> |S| + m(G - S). \ (5) \end{aligned}$$

Therefore, from the above discussion and (4) and (5), it follows that $|S| + m(G - S)$ is minimum. Hence, from equation (3) and the minimality of $|S| + m(G - S)$ we have,

$$\begin{aligned} HI(K_2[P_p]) &= min\{|X| + m(G - X) : X\ is\ a\ hub\ set\} \\ &= |S| + m(G - S). \\ &= p + \lceil 2\sqrt{p+1} \rceil - 2. \end{aligned}$$

□

Theorem 2.4. $HI(K_2[C_p]) = p + \lceil 2\sqrt{p} \rceil - 1.$

Proof. The proof is similar to that of Theorem 2.3. □

Theorem 2.5. $HI(K_2[K_{1,n}]) = n + 3.$

Proof. The proof is similar to that of Theorem 2.3. □

Theorem 2.6. $HI(K_2[K_p]) = 2p.$

Proof. The proof is similar to that of Theorem 2.3. □

Theorem 2.7.

$$HI(P_p[K_2]) = \begin{cases} 4 & \textit{if } p = 2, 3, \\ 6 & \textit{if } p = 4, \\ 7 + 4i & \textit{if } p \geq 5 \textit{ and } p = 5 + 3i, \\ 8 + 4i & \textit{if } p \geq 5 \textit{ and } p = 6 + 3i, \\ 10 + 4i & \textit{if } p \geq 5 \textit{ and } p = 7 + 3i, \end{cases}$$

where $i \in Z^+ \cup \{0\}$.

Proof. Let P_p be a path with vertices $u_1, u_2, .., u_p$ and complete graph K_2 with vertices v_1, v_2. Let G be the graph $P_p[K_2]$. Then,

$$V(G) = \{(u_i, v_j)/1 \leq i \leq p, 1 \leq j \leq 2\}$$

and

$E(G) = \{(u_i, v_j)(u_{i+1}, v_j)/1 \leq i \leq p-1, 1 \leq j \leq 2\} \cup \{(u_i, v_1)(u_{i+1}, v_2)/1 \leq i \leq p-1\} \cup \{(u_i, v_2)(u_{i+1}, v_1)/1 \leq i \leq p-1\}$.
Without loss of generality, we denote vertices (u_i, v_1) *by* $w_{i1}, 1 \leq i \leq p$ and (u_i, v_2) *by* $w_{i2}, 1 \leq i \leq p$. The graph $P_6[K_2]$ is shown in Figure 2 for better understanding of the notations and arrangement of vertices.

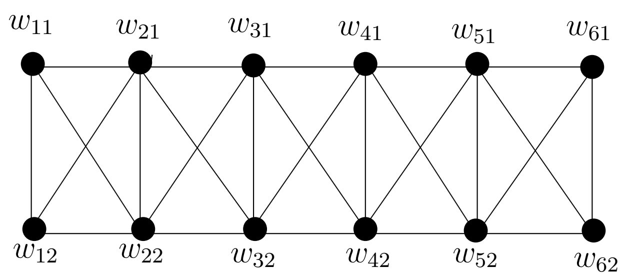

Figure 2

We consider the following two cases:
Case 1: $2 \leq p \leq 4$.
For $p = 2$, $P_2[K_2]$ is isomorphic to a complete graph K_4. By Proposition 1.7, $HI(P_2[K_2]) = 4$.
For $p = 3$, consider $S = \{w_{21}, w_{22}\}$, which is a hub set of $P_3[K_2]$ and $m(G-S) = 2$. There does not exist any hub set S_1 of G such that $|S_1| + m(G - S_1) < |S| + m(G - S)$. Hence, $HI(P_3[K_2]) = 4$.
For $p = 4$, consider $S = \{w_{21}, w_{22}, w_{32}\}$, which is a hub set for $P_4[K_2]$ and $m(G - S) = 3$. Moreover, for any hub set S_1 of G we have, $|S_1| + m(G - S_1) \geq |S| + m(G - S)$. Hence, $HI(P_4[K_2]) = 6$.
Case 2: $p \geq 5$. We consider a subset S of $V(G)$ as below :

- If $p = 5 + 3i$, where $i = 0, 1, 2, ...$, then consider $S = \{w_{(2+3j)1}, w_{(2+3j)2}/0 \leq j \leq i\} \cup \{w_{(3j)2}, w_{(3j+1)2}/1 \leq j \leq i+1\} \cup \{w_{(p-1)1}\}$.
  Then, $|S| = 5 + 4i$.
- If $p = 6 + 3i$, where $i = 0, 1, 2, ...$, then consider $S = \{w_{(2+3j)1}, w_{(2+3j)2}/0 \leq j \leq i+1\} \cup \{w_{(3j)2}, w_{(3j+1)2}/1 \leq j \leq i+1\}$.
  Then, $|S| = 6 + 4i$.

- If $p = 7 + 3i$, where $i = 0, 1, 2, ...$, then consider $S = \{w_{(2+3j)1}, w_{(2+3j)2}/0 \leq j \leq i+1\} \cup \{w_{(3j)2}, w_{(3j+1)2}/1 \leq j \leq i+2\}$.
  Then, $|S| = 8 + 4i$.

Moreover, in all above three cases $m(G - S) = 2$.
Now, we discuss the minimality of $|S|+m(G-S)$. If we consider any hub set S_1 of G such that $|S_1| < |S|$, then due to construction of G (i.e., to convert $G - S_1$ into disconnected graph we must include vertices w_{i1} and w_{i2} in S_1), it generates large value of $m(G - S_1)$ such that

$$|S| + m(G - S) \leq |S_1| + m(G - S_1). \ (6)$$

Let S_2 be any hub set of G such that $m(G - S_2) = 1$, then,

$$|S| + m(G - S) < |S_2| + m(G - S_2). \ (7)$$

Thus, from above discussion and (6) and (7), $|S| + m(G - S)$ is minimum. So, in both the cases we have,

$$\begin{aligned} |S| + m(G - S) &= min\{|X| + m(G - X) : X \ is \ a \ hub \ set\} \\ &= HI(G). \end{aligned}$$

Hence,

$$HI(P_p[K_2]) = \begin{cases} 4 & \text{if } p = 2, 3, \\ 6 & \text{if } p = 4, \\ 7 + 4i & \text{if } p \geq 5 \text{ and } p = 5 + 3i, \\ 8 + 4i & \text{if } p \geq 5 \text{ and } p = 6 + 3i, \\ 10 + 4i & \text{if } p \geq 5 \text{ and } p = 7 + 3i. \end{cases}$$

□

Definition 2.8. [9] *The (Cartesian)product $G \times H$ of graphs G and H has $V(G) \times V(H)$ as its vertex set and (u_1, u_2) is adjacent to (v_1, v_2) if either $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1 .*

Theorem 2.9.

$$HI(K_2 \times P_p) = \begin{cases} 3 & \textit{if } p = 2, \\ 4 & \textit{if } p = 3, \\ 6 & \textit{if } p = 4, \\ 7 + 4i & \textit{if } p \geq 5 \textit{ and } p = 5 + 3i, \\ 8 + 4i & \textit{if } p \geq 5 \textit{ and } p = 6 + 3i, \\ 10 + 4i & \textit{if } p \geq 5 \textit{ and } p = 7 + 3i, \end{cases}$$

where $i \in Z^+ \cup \{0\}$.

Proof. Let P_p be a path with vertices $u_1, u_2, .., u_p$ and complete graph K_2 with vertices v_1, v_2. Let G be the graph $K_2 \times P_p$, it has $2p$ vertices and $3p-2$ edges.
For the sake of convenience, we denote vertices (v_1, u_j) *by* $w_{1j}, 1 \leq j \leq p$ and (v_2, u_j) *by* $w_{2j}, 1 \leq j \leq p$. The graph $K_2 \times P_7$ is shown in Figure 3 for better understanding of the notations and arrangement of vertices. Moreover, P_p is a subgraph of G and $HI(P_p) = p-1, HI(G) > p-1$.

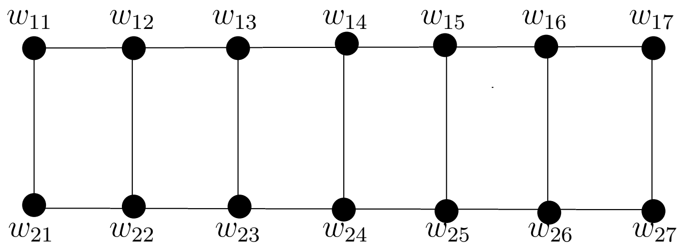

Figure 3

We consider the following two cases :
Case 1 : $2 \leq p \leq 4$.
For $p = 2$, $K_2 \times P_2$ is isomorphic to a cycle C_4. By Proposition 1.7, $HI(K_2 \times P_2) = 3$.
For $p = 3$, consider $S = \{w_{12}, w_{22}\}$, which is a hub set for $K_2 \times P_3$. and $m(G-S) = 2$. There does not exist any hub set S_1 of G such that $|S_1| + m(G-S_1) < |S| + m(G-S)$. Hence, $HI(K_2 \times P_3) = 4$.
For $p = 4$, consider $S = \{w_{12}, w_{22}, w_{14}, w_{24}\}$ or $S = \{w_{11}, w_{21}, w_{13}, w_{23}\}$, which is a hub set for $K_2 \times P_4$ and $m(G-S) = 2$. Therefore $|S| + m(G-S) = 6$. It is easy to observe that there does not exist a hub set S for which $|S| + m(G-S) < 6$. Therefore, $HI(K_2 \times P_4) = 6$.
Case 2 : $p \geq 5$. We consider a subset S of $V(G)$ as below :

- If $p = 5 + 3i$, where $i = 0, 1, 2, ...$, then consider

  $$S = \{w_{1(2+3j)}, w_{2(2+3j)} / 0 \leq j \leq i\} \cup \{w_{2(3j)}, w_{2(3j+1)} / 1 \leq j \leq i+1\} \cup \{w_{1p}\}.$$

  Then, $|S| = 5 + 4i$.
- If $p = 6 + 3i$, where $i = 0, 1, 2, ...$, then consider

  $$S = \{w_{1(2+3j)}, w_{2(2+3j)} / 0 \leq j \leq i+1\} \cup \{w_{2(3j)}, w_{2(3j+1)} / 1 \leq j \leq i+1\}.$$

  Then, $|S| = 6 + 4i$.
- If $p = 7 + 3i$, where $i = 0, 1, 2, ...$, then consider

  $$S = \{w_{1(2+3j)}, w_{2(2+3j)} / 0 \leq j \leq i+1\} \cup \{w_{2(3j)}, w_{2(3j+1)} / 1 \leq j \leq i+2\}.$$

Then, $|S| = 8 + 4i$.

In all the above cases, S is a hub set for $(K_2 \times P_p)$ and $m(G - S) = 2$.
Now we discuss the minimality of $|S|+m(G-S)$. If we consider any hub set S_1 of G such that $|S_1| \leq |S|$, then due to construction of G (i.e., to convert $G - S_1$ into disconnected graph we must include vertices w_{1i} and w_{2i} in S_1), it generates large value of $m(G - S_1)$ such that

$$|S| + m(G - S) \leq |S_1| + m(G - S_1). \ (8)$$

Let S_2 be any hub set of G such that $m(G - S) = 1$, then, for $p \geq 5$,

$$|S| + m(G - S) < |S_2| + m(G - S_2). \ (9)$$

Thus, from above discussion and (8) and (9), $|S| + m(G - S)$ is minimum. So, in both the cases we have,

$$\begin{aligned} |S| + m(G - S) &= min\{|X| + m(G - X) : X \ is \ a \ hub \ set\} \\ &= HI(G). \end{aligned}$$

Hence,

$$HI(K_2 \times P_p) = \begin{cases} 3 & \text{if } p = 2, \\ 4 & \text{if } p = 3, \\ 6 & \text{if } p = 4, \\ 7 + 4i & \text{if } p \geq 5 \text{ and } p = 5 + 3i, \\ 8 + 4i & \text{if } p \geq 5 \text{ and } p = 6 + 3i, \\ 10 + 4i & \text{if } p \geq 5 \text{ and } p = 7 + 3i. \end{cases}$$

□

Theorem 2.10.

$$HI(K_2 \times C_p) = \begin{cases} 5 & \text{if } p = 3, 4, \\ p + 2 & \text{if } p = 5, 6, \\ p + 3 & \text{if } p = 7, 8, \\ 13 & \text{if } p = 9. \end{cases}$$

Proof. Let K_2 be a complete graph with vertices v_1, v_2 and C_p be a cycle with vertices $u_1, u_2, ..., u_p$. Let G be the graph $K_2 \times C_p$, with $2p$ vertices and $3p$ edges for $p > 2$. Without loss of generality, we denote vertices $(v_1, u_i) \ by \ w_{1i}, 1 \leq i \leq p$ and $(v_2, u_i) \ by \ w_{2i}, 1 \leq i \leq p$. The graph $K_2 \times C_4$ is shown in Figure 4 for better understanding of the notations and arrangement of vertices.

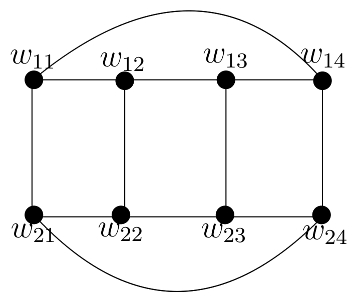

Figure 4

For $p = 3$, consider $S = \{w_{11}, w_{21}, w_{12}, w_{23}\}$, which is a hub set for $K_2 \times C_3$. and $m(G - S) = 1$. For $S = \{w_{11}, w_{21}, w_{22}, w_{13}\}$ also, we have $m(G - S) = 1$. There does not exist any hub set S_1 of G such that $|S_1| + m(G - S_1) < |S| + m(G - S)$. Hence, $HI(K_2 \times C_3) = 5$.

For $p = 4$, consider $S = \{w_{11}, w_{13}, w_{22}, w_{24}\}$, which is a hub set for $K_2 \times C_4$. and $m(G - S) = 1$. There does not exist any hub set S_1 of G such that $|S_1| + m(G - S_1) < |S| + m(G - S)$. Hence, $HI(K_2 \times C_4) = 5$.

For $p = 5$, consider $S = \{w_{11}, w_{13}, w_{15}, w_{22}, w_{24}\}$, which is a hub set for $K_2 \times C_5$. and $m(G - S) = 2$. There does not exist any hub set S_1 of G such that $|S_1| + m(G - S_1) < |S| + m(G - S)$. Hence, $HI(K_2 \times C_5) = 7$.

For $p = 6$, consider $S = \{w_{11}, w_{13}, w_{14}, w_{15}, w_{22}, w_{24}, w_{26}\}$, which is a hub set for $K_2 \times C_6$. and $m(G - S) = 1$. For $S = \{w_{11}, w_{12}, w_{14}, w_{16}, w_{21}, w_{23}, w_{25}\}$, also $m(G - S) = 1$. There does not exist any hub set S_1 of G such that $|S_1| + m(G - S_1) < |S| + m(G - S)$. Hence, $HI(K_2 \times C_6) = 8$.

For $p = 7$, consider $S = \{w_{11}, w_{12}, w_{13}, w_{15}, w_{17}, w_{21}, w_{22}, w_{24}, w_{26}\}$, which is a hub set for $K_2 \times C_7$. and $m(G - S) = 1$. There does not exist any hub set S_1 of G such that $|S_1| + m(G - S_1) < |S| + m(G - S)$. Hence, $HI(K_2 \times C_7) = 10$.

For $p = 8$, consider $S = \{w_{11}, w_{13}, w_{14}, w_{15}, w_{16}, w_{17}, w_{22}, w_{24}, w_{26}, w_{28}\}$, which is a hub set for $K_2 \times C_8$. and $m(G - S) = 1$.
For $S = \{w_{11}, w_{12}, w_{13}, w_{15}, w_{17}, w_{18}, w_{22}, w_{24}, w_{26}, w_{28}\}$, also $m(G - S) = 1$. There does not exist any hub set S_1 of G such that $|S_1| + m(G - S_1) < |S| + m(G - S)$. Hence, $HI(K_2 \times C_8) = 11$.

For $p = 9$, consider $S = \{w_{11}, w_{12}, w_{13}, w_{14}, w_{16}, w_{18}, w_{19}, w_{21}, w_{23}, w_{25}, w_{27}, w_{29}\}$, which is a hub set for $K_2 \times C_9$. and $m(G-S) = 1$. There does not exist any hub set S_1 of G such that $|S_1| + m(G - S_1) < |S| + m(G - S)$. Hence, $HI(K_2 \times C_9) = 13$.

□

Theorem 2.11.

$$HI(K_2 \times C_p) = \begin{cases} \frac{4p}{3} + 2 & \text{if } p \geq 10 \text{ and } p \equiv 0(mod\ 3), \\ \frac{4(p-1)}{3} + 3 & \text{if } p \geq 10 \text{ and } p \equiv 1(mod\ 3), \\ \frac{4(p+1)}{3} & \text{if } p \geq 10 \text{ and } p \equiv 2(mod\ 3). \end{cases}$$

Proof. Let K_2 be a complete graph with vertices v_1, v_2 and C_p be a cycle with vertices $u_1, u_2, ..., u_p$. Let G be the graph $K_2 \times C_p$, with $2p$ vertices and $3p$ edges for $p \geq 10$.
We consider a subset S of $V(G)$ as below:

- If $p \equiv 0(mod\ 3)$ then $p = 3k$ for some $k \in Z^+$. We have

$$S = \{w_{1(1+3i)}, w_{2(1+3i)}/0 \leq i \leq k-1\} \cup \{w_{2(3i)}/0 < i \leq k\} \cup \{w_{2(2+3i)}/0 \leq i \leq k-1\}$$

  and $|S| = 4k$. So, $|S| = \frac{4p}{3}\ for\ p \equiv 0(mod\ 3)$.
- If $p \equiv 1(mod\ 3)$ then $p = 3k+1$ for some $k \in Z^+$. We have $S = \{w_{1(1+3i)}, w_{2(1+3i)}/0 \leq i \leq k-1\} \cup \{w_{1p}\} \cup \{w_{2(3i)}/0 < i \leq k\} \cup \{w_{2(2+3i)}/0 \leq i \leq k-1\}\ and\ |S| = 4k+1$. So, $|S| = \frac{4(p-1)}{3} + 1\ for\ p \equiv 1(mod\ 3)$.
- If $p \equiv 2(mod\ 3)$ then $p = 3k-1$ for some $k \in Z^+$. We have $S = \{w_{1(1+3i)}, w_{2(1+3i)}/0 \leq i \leq k-1\} \cup \{w_{2(3i)}/0 < i \leq k-1\} \cup \{w_{2(2+3i)}/0 \leq i \leq k-2\}\ and\ |S| = 4k-2$. So, $|S| = \frac{4(p+1)}{3} - 2\ for\ p \equiv 2(mod\ 3)$.

In all the above cases, S is a hub set for G and $m(G-S) = 2$.
Now we discuss the minimality of $|S|+m(G-S)$. If we consider any hub set S_1 of G such that $|S_1| < |S|$, then due to construction of G (i.e., to convert $G - S_1$ into disconnected graph), it generates large value of $m(G - S_1)$ such that

$$|S| + m(G-S) \leq |S_1| + m(G - S_1).\ (10)$$

Let S_2 be any hub set of G such that $m(G-S) = 1$, then, for $p \geq 10$,

$$|S| + m(G-S) < |S_2| + m(G - S_2).\ (11)$$

Thus, from above discussion and (10) and (11), $|S|+m(G-S)$ is minimum. So, in both the cases we have,

$$\begin{aligned} |S|+m(G-S) &= min\{|X|+m(G-X) : X \ is \ a \ hub \ set\} \\ &= HI(G). \end{aligned}$$

Hence,

$$HI(K_2 \times C_p) = \begin{cases} \frac{4p}{3}+2 & \text{if } p \geq 10 \text{ and } p \equiv 0 (mod\ 3), \\ \frac{4(p-1)}{3}+3 & \text{if } p \geq 10 \text{ and } p \equiv 1 (mod\ 3), \\ \frac{4(p+1)}{3} & \text{if } p \geq 10 \text{ and } p \equiv 2 (mod\ 3). \end{cases}$$

□

Theorem 2.12. $HI(K_2 \times K_{1,n}) = 4$.

Proof. Consider the graph $K_2 \times K_{1,n}$. The number of vertices in $K_2 \times K_{1,n}$ is $2n+2$. Let S be a hub set of $K_2 \times K_{1,n}$. We choose $S = \{u, v\}$ as shown in Figure 5. If we remove the set S and all its adjacent vertices, we get n components of order 2, i.e $m(K_2 \times K_{1,n} - S) = 2$. So, $|S| + m(K_2 \times K_{1,n} - S) = 4$. Therefore, $HI(K_2 \times K_{1,n}) = 4$. If S_1 is any hub set of $K_2 \times K_{1,n}$ other than S with $m(K_2 \times K_{1,n} - S_1) = 1$, then $|S_1| \geq n+2$. This implies that $|S_1| + m(K_2 \times K_{1,n} - S_1) \geq n+3 > 4$.
If $m(K_2 \times K_{1,n} - S_1) \geq 3$, then trivially $|S_1| + m(K_2 \times K_{1,n} - S_1) > 4$. Thus $HI(K_2 \times K_{1,n}) = 4$.

1

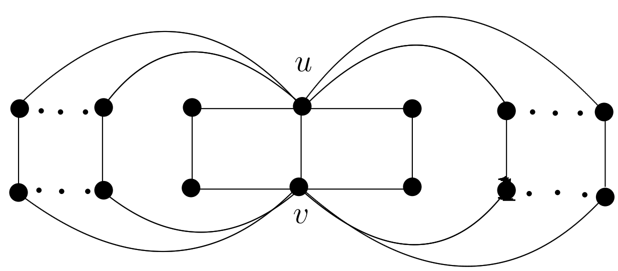

Figure 5

□

The join of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, denoted $G_1 + G_2$ consists of vertex set $V = V_1 \cup V_2$, and edge set $E = E_1 \cup E_2$ and all edges joining V_1 with V_2 [9].

Theorem 2.13. $HI(K_2 + P_p) = \lceil 2\sqrt{p+1} \rceil$.

Proof. Let K_2 be a complete graph with vertices u_1, u_2 and P_p, a path with $v_1, v_2, ..., v_p$. Let G be the graph $K_2 + P_p$. Then, $V(G) = \{u_1, u_2, v_1, ..., v_p\}$, $|V(G)| = p+2$, and $|E(G)| = 3p$.

The graph $K_2 + P_5$ is shown in Figure 6 for better understanding of the notation and arrangement of vertices. Consider $S_1 = \{u_1, u_2\}$, $|S_1| = 2$. Then, S_1 is a hub set of G and $G - S_1 = P_p$, so that $m(G - S_1) = p$.

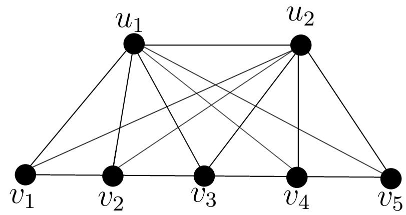

Figure 6

Let $S_2 = \{v_k/v_k \in I - set\ of\ P_p\}$. Take $V_1 = \{v_k/v_k \in I - set\ of\ P_p\}$ so that $|S_2| = |V_1|$. Consider $S = S_1 \cup S_2$. Then, S is also a hub set of G as $S_1 \subset S$. Thus,

$$|S| = |S_1| + |S_2| = |S_1| + |V_1|\ and\ G - S = P_p - V_1,\ so\ m(G - S) = m(P_p - V_1).$$

By Theorem 1.6, we have

$$\begin{aligned} |S| + m(G - S) &= |S_1| + |V_1| + m(P_p - V_1) \\ &= |S_1| + I(P_p). \\ &= 2 + \lceil 2\sqrt{p+1} \rceil - 2. \end{aligned}$$

Hence,

$$|S| + m(G - S) = \lceil 2\sqrt{p+1} \rceil. \ (12)$$

Now we discuss the minimality of $|S| + m(G - S)$. If S_3 is any hub set of G which is not containing S_1 or S_2 as a proper subset and $|S_3| = k < 2 + p$. Then, due to construction of G, (u_i *is adjacent to* v_k *for* $1 \leq i, k \leq p$),

$$|S_3| + m(G - S_3) = k + 2 + p - k = 2 + p > |S| + m(G - S). \ (13)$$

Let S_5 be another hub set of G such that $S_5 = S_4 \cup S_2$, where $S_4 \subset S_1$ with $|S_4| < 2$. In G, u_i is adjacent to v_k for $1 \leq i, k \leq p$. Therefore,

$$m(G - S_5) = |S_2| + p - |S_4|.$$

Hence,

$$\begin{aligned} |S_5| + m(G - S_5) &= |S_2| + |S_4| + |S_2| + p - |S_4| \\ &= 2|S_2| + p. \\ &> |S| + m(G - S). \ (14) \end{aligned}$$

Therefore, from the above discussion and (13) and (14), $|S| + m(G - S)$ is minimum. Hence, from equation (12) and the minimality of $|S| + m(G - S)$ we have,

$$HI(K_2 + P_p) = min\{|X| + m(G - X) : X \ is \ a \ hub \ set\}$$
$$= |S| + m(G - S).$$
$$= \lceil 2\sqrt{p+1} \rceil.$$

□

Theorem 2.14. $HI(K_2 + C_p) = \lceil 2\sqrt{p} \rceil + 1.$

Proof. The proof is similar to that of the Theorem 2.13. □

Theorem 2.15. $HI(K_2 + K_p) = p + 2.$

Proof. Since $K_2 + K_p = K_{p+2}$, a complete graph of order $p + 2$, by Proposition 1.7 , we get $HI(K_2 + K_p) = HI(K_{(p+2)}) = p + 2.$ □

Proposition 2.16. *For any two graphs* G *and* H*,* $HI(G + H) = I(G + H).$

Proof. Let $S = V(H)$, and let T be an I-set of G. Then $S \cup T$ is a hub set of $G + H$. Then

$$HI(G + H) \leq |S \cup T| + m((G + H) - (S \cup T))$$
$$= |V(H)| + |T| + m(G - T) = |V(H)| + I(G).$$

Also, we have $HI(G + H) \leq |V(G)| + I(H)$.
By Theorem 1.5 we get , $HI(G+H) \leq min\{I(G)+|V(H)|, I(H)+|V(G)|\} = I(G+H) \leq HI(G + H)$.
Hence, $HI(G + H) = I(G + H)$.

□

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