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n-ROOT NEAR FRACTIONS

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ABSTRACT. n-root near fraction are defined, some basic result about them are proved and conjectures are formulated.

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INTRODUCTION

The old computational problem is to approximate an irrational number by a close rational number. The reader certainly knows the popular approximation of the Ludolphian number π by $\frac{22}{7}$; the famous mathematical result is that $\frac{22}{7}$ is greater than π, proofs of it date back to antiquity. It is also well known that the theory of continued fractions allows one to find good rational approximations of any irrational number. By such a research, we can also try to somehow affect certain problems of theoretical computer science, because it is, of course, about representations of numbers. (Regardless of how powerful a computer may be, it is still finite. This means that the computer takes a finite time to complete an operation and, what is important for our considerations, that it has only a finite amount of space in which to store data. Integers have a finite representation. In fact, even rational numbers can have a finite representation as can be expressed as a pair of integers: the numerator and the denominator. Mathematicians and computer scientists also search for systems representing arithmetic of rational numbers in a finite and effective form, see e.g. [1].) Nevertheless, we formulate a very special problem in this paper, which has essentially a mathematical character and perhaps hardly reasonably applicable at this moment. We will deal with square roots, cubic roots, in general n-th order roots. We can find fractions closed to these irrational numbers: from up or from down. In the paper, we introduce a completely new concept of so-called *n-root near fraction*. It is a fraction, which is in some meaning clarified below, simultaneously upper and lower bound of the n-th root of its denominator.

Some properties of these fractions and conjectures are discussed in the paper. We also demonstrate graphically computer experiments.

1. Basic definitions and the formulation of problems

Let $n, k \in \mathbb{N}$ and let us consider rational numbers

$$u_{(n,k)} = \min\left\{\frac{j}{k};\ j \in \mathbb{N},\ \frac{j}{k} \geq \sqrt[n]{k}\right\}$$

and

$$l_{(n,k)} = \max\left\{\frac{j}{k};\ j \in \mathbb{N},\ \frac{j}{k} \leq \sqrt[n]{k}\right\}.$$

The question is, for a given n, whether there exist two different natural numbers $k_1 \neq k_2$ such that $u_{(n,k_1)} = l_{(n,k_2)} \notin \mathbb{N}$. In such a case, we will call the non-integer $u_{(n,k_1)} = l_{(n,k_2)}$ the *n-root near fraction*.

Example 1. Let $n = 3$. We compute $u_{(3,2)} = \frac{3}{2}$, because the set of numbers of the form $\frac{j}{2}$ which are greater or equal than $\sqrt[3]{2}$ has its minimum $\frac{3}{2}$. On the other hand, $l_{(3,4)} = \frac{3}{2}$, because the set of numbers of the form $\frac{j}{4}$ which are lower or equal than $\sqrt[3]{4}$ has its maximum $\frac{6}{4} = \frac{3}{2}$. Thus, $\frac{3}{2}$ represents the 3-root near fraction. We left to reader a verification of that $\frac{5}{3}$ also represents the 3-root near fraction.

Now, we see that there are many questions connected with the definition of the n-root near fraction. For example:

(i) What we can say about cardinalities of sets of all n-root near fractions?
(ii) Is every fraction greater than one a n-root near fraction for a suitable n?
(iii) Do exist fractions which are, for a certain N, n-root near fractions for all $n \geq N$?

2. Results and conjectures

Proposition 1. *Let $a = u_{(n,k_1)} = l_{(n,k_2)}$ be a n-root near fraction. Then $k_1 < k_2$.*

Proof. Let $u_{(n,k_1)} = \frac{j_1}{k_1}$, $l_{(n,k_2)} = \frac{j_2}{k_2}$.

If $k_1 = k_2$, we have also $j_1 = j_2$ and then $\frac{j_1}{k_1} \geq \sqrt[n]{k_1} \geq \frac{j_1}{k_1}$. It yields $\sqrt[n]{k_1} = \frac{j_1}{k_1}$ and it means $\sqrt[n]{k_1}$ is rational. The only possibility for this is $\sqrt[n]{k_1}$ is an integer. This is contradiction because n-root near fractions are defined as non-integers.

If $k_1 > k_2$, we have $\frac{j_1}{k_1} \geq \sqrt[n]{k_1} > \sqrt[n]{k_2} \geq \frac{j_2}{k_2}$, so $\frac{j_1}{k_1} > \frac{j_2}{k_2}$ but this is contradiction, too: we have $\frac{j_1}{k_1} = \frac{j_2}{k_2}$. $\square$

Proposition 2. *Let $h \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ there exist $k_1, k_2 \in \mathbb{N}$ such that $h = u_{(n,k_1)}$ and $h = l_{(n,k_2)}$.*

Proof. One can take $k_1 = k_2 = h^n$. $\square$

Remark 1. The previous proposition explains why the case of "n-root near integers" is trivial. Nevertheless, it is possible to find also another $k_1, k_2 \in \mathbb{N}$ such that $h = u_{(n,k_1)}$ and $h = l_{(n,k_2)}$; for example $u_{(2,3)} = u_{(2,4)} = 2$ and $l_{(2,1)} = l_{(2,2)} = 1$.

We also remark that an omitting of the property that natural numbers k_1, k_2 must be different leads to the trivial case, see the proof of the Proposition 1.

Remark 2. Of course, if we express $u_{(n,k)}$ or $l_{(n,k)}$ in a form $\frac{j}{k}$, then the fraction $\frac{j}{k}$ is not cancelled, in general. Let us write $\frac{p}{q}$ for its cancelled form, i.e. $j = ip$, $k = iq$, $i \in \mathbb{N}$ and p and q are coprime natural numbers. It follows from the definition that $q \in \mathbb{N} - \{1\}$.

(From here, we will write p's and q's just for described cancelled form $\frac{p}{q}$.)

Proposition 3. *Let $n = 1$. Then 1-root near fractions do not exist.*

Proof. Trivially both $u_{(1,k)}$ and $l_{(1,k)}$ are only integers: $u_{(1,k)} = l_{(1,k)} = k$. $\square$

Remark 3. We remark that one can to find the numerator j as the bound of integer solutions of the inequality $k^{n+1} \leq j^n$ ($k^{n+1} \geq j^n$, respectively). We do not know whether it would be beneficial to explore relations with the equation $k^{n+1} = j^n + c$, $c \in \mathbb{Z}$ (if $n = 2$ for instance, this is the famous Mordell equation and thus a very non-trivial number theoretical problem).

Conjecture 1. *Let $n = 2$. Then 2-root near fractions do not exist.*

We can reformulate this conjecture by the previous remark as follows. Let j_1 be the least positive integer solution of the inequality $k_1^3 \leq j_1^2$ for some $k_1 \in \mathbb{N}$ and j_2 be the greatest positive integer solution of the inequality $k_2^3 \geq j_2^2$ for some $k_2 \in \mathbb{N}$, $k_2 \neq k_1$, such that $\frac{j_1}{k_1} = \frac{j_2}{k_2}$. Under these assumptions, the conjecture reads as j_1 is necessarily an integer multiple of k_1 and j_2 an integer multiple of k_2.

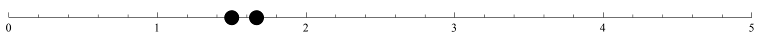
We add one more statement, which probably leads to the fact that the number of n-root near fractions is finite.

Proposition 4. *For a given n, n-root near fractions form a finite set or an infinite divergent sequence.*

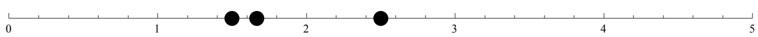
Proof. We will prove that it is not possible that a convergent (infinite) sequence of different n-root near fractions exist. Such a sequence can be considered as increasing. Let its limit be a proper number a. But it impossible because $u_{(n,k)} \geq \sqrt[n]{k}$ and $\lim_{k\to\infty} \sqrt[n]{k} = \infty$. $\square$

3. Some empirical data

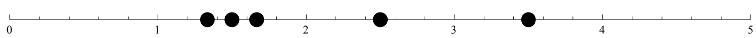
In the final section, we add a few graphically representated results which we have found using a computer.



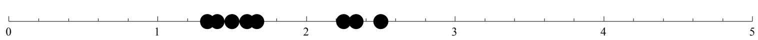
$n = 3: \quad \left\{\frac{3}{2}, \frac{5}{3}\right\}$ (2 fractions)



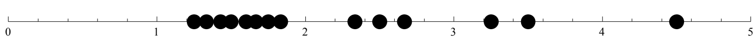
$n = 4: \quad \left\{\frac{3}{2}, \frac{5}{3}, \frac{5}{2}\right\}$ (3 fractions)



$n = 5: \quad \left\{\frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{5}{2}, \frac{7}{2}\right\}$ (5 fractions)



$n = 6: \quad \left\{\frac{4}{3}, \frac{7}{5}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{9}{4}, \frac{7}{3}, \frac{5}{2}\right\}$ (8 fractions)



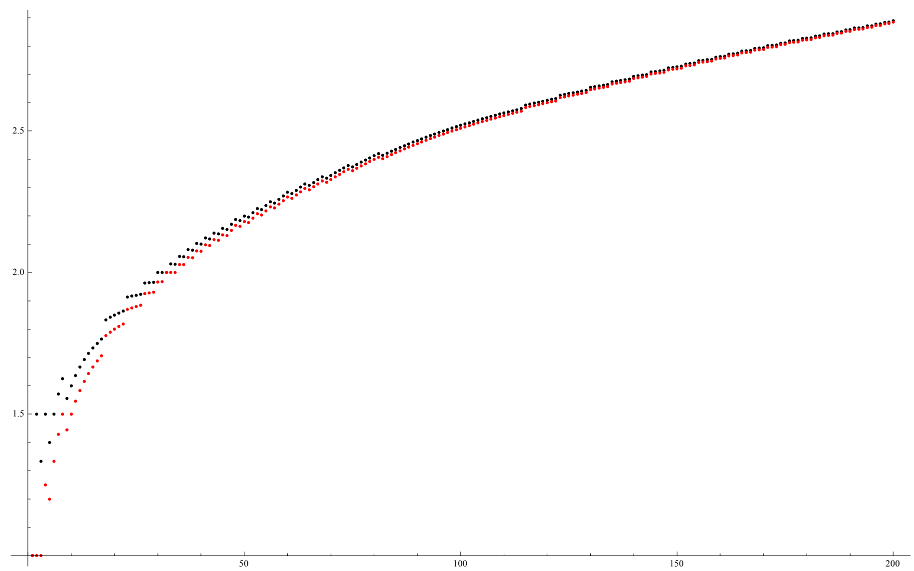
$n = 7: \quad \left\{\frac{5}{4}, \frac{4}{3}, \frac{10}{7}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{7}{4}, \frac{11}{6}, \frac{7}{3}, \frac{5}{2}, \frac{8}{3}, \frac{13}{4}, \frac{7}{2}, \frac{9}{2}\right\}$ (14 fractions)

$n = 8$: 21 fractions, $n = 9$: 25 fractions, etc. ...

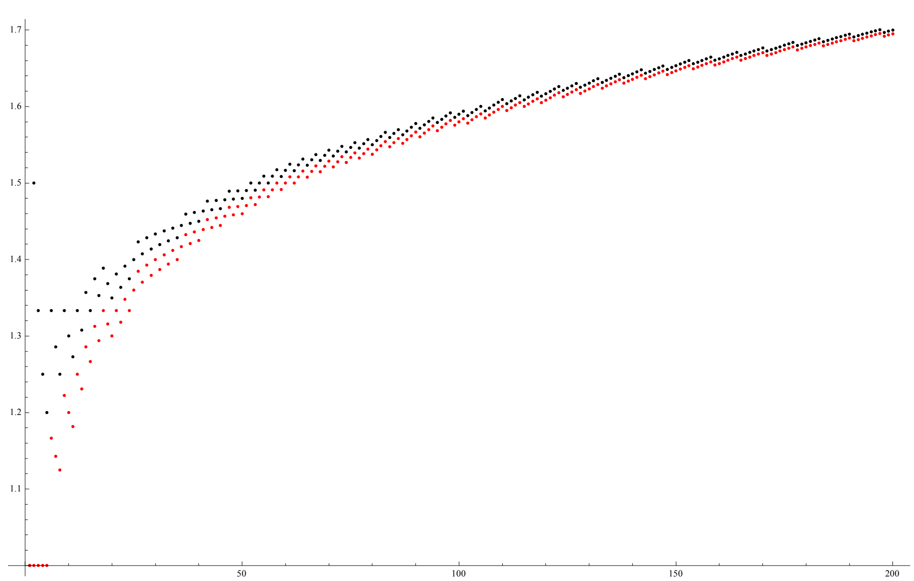
Conjecture 2. *For $3 \leq n \leq 9$, the numbers of n-root near fractions stated above are already the final numbers.*

We also observe that if a is a n-root near fraction, then it need not be $(n+1)$-root near fraction; we see it e.g. for $n = 5$ and $a = \frac{7}{2}$.

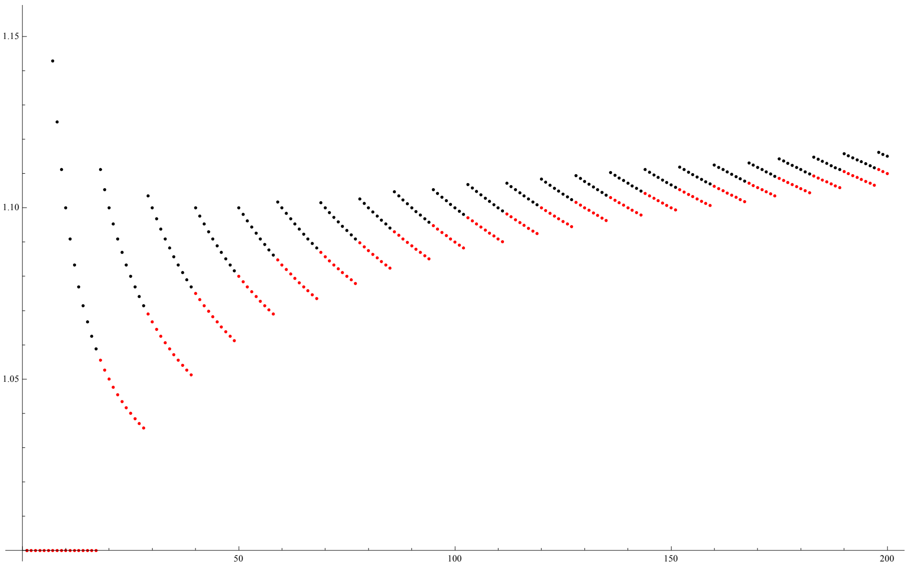
Now we add a few illustrative pictures to see distribution of points $u_{(n,k)}$ and $l_{(n,k)}$ for different n.



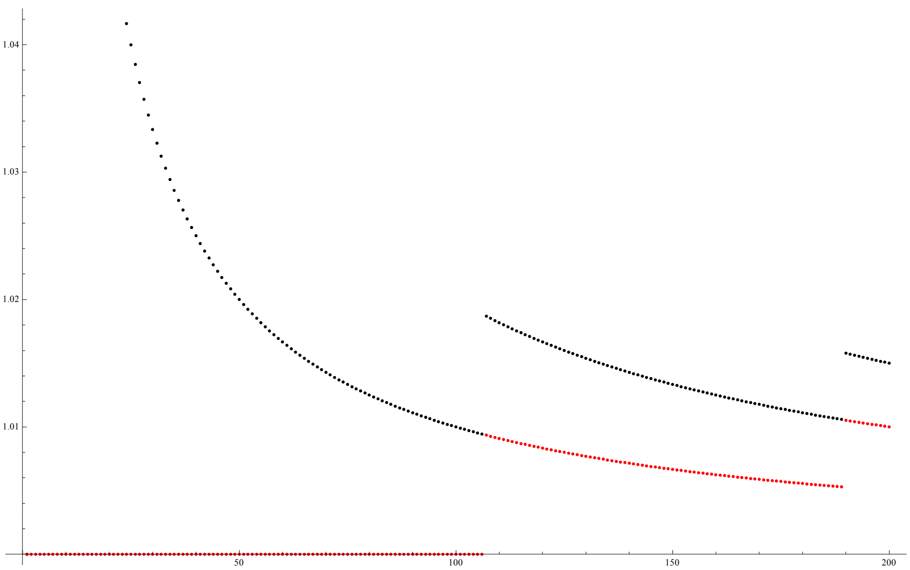
Points $u_{(5,k)}$ (black) and $l_{(5,k)}$ (red); for k from 1 to 200.



Points $u_{(10,k)}$ (black) and $l_{(10,k)}$ (red); for k from 1 to 200.



Points $u_{(50,k)}$ (black) and $l_{(50,k)}$ (red); for k from 1 to 200.



Points $u_{(500,k)}$ (black) and $l_{(500,k)}$ (red); for k from 1 to 200.

Acknowledgement

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References

[1] Hehner, E. C. R. and Horspool, R. N. S., A new representation of the rational numbers for fast easy arithmetic, SIAM J. Comput., Vol. 8, No. 2 (1979), pp. 124–134