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CERTAIN THIRD ORDER DIFFERENTIAL SUBORDINATION RESULTS OF MEROMORPHIC MULTIVALENT FUNCTIONS

H. AAISHA FARZANA[1,*], B. ADOLF STEPHEN[1] AND M. P. JEYARAMAN[2]

[1]Department of Mathematics, Madras Christian College, Tambaram, Chennai 600 059, TamilNadu, India

[2]Department of Mathematics, L. N. Government College, Ponneri, Chennai 601 204, TamilNadu, India

*Corresponding author

Abstract. Certain applications of third order differential subordination results are obtained for meromorphic function involving Liu-Srivastava linear operator. These results are obtained by investigating appropriate classes of admissible function.

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1. Introduction, Definitions and Preliminaries

Let $\mathcal{H}(U)$ denote the class of analytic functions in the open unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$. For $n \in \mathbb{N} = \{1, 2, 3, ...\}$ and $a \in \mathbb{C}$, let $\mathcal{H}[a, n] = \{f : f \in \mathcal{H}(U) \quad and \quad f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}$, with $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$ and $\mathcal{H} \equiv \mathcal{H}[1, 1]$. Let Σ_p denote the class of all p-valent functions of the form

$$(1.1) \quad f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \qquad (z \in U^\star := \{z \in \mathbb{C} : 0 < |z| < 1\} \quad and \quad p \in \mathbb{N} := \{1, 2, 3, ...\}).$$

Let $f, F \in \mathcal{H}(U)$, then the function f is said to be subordinate to F or F is said to be superordinate to f, if there exists a function w analytic in U with $w(0) = 0$ and $|w(z)| < 1 \quad (z \in U)$, such that

$$f(z) = F(w(z)).$$

In such a case we write $f(z) \prec F(z)$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Let Q denote the set of all functions q that are analytic and injective on $\partial U \backslash E(q)$, where

$$E(q) = \{\zeta \in \partial U : \lim_{z\to\zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(q)$. Further, let the subclass of Q for $Q(0) \equiv a$ be denoted by $Q(a)$ and $Q(1) \equiv Q_1$.

For two functions $f, g \in \Sigma_p$ the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k b_k z^k := (g * f)(z),$$

where

$$g(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} b_k z^k.$$

For $\alpha_j \in \mathbb{C}(j = 1, 2, ..., l)$ and $\beta_k \in \mathbb{C}\backslash\{0, -1, -2, ...\}(k = 1, 2, 3, ...m)$ the generalized hypergeometric function ${}_lF_m(\alpha_1, \alpha_2, ..., \alpha_l; \beta_1, \beta_2, ..., \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, \alpha_2, ..., \alpha_l; \beta_1, \beta_2, ..., \beta_m; z) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k ...(\alpha_l)_k}{(\beta_1)_k(\beta_2)_k ...(\beta_m)_k} \frac{z^k}{k!}$$

$$(l \leq m + 1; l, m \in \mathbb{N}_0 := \{1, 2, 3, ...\} \cup \{0\}),$$

where $(\eta)_k$ is the Pochammer symbol defined in terms of the Gamma function by,

$$(\eta)_k = \frac{\Gamma(\eta + k)}{\Gamma(\eta)} \begin{cases} 1, & \text{if} \quad k = 0, \ \eta \in \mathbb{C}^*, \\ \eta(\eta + 1) \ldots (\eta + k - 1), & \text{if} \quad k \in \mathbb{N}, \ \eta \in \mathbb{C}. \end{cases}$$

Corresponding to the function

$$h_p(\alpha_1, \alpha_2, ..., \alpha_l; \beta_1, \beta_2, ..., \beta_m; z) := z^{-p}{}_lF_m(\alpha_1, \alpha_2, ..., \alpha_l; \beta_1, \beta_2, ..., \beta_m; z),$$

the Liu-Srivastava operator $H_p^{l,m}(\alpha_1, \alpha_2, ..., \alpha_l; \beta_1, \beta_2, ..., \beta_m) : \Sigma_p \to \Sigma_p$ defined by the Hadamard product

$$\begin{aligned} H_p^{l,m}(\alpha_1, \alpha_2, ..., \alpha_l; \beta_1, \beta_2, ..., \beta_m)f(z) &:= h_p(\alpha_1, \alpha_2, ..., \alpha_l; \beta_1, \beta_2, ..., \beta_m; z) * f(z) \\ &= \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \frac{(\alpha_1)_{k+p}(\alpha_2)_{k+p} ...(\alpha_l)_{k+p}}{(\beta_1)_{k+p}(\beta_2)_{k+p} ...(\beta_m)_{k+p}} \frac{a_k z^k}{(k + p)!}. \end{aligned}$$

We denote $H_p^{l,m}(\alpha_1, \alpha_2, ..., \alpha_l; \beta_1, \beta_2, ..., \beta_m)f(z)$ by $H_p^{l,m}[\alpha_1]f(z)$. The Liu-Srivastava operator was considered by Liu- Srivastava [1]. Also, the Liu-Srivastava operator is the meromorphic analogue of the Dziok-Srivastava operator [5]. The meromorphic analogue of the Carlson- Shaffer linear operator $L_p(a, c) := H_p^{(2,1)}(1, a; c)$ is a special case of Liu-Srivastava operator [2, 3, 4].

In order to prove our main subordination results, we shall make use of following definition and a known result.

Definition 1.1. [8] *Let* $\phi : \mathbb{C}^4 \times U \to \mathbb{C}$ *and* h *be univalent in* U. *If* p *is analytic in* U *and satisfies the following (third-order) differential subordination:*

$$\phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec h(z) \quad (z \in U), \tag{1.2}$$

then p *is called a solution of the differential subordination. The univalent function* q, *is called a dominant of the solutions of the differential subordination if*

$$p \prec q,$$

for all p *satisfying (1.2). A dominant* $\tilde{q}$ *that satisfies,*

$$\tilde{q} \prec q,$$

for all of dominants q *of (1.2) is said to be the best dominant.*

Definition 1.2. [8] *Let* Ω *be a set in* $\mathbb{C}$, $q \in Q$ *and* $n \geq 2$. *The class of admissible functions* $\Psi_n[\Omega, q]$ *consists of those functions*

$$\psi : \mathbb{C}^4 \times U \to \mathbb{C}$$

that satisfy the following admissibility condition:

$$\psi(a, b, c, d; z) \notin \Omega$$

whenever,

$$a = q(\zeta), b = n\zeta q'(\zeta),$$
$$\Re\left\{\frac{c}{b} + 1\right\} \geq n\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\} \text{ and } \Re\left\{\frac{d}{b}\right\} \geq n^2\Re\left\{\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right\}.$$
$$(z \in U; \zeta \in \partial U \backslash E(q)).$$

Lemma 1.1. [8] *Let* $p \in \mathcal{H}[a, n]$ *with* $n \geq 2$ *and let* $q \in Q(a)$ *satisfy*

$$\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\} \geq 0 \text{ and } \left|\frac{zp'(z)}{q'(\zeta)}\right| \leq n,$$

where $z \in U$ *and* $\zeta \in \partial U \backslash E(q)$. *If* Ω *is a set in* $\mathbb{C}$, $\psi \in \Psi_n[\Omega, q]$ *and*

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega,$$

then

$$p(z) \prec q(z) \quad (z \in U).$$

The theory of differential subordination in $\mathbb{C}$ is the generalization of differential inequality in $\mathbb{R}$. Many of the significant works on differential subordination have been pioneered by Miller and Mocanu, and their monograph [6] compiled their great efforts in introducing and developing the same. The theory of first and second order differential subordination has been successfully applied to address many important problems

in this field.
Recently, Antonio and Miller [8] obtained some general resuts of third order differential inequality and subordination. It should be remarked in passing that only few articles [7, 8] deal with the very narrow classes of third order differential subordination.
Very recently, by making use of third order differential subordination of Miller [8], Jeyaraman et. al., obtained certain applications of third order differential subordination of analytic function involving Schwarzian derivative. Also, certain third order differential subordination results involving fractional derivative have been obtained by authors in [11].
Motivated by the aforementioned work, in the present investigation, by making use of the differential subordination and results obtained in [8] certain classes of admissible functions are determined so that subordination implications of functions associated with the Liu-Srivastava linear operator $H_p^{l,m}$ hold.

2. The Main Subordination Results

We first define the following class of admissible functions that are required in our result.

Definition 2.1. *Let Ω be a set in $\mathbb{C}$ and $q \in Q_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_H[\Omega, q]$ consist of those functions $\phi : \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u, v, w, x; z) \notin \Omega$$

whenever

$$u = q(\zeta),$$

$$v = \frac{\alpha_1 q(\zeta) + n\zeta q'(\zeta)}{\alpha_1},$$

$$\Re\left(\frac{(\alpha_1+1)(w-u)}{v-u} - (2\alpha_1+1)\right) \geq n\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right)$$

and

$$\Re\left(\frac{(\alpha_1+1)(\alpha_1+2)(x-3w+3v-u)}{v-u}\right) \geq n^2\Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

$$(z \in U, \zeta \in \partial U \backslash E(q) \text{ and } n \geq 2).$$

Theorem 2.1. *Let $\phi \in \Phi_H[\Omega, q]$. If $f \in \Sigma_p$ and $q \in Q_1 \cap \mathcal{H}$ with*

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0$$

and

$$|pz^p H_p^{l,m}[\alpha_1]f(z) + z^{p+1}[H_p^{l,m}[\alpha_1]f(z)]'| \leq n|q'(\zeta)|,$$

$$(z \in U, \zeta \in \partial U \backslash E(q) \ and \ n \geq 2),$$

satisfies

(2.1)
$$\Big\{\phi\Big(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1+1]f(z),$$
$$z^p H_p^{l,m}[\alpha_1+2]f(z), z^p H_p^{l,m}[\alpha_1+3]f(z); z\Big) : z \in U\Big\} \subset \Omega,$$

then

$$z^p H_p^{l,m}[\alpha_1]f(z) \prec q(z)$$

Proof. Define the analytic function g in U by

(2.2)
$$g(z) := z^p H_p^{l,m}[\alpha_1]f(z).$$

(2.3)
$$\alpha_1 H_p^{l,m}[\alpha_1+1]f(z) = z[H_p^{l,m}[\alpha_1]f(z)]' + (\alpha_1+p)H_p^{l,m}[\alpha_1]f(z)$$

A simple calculation using (2.3) yields,

(2.4)
$$z^p H_p^{l,m}[\alpha_1+1]f(z) = \frac{1}{\alpha_1}[\alpha_1 g(z) + zg'(z)].$$

Further computations show that,

(2.5)
$$z^p H_p^{l,m}[\alpha_1+2]f(z) = \frac{1}{\alpha_1(\alpha_1+1)}[z^2 g''(z) + 2(\alpha_1+1)zg'(z)] + g(z)$$

and

(2.6)
$$z^p H_p^{l,m}[\alpha_1+3]f(z) = \frac{1}{\alpha_1(\alpha_1+1)(\alpha_1+2)}[z^3 g'''(z) + 3(\alpha_1+2)z^2 g''(z)$$
$$+ 3(\alpha_1+1)(\alpha_1+2)zg'(z)] + g(z).$$

We now define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by,

$$\psi(a,b,c,d;z) = \phi(u,v,w,x;z),$$

where

(2.7)
$$u = a, v = \frac{\alpha_1 a + b}{\alpha_1}, w = \frac{1}{\alpha_1(\alpha_1+1)}[c + 2(\alpha_1+1)b] + a$$
$$and \quad x = \frac{1}{\alpha_1(\alpha_1+1)(\alpha_1+2)}[d + 3(\alpha_1+2)c + 3(\alpha_1+1)(\alpha_1+2)b] + a.$$

Let

$$(2.8)\quad \psi(a,b,c,d;z) = \phi(u,v,w,x;z)$$
$$= \phi\bigg(a, \frac{\alpha_1 a + b}{\alpha_1}, \frac{1}{\alpha_1(\alpha_1+1)}[c + 2(\alpha_1+1)b] + a,$$
$$\frac{1}{\alpha_1(\alpha_1+1)(\alpha_1+2)}[d + 3(\alpha_1+2)c + 3(\alpha_1+1)(\alpha_1+2)b] + a; z\bigg).$$

Using Lemma 1.1, (2.2)-(2.6) and (2.7) from (2.8) we obtain,

$$(2.9)\quad \psi(g(z), zg'(z), z^2g''(z), z^3g'''(z); z) = \bigg\{\phi\bigg(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1+1]f(z),$$
$$z^p H_p^{l,m}[\alpha_1+2]f(z), z^p H_p^{l,m}[\alpha_1+3]f(z); z\bigg)\bigg\}.$$

Hence (2.1) becomes,

$$\psi(g(z), zg'(z), z^2g''(z), z^3g'''(z); z) \in \Omega.$$

A computation using (2.7) yields,

$$\frac{c}{b} + 1 = \frac{(\alpha_1+1)(w-u)}{(v-u)} - (2\alpha_1+1)$$

and

$$\frac{d}{b} = \frac{(\alpha_1+1)(\alpha_1+2)(x - 3w + 3v - u)}{(v-u)}.$$

Thus, the admissibility condition for $\phi \in \Phi_H[\Omega, q]$ in the Definition 2.1 is equivalent to the admissibility for ψ given in Definition 1.2. Hence, $\psi \in \Psi[\Omega, q]$ and by Lemma 1.1, we have

$$g(z) \prec q(z).$$

(or) equivalently

$$z^p H_p^{l,m}[\alpha_1]f(z) \prec q(z),$$

which completes the proof of Theorem 2.1. □

If $\Omega \neq \mathbb{C}$ is a simply connected doma in, then $\Omega = h(U)$ for some conformal mapping h of U onto Ω. In this case, $\Phi_H[h(U), q]$ is written as $\Phi_H[h, q]$.
The following Theorem is an immediate consequence of Theorem 2.1

Theorem 2.2. *Let $\phi \in \Phi_H[h, q]$. If $f \in \Sigma_p$ and $q \in Q_1 \cap \mathcal{H}$ with*

$$\Re\bigg(\frac{\zeta q''(\zeta)}{q'(\zeta)}\bigg) \geq 0$$

and

$$|pz^p H_p^{l,m}[\alpha_1]f(z) + z^{p+1}[H_p^{l,m}[\alpha_1]f(z)]'| \leq n|q'(\zeta)|,$$

$$(z \in U, \zeta \in \partial U \backslash E(q) \ and \ n \geq 2).$$

Let

$$\phi\bigg(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1+1]f(z), z^p H_p^{l,m}[\alpha_1+2]f(z), z^p H_p^{l,m}[\alpha_1+3]f(z); z\bigg)$$

be analytic in U, then

$$\phi\bigg(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1+1]f(z), z^p H_p^{l,m}[\alpha_1+2]f(z), z^p H_p^{l,m}[\alpha_1+3]f(z); z\bigg) \prec h(z)$$

implies

$$z^p H_p^{l,m}[\alpha_1]f(z) \prec q(z) \qquad (z \in U).$$

When the behaviour of q is not known on ∂U, we obtain the similar arguments as in [9, Corollary 1.1].

Corollary 2.1. *Let* $\Omega \subset \mathbb{C}$ *and let* $q(z)$ *be univalent in U with* $q(0) = 1$ *and for* $\varrho \in (0,1)$ *set* $q_\varrho(z) = q(\varrho z)$*. Let* $\phi \in \Phi_H[h, q_\varrho]$*. If* $f \in \Sigma_p$, $q_\varrho \in Q_1 \cap \mathcal{H}$ *with*

$$\Re\bigg(\frac{\zeta q_\varrho''(\zeta)}{q_\varrho'(\zeta)}\bigg) \geq 0 \ and \ |p z^p H_p^{l,m}[\alpha_1]f(z) + z^{p+1}[H_p^{l,m}[\alpha_1]f(z)]'| \leq n|q_\varrho'(\zeta)|,$$

$$(z \in U, \zeta \in \partial U \backslash E(q) \ and \ n \geq 2).$$

Let

$$\phi\bigg(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1+1]f(z), z^p H_p^{l,m}[\alpha_1+2]f(z), z^p H_p^{l,m}[\alpha_1+3]f(z); z\bigg)$$

be analytic in U,

$$\phi\bigg(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1+1]f(z), z^p H_p^{l,m}[\alpha_1+2]f(z), z^p H_p^{l,m}[\alpha_1+3]f(z); z\bigg) \prec h(z)$$

implies

$$z^p H_p^{l,m}[\alpha_1]f(z) \prec q(z) \qquad (z \in U).$$

The following theorem gives a relation between the best dominant of the differential subordination and the solution of the corresponding differential equation.

Theorem 2.3. *Let* $\phi \in \Phi_H[h, q_\varrho]$, $\phi : \mathbb{C}^4 \times U \to \mathbb{C}$ *and* $\phi(q(z), T_1(z), T_2(z), T_3(z); z)$ *be analytic in* U *where,*

$$T_1(z) = \frac{\alpha_1 q(z) + zq'(z)}{\alpha_1}, \tag{2.10}$$

$$T_2(z) = \frac{1}{\alpha_1(\alpha_1+1)}[z^2q''(z) + 2(\alpha_1+1)zq'(z)] + q(z)$$

and

$$T_3(z) = \frac{1}{\alpha_1(\alpha_1+1)(\alpha_1+2)}[z^3q'''(z) + 3(\alpha_1+2)z^2q''(z) + 3(\alpha_1+1)(\alpha_1+2)zq'(z)] + q(z).$$

Let h *be univalent in* U *and suppose the differential equation*

$$\phi(q(z), T_1(z), T_2(z), T_3(z); z) = h(z) \tag{2.11}$$

has a solution $q \in Q_1 \cap \mathcal{H}$*. If* $f \in \Sigma_p$ *satisfies*

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0 \text{ and } |pz^pH_p^{l,m}[\alpha_1]f(z) + z^{p+1}[H_p^{l,m}[\alpha_1]f(z)]'| \leq n|q'(\zeta)|,$$

$$(z \in U, \zeta \in \partial U \backslash E(q) \text{ and } n \geq 2), \text{then}$$

$$\phi\Big(z^pH_p^{l,m}[\alpha_1]f(z), z^pH_p^{l,m}[\alpha_1+1]f(z), z^pH_p^{l,m}[\alpha_1+2]f(z), z^pH_p^{l,m}[\alpha_1+3]f(z); z\Big) \prec h(z) \tag{2.12}$$

implies

$$z^pH_p^{l,m}[\alpha_1]f(z) \prec q(z) \qquad (z \in U).$$

and $q(z)$ *is the best dominant.*

Proof. Following the same argument as in [9, Theorem 3, page 451] by applying Theorem 2.1, we see that q is a dominant of (2.12). Since q is satisfies (2.11), it is also a solution of differential subordination (2.12) and therefore q will be dominated by all the dominants of (2.12). Hence q is the best dominant of (2.11). □

We will apply Theorem 2.1 to a specific case for $q(z) = 1 + Mz, M > 0$.
In the particular case $q(z) = 1 + Mz, M > 0$ and in view of Definition 2.1, the class of admissible functions $\Phi_H[\Omega, q]$ denoted by $\Phi_H[\Omega, M]$, can be expressed in the following form.

Definition 2.2. *Let Ω be a set in $\mathbb{C}$. The class of admissible functions $\Phi_H[\Omega, M]$ consist of those functions $\phi : \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi\left(1 + Me^{i\theta}, 1 + \frac{(\alpha_1 + n)Me^{i\theta}}{\alpha_1}, 1 + \frac{L + (\alpha_1 + 2n)(\alpha_1 + 1)Me^{i\theta}}{\alpha_1(\alpha_1 + 1)}, 1 + \frac{N + 3(\alpha_1 + 2)L + (\alpha_1 + 1)(\alpha_1 + 2)(3n + \alpha_1)Me^{i\theta}}{\alpha_1(\alpha_1 + 1)(\alpha_1 + 2)}; z\right) \notin \Omega \tag{2.13}$$

whenever $z \in U, \Re(Le^{-i\theta}) \geq n(n-1)M, \Re(Ne^{i\theta}) \geq 0$ for all $\theta \in [0, 2\pi]$, $\alpha_1 \in \mathbb{C}\backslash\{0, -1, -2\}$ and $n \geq 2$.

Corollary 2.2. *Let $\phi \in \Phi_H[\Omega, M]$. If $f \in \Sigma_p$ and $q \in Q_1 \cap \mathcal{H}$, with*

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0$$

and

$$|pz^p H_p^{l,m}[\alpha_1]f(z) + z^{p+1}[H_p^{l,m}[\alpha_1]f(z)]'| \leq nM,$$

$$(z \in U, \zeta \in \partial U\backslash E(q) \text{ and } n \geq 2),$$

satisfies

$$\phi\left(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1 + 1]f(z), z^p H_p^{l,m}[\alpha_1 + 2]f(z), z^p H_p^{l,m}[\alpha_1 + 3]f(z); z\right) \in \Omega$$

then

$$|z^p H_p^{l,m}[\alpha_1]f(z) - 1| < M.$$

When $\Omega = q(U) = \{w : |w - 1| < M\}$, we denote the class $\Phi_H[\Omega, M]$ as $\Phi_H[M]$. Then the above Corollary can be written as follows,

Corollary 2.3. *Let $\phi \in \Phi_H[\Omega, M]$. If $f \in \Sigma_p$ and $q \in Q_1 \cap \mathcal{H}$, with*

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0 \text{ and } |pz^p H_p^{l,m}[\alpha_1]f(z) + z^{p+1}[H_p^{l,m}[\alpha_1]f(z)]'| \leq nM,$$

$$(z \in U, \zeta \in \partial U\backslash E(q) \text{ and } n \geq 2),$$

satisfies

$$\left|\phi\left(z^p H_p^{l,m}[\alpha_1]f(z), z^p H_p^{l,m}[\alpha_1 + 1]f(z), z^p H_p^{l,m}[\alpha_1 + 2]f(z), z^p H_p^{l,m}[\alpha_1 + 3]f(z); z\right) - 1\right| < M$$

then

$$|z^p H_p^{l,m}[\alpha_1]f(z) - 1| < M \quad M > 0.$$

Example Consider the function $\phi(u,v,w,x;z) = \lambda(z)(u-1)+v$, where $\Re(\lambda(z)) \geq 0 \ for \ z \in U$ which satisfies the admissibility criteria (2.13) when $\Re(\alpha_1) \geq \frac{-1}{2}$ and hence Corollary (2.3) yields

$$|\lambda(z)(z^p H_p^{l,m}[\alpha_1]f(z)-1) + z^p H_p^{l,m}[\alpha_1+1]f(z)-1| < M$$

implies

$$|z^p H_p^{l,m}[\alpha_1]f(z)-1| < M \quad (M>0).$$

For $\lambda(z)=0$, we have $\phi(u,v,w,x;z)=v$, if $n \geq 1$ in Corollary 2.3 then

$$|z^p H_p^{l,m}[\alpha_1+1]f(z)-1| < M$$

implies

$$|z^p H_p^{l,m}[\alpha_1]f(z)-1| < M.$$

This result was obtained in [9, Corollary 2.4, page 199].

3. Special Case

Now, we specialize the class of admissible functions and corresponding theorem of $q(U)$ being the half-plane $\Delta = \{w : \Re\{w\} > 0\}$. The function

$$q(z) = \frac{a+\bar{a}z}{1-z} \qquad (z \in U) \tag{3.1}$$

where $\Re\{a\} > 0$, is univalent in $\partial U \backslash \{1\}$ and satisfies $q(U)=\Delta, q(0)=a$ and $q \in Q$.

Lemma 3.1. [10] *Let q be given by (3.1) and* $p(z) = a + a_n z^n + a_{n+1}z^{n+1} + ...$ *be analytic in U, with* $p(z) \neq a$ *and* $n \geq 2$. *If there exist points* $z \in U$ *and* $\zeta_0 \in \partial U \backslash \{1\}$ *such that* $p(z_0) = q(\zeta_0), p(U_{r_0}) \subset q(U)$, *where* $r_0 = |z_0|$, *and*

$$|zp'(z)||1-\zeta|^2 \leq 2n|\Re\{a\}|,$$

then

$$z_o p'(z_0) = -\frac{n}{2}\frac{(a-q(\zeta_0))(\bar{a}+q(\zeta_0))}{\Re\{a\}},$$

$$\Re\left(\frac{z_0 p''(z_0)}{p'(z_0)}+1\right) \geq 0$$

and

$$\Re\left(\frac{z_0^2 p'''(z_0)}{p'(z_0)}\right) \geq \frac{3n^2}{2}\frac{|a-q(\zeta_0)|^2}{(\Re\{a\})^2}.$$

We will use Lemma 3.1 and Definition 2.1 to define the class of admissible functions for the specific function q defined in (3.1). We denote $\Phi_H[\Omega, q]$ by $\Phi_H[\Omega, a]$ and when $\Omega = \Delta$, denote the class by $\Phi_H[a]$.

Definition 3.1. *Let Ω be a set in $\mathbb{C}$, let q be given by (3.1), and let $n \geq 2$. The class of admissible functions $\Phi_H[\Omega, a]$ consists of those funtions $\phi : \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the admissibility condition,*

$$\phi(u, v, w, x; z) \notin \Omega$$

whenever

$$u = i\rho, v = i\rho - \frac{n|a - i\rho|^2}{2\alpha_1 \Re\{a\}},$$

$$\Re\left(\frac{(\alpha_1 + 1)(w - u)}{v - u} - (2\alpha_1 + 1) \right) \geq 0$$

and

$$\Re\left(\frac{(\alpha_1 + 1)(\alpha_1 + 2)(x - 3w + 3v - u)}{v - u} \right) \geq \frac{3n^2}{2} \frac{|a - i\rho|^2}{(\Re\{a\})^2}.$$

$$(z \in U, \rho \in \mathbb{R}; a \in \mathbb{C}).$$

In this particular case, Theorem (2.1) can be rephrased in the following form,

Theorem 3.1. *Let q be given by* (3.1) *and let $f \in \Sigma_p$ satisfy*

$$|pz^p H_p^{l,m}[\alpha_1] f(z) + z^{p+1} [H_p^{l,m}[\alpha_1] f(z)]'||1 - \zeta|^2 \leq 2n\Re\{a\},$$

$$(z \in U, \zeta \in \partial U \backslash E(q) \text{ and } n \geq 2).$$

(i) *If Ω is a set in $\mathbb{C}$ and $\phi \in \Phi_H[\Omega, a]$ then*

$$\phi\left(z^p H_p^{l,m}[\alpha_1] f(z), z^p H_p^{l,m}[\alpha_1 + 1] f(z), z^p H_p^{l,m}[\alpha_1 + 2] f(z), z^p H_p^{l,m}[\alpha_1 + 3] f(z); z \right) \in \Omega$$

implies

$$\Re(z^p H_p^{l,m}[\alpha_1] f(z)) > 0.$$

(ii) *If $\phi \in \Phi_H[a]$ then*

$$\Re\phi\left(z^p H_p^{l,m}[\alpha_1] f(z), z^p H_p^{l,m}[\alpha_1 + 1] f(z), z^p H_p^{l,m}[\alpha_1 + 2] f(z), z^p H_p^{l,m}[\alpha_1 + 3] f(z); z \right) > 0$$

implies

$$\Re(z^p H_p^{l,m}[\alpha_1] f(z)) > 0.$$

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