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NONLINEAR ELLIPTIC PROBLEMS INVOLVING THE ANISOTROPIC $(p(\vec{x}), q(\vec{x}))$ SYSTEM

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Abstract. In this article, we study the existence of multi solutions of some elliptic problems involving anisotropic system with Dirichlet boundary conditions, by applying a result by B. Ricceri " A further three critical points theorem" published in " Nonlinear Anal, 71, 9: 4151–4157, 2009".

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1. Introduction

In this article, we study the following two nonlinear problems with Dirichlet boundary condition:

$$(1.1)\qquad \begin{gathered} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) = \lambda f(x, u) + \mu g(x, u), \quad \text{for } x \in \Omega \\ u(x) = 0 \quad on \quad \partial\Omega \end{gathered}$$

and

$$(1.2)\qquad \begin{gathered} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) = \lambda f_u(x, u, v) + \mu g_u(x, u, v), \quad \text{for } x \in \Omega \\ -\sum_{i=1}^{N} \partial_{x_i} b_i(x, \partial_{x_i} v) = \lambda f_v(x, u, v) + \mu g_v(x, u, v), \quad \text{for } x \in \Omega \\ u(x) = 0, v(x) = 0 \quad on \quad \partial\Omega \end{gathered}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $f, g \in C(\Omega \times \mathbb{R}, \mathbb{R})$ are two given functions that satisfy certain properties, pi and $a_i(x, s)$ are continuous functions will be described in Section 3. $2 \leq p_i(x) < N$ for each $x \in \Omega$ and every $i \in \{1, 2, ..., N\}$, λ, μ are positive real numbers. We mention that the assumptions that will be imposed on functions a_i allow us to take $a_i(x, s) = |s|^{p_i(x)-2}s$ for all $i \in \{1, \dots, N\}$, the operator becomes in particular $\sum_{i=1}^{N} \partial_{x_i} a_i \left(x, \partial_{x_i} u\right)$ the $\vec{p}(\cdot)$ - Laplace operator

$$\Delta_{\vec{p}(x)}(u) = \sum_{i=1}^{N} \partial_{x_i} \Big(|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \Big).$$

The differential operator involved in this article has as frame work a subspace of the anisotropic Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$, where $\vec{p}(\cdot) = \big(p_1(\cdot), \ldots, p_N(\cdot)\big)$ is a vector with variable components. Since the classical Sobolev space with fix exponent can not be used in in modeling some physical phenomena, like the electrorheological fluids or to the thermorheological fluids that have multiple applications to hydraulic valves and clutches, brakes, shock absorbers, robotics, space technology, tactile displays etc (see for example [6], [12], [14], [15], [16]). As a result, the anisotropic spaces with variable exponent were introduced, see [8],[10],[13]. Problems involving $\vec{p(x)}$-growth conditions $-\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) = \lambda f(x, u)$ was studies by boureau in [4] where the poof the existence of weak solutions is mainly based on a variant of the mountain pass theorem of Ambrosetti and Rabinowitz [28]. In our present work we will be using a new variational approach known as the Riceri principle.
Since the appearance of the abstract result proved by Ricceri in [3] and its revisited note established in [2] dealing with variational equations with both Dirichlet and Neumann conditions, they have extensively been investigated and have widely been applied for the study of the existence of multiple nontrivial solutions and in recent years a lot of papers has been appeared in the scalar case and systems of elliptic equations. We can cite, among others, the articles [23, 24, 25, 26, 27] and the references therein. In [1], Ricceri obtained a general three critical points theorem, that has been applied for a class of elliptic operators involving nonlinearities of polynomial growth.
When dealing with problem like 1.1 new difficulties occur, first we are dealing with tow eigenvalues nonlinear problem instead the one eigenvalue degenerate case, this can not be solved using classic methods, to overcame this difficulty we adopt a recent methods proved by B.Ricceri. Second difficulty occur when trying to prove some topological proprieties like the coercivety and the existence of compact inverse of some operators in the anisotropic case, this mainly due to the nature of the the norm involved. To overcome these difficulties, we used a recent techniques that appeared when treating anisotropic problems with variable exponents.
The purpose in his article is to investigate the tow-eigenvalue anisotropic problem, consider nonlinearities with polynomial growth and prove the existence of multiple

nontrivial solutions to Dirichlet boundary value problems. Precisely, we are interested in extending some results to a more general class of elliptic equations and systems by making also use of the variational principle of Ricceri [1].

This article is organized as follows. In section 2 we introduce the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega), W^{1,p(x)}(\Omega)$ and $W^{1,p(\vec{x})}(\Omega)$, and some embedding results. In section 3 we treat the the case of the one elliptic equation involving p(x)-Laplacian (1.1) and we will prove multiple results by applying Ricceri's principle in [1]. Finally, in section 4, we extend the result of the last section to general elliptic systems of two nonlinear partial differential equations governed essentially by the $(p(\vec{x}), q(\vec{x})) - Laplacian$(1.2). Let us first recall the crucial theorem below. First we need to introduce the following notation. If X is a real Banach space, we denote by W_X the class of all functionals $\Phi : X \to \mathbb{R}$ possessing the following property: if (u_n) is a sequence in X converging weakly to $u \in X$ and $\liminf_{n\to\infty} \Phi(u_n) \leq \Phi(u)$, then (u_n) has a subsequence converging strongly to u.

Theorem 1.1. [1] *Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbb{R}$, a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of X, and whose derivative admits a continuous inverse on X^*; $J : X \to \mathbb{R}$ a C^1 functional with compact derivative. Assume that Φ has a strict local minimum x_0 with $\Phi(x_0) = J(x_0) = 0$. Finally, set*

$$\alpha = \max\big\{0, \limsup_{\|x\|\to+\infty} \frac{J(x)}{\Phi(x)}, \limsup_{x\to x_0} \frac{J(x)}{\Phi(x)}\big\},$$

$$\beta = \sup_{x\in\Phi^{-1}(]0,+\infty[)} \frac{J(x)}{\Phi(x)},$$

and assume that $\alpha < \beta$. Then, for each compact interval $[a, b] \subset]1/\beta, 1/\alpha[$ (with the conventions $1/0 = +\infty$, $1/\infty = 0$) there exists $r > 0$ with the following property: for every $\lambda \in [a, b]$, and every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\sigma > 0$ such that for each $\mu \in [0, \sigma]$, the equation

$$\Phi'(x) = \lambda J'(x) + \mu\Psi'(x)$$

has at least three solutions in X whose norms are less than r.

2. Preliminaries

In what follows, we will recall the definition and the main properties of the spaces with variable exponents together with some results that are needed for the proof of our main results.

For $r \in C_+(\overline{\Omega})$, we introduce the Lebesgue space with variable exponent defined by

$$L^{r(\cdot)}(\Omega) = \{u : u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{r(x)}\,dx < \infty\},$$

where

$$C_+(\overline{\Omega}) = \{r \in C(\overline{\Omega};\mathbb{R}) : \inf_{x\in\Omega} r(x) > 1\}.$$

This space, endowed with the Luxemburg norm,

$$\|u\|_{L^{r(\cdot)}(\Omega)} = \inf\{\mu > 0 : \int_\Omega |\frac{u(x)}{\mu}|^{r(x)}\,dx \le 1\},$$

is a separable and reflexive Banach space [11, Theorem 2.5, Corollary 2.7].

Furthermore, the Hölder-type inequality

$$\Big|\int_\Omega u(x)v(x)\,dx\Big| \le 2\|u\|_{L^{r(\cdot)}(\Omega)}\|v\|_{L^{r'(\cdot)}(\Omega)} \tag{2.1}$$

holds for all $u \in L^{r(\cdot)}(\Omega)$ and $v \in L^{r'(\cdot)}(\Omega)$ (see [11, Theorem 2.1]), where we denoted by $L^{r'(\cdot)}(\Omega)$ the conjugate space of $L^{r(\cdot)}(\Omega)$, obtained by conjugating the exponent point wise; that is, $1/r(x) + 1/r'(x) = 1$ (see [11, Corollary 2.7]). Moreover, we denote

$$r^+ = \sup_{x\in\Omega} r(x), \quad r^- = \inf_{x\in\Omega} r(x)$$

and for $u \in L^{r(\cdot)}(\Omega)$, we have the following properties (see for example [9, Theorem 1.3, Theorem 1.4]):

$$\|u\|_{L^{r(\cdot)}(\Omega)} < 1\ (= 1;\ > 1) \ \Leftrightarrow\ \int_\Omega |u(x)|^{r(x)}\,dx < 1\ (= 1;\ > 1); \tag{2.2}$$

$$\|u\|_{L^{r(\cdot)}(\Omega)} > 1 \ \to\ \|u\|^{r^-}_{L^{r(\cdot)}(\Omega)} \le \int_\Omega |u(x)|^{r(x)}\,dx \le \|u\|^{r^+}_{L^{r(\cdot)}(\Omega)}; \tag{2.3}$$

$$\|u\|_{L^{r(\cdot)}(\Omega)} < 1 \ \to\ \|u\|^{r^+}_{L^{r(\cdot)}(\Omega)} \le \int_\Omega |u(x)|^{r(x)}\,dx \le \|u\|^{r^-}_{L^{r(\cdot)}(\Omega)}; \tag{2.4}$$

$$\|u\|_{L^{r(\cdot)}(\Omega)} \to 0\ (\to \infty) \ \Leftrightarrow\ \int_\Omega |u(x)|^{r(x)}\,dx \to 0\ (\to \infty). \tag{2.5}$$

To recall the definition of the isotropic Sobolev space with variable exponent, $W^{1,r(\cdot)}(\Omega)$, we set

$$W^{1,r(\cdot)}(\Omega) = \{u \in L^{r(\cdot)}(\Omega) : \partial_{x_i} u \in L^{r(\cdot)}(\Omega) \text{ for all } i \in \{1,\dots,N\}\},$$

endowed with the norm

$$\|u\|_{W^{1,r(\cdot)}(\Omega)} = \|u\|_{L^{r(\cdot)}(\Omega)} + \sum_{i=1}^N \|\partial_{x_i} u\|_{L^{r(\cdot)}(\Omega)}.$$

The space $\left(W^{1,r(\cdot)}(\Omega), \|\cdot\|_{W^{1,r(\cdot)}(\Omega)}\right)$ is a separable and reflexive Banach space (see [11, Theorem 1.3]). To pass to the anisotropic spaces with variable exponent, everywhere below we consider $\vec{p}:\overline{\Omega}\to\mathbb{R}^N$ to be the vectorial function

$$\vec{p}(x) = (p_1(x), \ldots, p_N(x))$$

with $p_i \in C_+(\overline{\Omega})$ for all $i \in \{1, \ldots, N\}$ and we put

$$\vec{P}^+ = (p_1^+, \ldots, p_N^+), \quad \vec{P}^- = (p_1^-, \ldots, p_N^-)$$

and

$$P_+^+ = \max\{p_1^+, \ldots, p_N^+\}, \quad P_-^- = \min\{p_1^-, \ldots, p_N^-\},, \quad P_-^+ = \max\{p_1^-, \ldots, p_N^-\}$$

Throughout this paper we assume that:

$$\sum_1^N \frac{1}{p_i^-} > 1 \tag{2.6}$$

we define P_-^* and $P_{-,\infty}$ by

$$P_-^* = \frac{N}{\sum_1^N \frac{1}{p_i^-} - 1}, \quad P_{-,\infty} = \max\{P_-^*, P_-^+\}$$

An important subspace of $W^{1,\vec{p}(\cdot)}(\Omega)$ is $W_0^{1,\vec{p}(\cdot)}(\Omega)$, that is, the subspace of the functions that are vanishing on the boundary. According to [13], the space $W_0^{1,\vec{p}(\cdot)}(\Omega)$, equip with the norm $\|u\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)}$ is a reflexive Banach space, where

$$\|u\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)} = \sum_{i=1}^N \|\partial_{x_i} u\|_{L^{p_i(\cdot)}(\Omega)}\,.$$

Let us also recall a compactness result that will be essential in our approach (see, [[19], Theorem 1] or [[19], Proposition 2.1]):

Theorem 2.1. *Assume that $\Omega \subset \mathbb{R}^N,\ (N \geq 3)$ is a bounded domain with smooth boundary. Assume relation (2.6) is fulfilled. For any $q \in C_+(\overline{\Omega})$ verifying*

$$1 < q(x) < P_{-,\infty} \ forall\ x \in \overline{\Omega}, \tag{2.7}$$

the embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact.

For further results, properties and applications regarding anisotropic variable exponent spaces the reader can also consult [[20][21][22].

3. $p(\vec{x})$- Laplace Anisotropic Equation

Consider the Dirichlet problem

$$-\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) = \lambda f(x,u) + \mu g(x,u), \quad \text{for } x \in \Omega \tag{3.1}$$
$$u(x) = 0 \quad on \quad \partial\Omega$$

where Ω is a bounded domain of $\mathbb{R}^N$ with C^1-boundary $\partial\Omega$, and

$A_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $i \in \{1, \ldots, N\}$, and by $F : \Omega \times \mathbb{R} \to \mathbb{R}$ the antiderivatives of the Carathéodory functions $a_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, respectively $f : \Omega \times \mathbb{R} \to \mathbb{R}$; that is,

$$A_i(x,s) = \int_0^s a_i(x,t)\,dt, \quad F(x,s) = \int_0^s f(x,t)\,dt, \quad G(x,s) = \int_0^s g(x,t)\,dt.$$

For every $i \in \{1, \ldots, N\}$, we work under the following hypotheses.

(A1) There exists a positive constant $\bar{c}_i$ such that a_i fulfills

$$|a_i(x,s)| \leq \bar{c}_i \left(1 + |s|^{p_i(x)-1}\right),$$

for all $x \in \Omega$ and all $s \in \mathbb{R}$.

(A2) There exists $k_i > 0$ such that

$$k_i |s|^{p_i(x)} \leq a_i(x,s)s \leq p_i(x)\ A_i(x,s),$$

for all $x \in \Omega$ and all $s \in \mathbb{R}$.

(A3) The monotonicity condition

$$[a_i(x,s) - a_i(x,t)](s-t) > 0$$

takes place for all $x \in \Omega$ and all $s, t \in \mathbb{R}$ with $s \neq t$.

(A4) $a_i(x,0) = 0$ for all $x \in \partial\Omega$.

(FG) the functions $f, g \in C(\Omega \times \mathbb{R}, \mathbb{R})$ such that:

$$|f(x,t)|, |g(x,t)| \leq C(1 + |t|^{\rho(x)-1}) \forall (x,t) \in \Omega \times \mathbb{R}$$

where C is a positive constant, $\rho : \Omega \to (1, \infty)$ is a continuous function and

$$P_+^+ < \rho^- \leq \rho^+ < P_\infty^-$$

Taking into consideration condition (A4) we can introduce the notion of weak solution to our problem.

Definition 3.1. *We define the weak solution for problem* (1.2) *as a function* $u \in V$ *satisfying:*

$$\int_{\Omega} \sum_{i=1}^{N} a_i(x, \partial_{x_i} u)\partial_{x_i} v \, dx = \lambda \int_{\Omega} f(x,u)v \, dx \; + \int_{\Omega} \mu g(x,u)v \, dx, = 0,$$

for all $v \in V$.

Theorem 3.2. *Let us suppose* $2 < p_-^- \leq p_+^+ < N$ *and* $(A1), (A2), (A3), (A4)$ *and* (FG). *Furthermore, suppose that:*

$$\max\{\limsup_{t \to 0} \frac{\sup_{x \in \Omega} F(x,t)}{|t|^{P_+^+}}, \limsup_{t \to \infty} \frac{\sup_{x \in \Omega} F(x,t)}{|t|^{P_-^-}}\} \leq 0 \tag{3.2}$$

and

$$\sup_{u \in W_0^{1,p(\vec{x})}(\Omega)} \int_{\Omega} F(x,t)dx > 0.$$

If we set:l

$$\omega = inf\{\frac{\sum_{i=1}^{N} \int_{\Omega} A_i(x,u)dx}{\int_{\Omega} F(x,t)dx} : \; u \in W_0^{1,p(\vec{x})}(\Omega), \; \int_{\Omega} F(x,t)dx > 0\}$$

Then for each compact interval $[a,b] \subset]\omega, +\infty[$*, there exists* $r_1 > 0$ *with the following property: for every* $\lambda \in [a,b]$, *and every function* g *satifying (F), there exists* $\delta > 0$ *such that for each* $\mu \in [0, \delta]$*, the one equation problem* (3.1) *has at least three nonzero weak solutions in* E *whose norms are less than* r_1.

To prove our result we need to define the following functionals $\Phi, J, \Psi : E \longrightarrow \mathbb{R}$,

$$\Phi(u) = \int_{\Omega} \sum_{i=1}^{N} A_i(x, \partial_{x_i} u) \, dx$$

and

$$J(u) = \int_{\mathbb{R}^n} F(x,u) \, dx$$

and

$$\Psi(u) = \int_{\mathbb{R}^n} G(x,u) \, dx$$

The idea is to apply a variational principle due to B. Ricceri [1], by proving that the functions $\Psi(u)$, $J(u)$, $\Phi(u)$ verify the assumptions of Theorem 1

Lemma 3.3. $\Phi(u)$ *is well defined function , coercive, bounded on each bounded subset of* E. $\Phi'(u) : E \longrightarrow E'$ *is well defined mapping, coercive, hemicontinuous, monotone and satisfies the property: for any sequence* $(u_n) \subset E$ *and any* $u \in E$ *such that* (u_n) *converges weakly to* $u \in E$ *and* $limsup_{n \longrightarrow \infty} \langle \Phi(u)'(u_n - u), u_n - u \rangle \leq 0$, *we have* (u_n) *converges strongly to* u *in* E.

Proof. Let prove that the function $\Phi(u)$ is coercive i.e. $\Phi(u)$ needs to go to ∞ when $\|u\|_{p(\vec{x})}$ goes to ∞. let for every $u \in E$ avec $\|u\|_{p(\vec{x})} > 1$, we definie

$$\delta_i \begin{cases} P_+^+ \text{ si } & |\partial_{x_i}|_{p_i(.)} < 1 \\ P_-^- \text{ si } & |\partial_{x_i}|_{p_i(.)} > 1 \end{cases}$$

on one hand the following equality holds $\forall u \in E$

$$\sum_{i=1}^{N} |\partial_{x_i}|_{p_i(.)}^{P_-^-} \geq N\left(\frac{\sum_{i=1}^{N} |\partial_{x_i}|_{p_i(.)}}{N}\right)^{P_-^-} = \frac{\|\partial_{x_i}\|_{\vec{p}(.)}^{P_-^-}}{N^{P_-^- - 1}}$$

on the other hand for $\|u\|_{p(\vec{.})} > 1$ we have :

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i}|^{P_i(x)} dx \geq \sum_{i=1}^{N} |\partial_{x_i}|_{p_i(.)}^{\delta_i} \geq \sum_{i=1}^{N} |\partial_{x_i}|_{p_i(.)}^{P_-^-} - \sum_{i, \delta_i = P_+^+} |\partial_{x_i}|_{p_i(.)}^{P_+^+} \frac{\|u\|_{\vec{p}(.)}^{P_-^-}}{N^{P_-^- - 1}} - N$$

Thus:

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i}|^{P_i(x)} dx \geq \frac{\|u\|_{\vec{p}(.)}^{P_-^-}}{N^{P_-^- - 1}} - N$$

the assumption (A2) gave us :

$$\sum_{i=1}^{N} \int_{\Omega} |\frac{P_i(x)}{k_i}| A_i(x,u) dx \geq \sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i}|^{P_i(x)} dx$$

$$\sum_{i=1}^{N} \int_{\Omega} |\frac{P_+^+}{k^+}| A_i(x,u) dx \geq \sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i}|^{P_i(x)} dx$$

and:

$$\sum_{i=1}^{N} \int_{\Omega} \frac{P_+^+}{k^+} A_i(x,u) dx \geq \frac{\|u\|_{\vec{p}(.)}^{P_-^-}}{N^{P_-^- - 1}} - N$$

then

$$\Phi(u) = \sum_{i=1}^{N} \int_{\Omega} A_i(x,u) dx \geq \frac{k^+}{P_+^+}\left(\frac{\|u\|_{\vec{p}(.)}^{P_-^-}}{N^{P_-^- - 1}} - N\right) \tag{3.3}$$

For all $u \in E$ with $\|u\|_{\vec{p}(.)} > 1$, then Φ is coercive.

A simple calculus leads us to the fact that Φ is well-defined on E and $\Phi \in C^1(E, \mathbb{R})$with the derivative given by:

$$\langle \Phi'(u), u \rangle = \sum_{i=1}^{N} \int_{\Omega} a_{x_i}(x, \partial_{x_i} u) \partial_{x_i} u \; dx$$

It is obvious that Φ bounded on each bounded of E. Next step we will prove that Φ' est coercive. hypotheses (A2) leads us to:

$$\langle \Phi'(u), u\rangle = \sum_{i=1}^{N}\int_{\Omega} |a_{x_i}(x, \partial_u)|\partial_{x_i} u dx \geq \sum_{i=1}^{N}\int_{\Omega} \frac{|\partial_{x_i}(u)|^{p_i(x)}}{k_i}\, dx$$

then :

$$\langle \Phi'(u), u\rangle \geq \frac{1}{k^+}\frac{\|u\|_{\vec{p}(.)}^{P_-^-}}{N^{P_-^- -1}}$$

hold for all $u \in E$ with $\|u\|_{\vec{p}(.)} > 1$, then Φ' is coercive.
the assumption (A3) assure that Φ' uniformly monotone.
We note that in [4] proved that if the condition (A2)(A3)(A4) are satisfied then for any sequence $(u_n) \subset E$ and any $u \in E$ such that (u_n) converges weakly to $u \in E$ and $limsup_{n\longrightarrow\infty}\langle \Phi(u)'(u_n - u), u_n - u\rangle \leq 0$, we have (u_n) converges strongly to u in E.

□

Proof of Theorem 2. From proposition1 we deduce that Φ is a coercive, sequentially weakly lower semicontinuous C1 functional, belonging to W_E and bounded on each bounded subset of E. Since Φ' is coercive, hemicontinuous and uniformly monotone on E, using [[18], Theorem 26.A (d)] we deduce that the inverse of Φ' is continuous. It is easy to see that J' and ψ are strongly continuous. Using Proposition 26.2(a) in [[18]] it follows that J' and ψ' are compact. Thus, J, ψ are C^1 functionals that admit compact derivative. Functional Φ has a strict local minimum at $u = 0$ with $\Phi(0) = J(0) = 0$.

We fix an arbitrary. $\varepsilon > 0$. From the condition (10) we deduce there exist des r_1, r_2 with $0 < r_1 < 1 < r_2$ such that $F(x,t) \leq \varepsilon |t|^{P_+^+}$ for all $(x,t) \in \Omega \times [-r_1, r_1]$ and

$$F(x,t) \leq \varepsilon |t|^{P_-^-},\ \ \forall,\ \ (x,t) \in \Omega \times (\mathbb{R} - [r_2, r_2]). \tag{3.4}$$

thus , we get $F(x,t) \leq \varepsilon |t|^{P_+^+}$ for all $(x,t) \in \Omega \times (\mathbb{R}\ ([-r_2, -r_1][r_1, r_2]))$.
The fact that F is bounded on each bounded subset of $\Omega \times \mathbb{R}$, we can choose a constant $C_\varepsilon > 0$ and s with $P_+^+ < s < P_{-,\infty}$, such that:

$$F(x,t) \leq \varepsilon |t|^{P_+^+} + C_\varepsilon |t|^s,\ \ \forall,\ \ (x,t) \in \Omega \times \mathbb{R}.$$

the relation (4) , for all $u \in E$ with $\|u\|_{\vec{p}(.)} < 1$, we obtient

$$\begin{aligned}\frac{\|u\|_{\vec{p}(.)}^{P_+^+}}{N^{P_+^+-1}} &= N\left(\frac{\sum_{i=1}^N |\partial_{x_i} u|_{p_i(.)}}{N}\right)^{P_+^+} \\ &\leq \sum_{i=1}^N |\partial_{x_i} u|_{P_+^+}^{p_i(.)} \\ &\leq \sum_{i=1}^N |\partial_{x_i} u|_{p_i^+}^{p_i(.)} \\ &\leq \sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)} \\ &\leq \sum_{i=1}^N p_i(x) A_i(x, \partial_{x_i} u) = P_+^+ \Phi(u)\end{aligned}$$

this give us

$$\Phi(u) \geq \frac{\|u\|_{\vec{p}_i(.)}^{P_+^+}}{P_+^+ N^{P_+^+-1}}, \quad \forall, \quad \|u\|_{\vec{p}_i(.)} < 1.$$

the theorem 2 and the relation (16) assure the existence of two constants C_1, C_2 such that:

(3.5) $$J(u) \leq C_1 \|u\|_{\vec{p}_i(.)}^{P_+^+} \varepsilon + C_2 \|u\|_{\vec{p}_i(.)}^{s} C_\varepsilon$$

the relation (17) give us

(3.6) $$\limsup_{u \to 0} \frac{J(u)}{\Phi(u)} \leq C_1 P_+^+ N^{P_+^+-1} \varepsilon.$$

the relations (15) et (12), for all $u \in E$ with $\|u\|_{\vec{p}(.)} > N$ we get

$$\frac{J(u)}{\Phi(u)} \leq P_+^+ N^{P_-^- -1}\left(\frac{\int_{\Omega\{|u| \leq r_2\}} F(x,u)dx}{\|u\|_{\vec{p}_i(.)}^{P_-^-} - N^{P_-^-}} + \frac{\int_{\Omega\{|u| > r_2\}} F(x,u)dx}{\|u\|_{\vec{p}_i(.)}^{P_-^-} - N^{P_-^-}}\right)$$

then the theorem give us

(3.7) $$\limsup_{\|u\|_{\vec{p}_i(.)} \to \infty} \frac{J(u)}{\Phi(u)} \leq C_3 P_+^+ N^{P_-^- -1} \varepsilon.$$

where C_3 is a positive constant.
since $\varepsilon > 0$ is arbitrate, with the relations (18) and (19) we conclude that:

$$max\left\{\limsup_{\|u\|_{\vec{p}(.)} \to 0} \frac{J(u)}{\Phi(u)}, \limsup_{\|u\|_{\vec{p}(.)} \to \infty} \frac{J(u)}{\Phi(u)}\right\} \leq 0$$

then α defined in theorem is nulle i.e. ($\alpha = 0$). we deduce that $\beta > 0$ of the theorem R.

4. (p(x),q(x))-Anisotropic System

In this section we consider the two-equations nonlinear problems involving the $(\vec{p}(x),\vec{q}(x))$-laplacian :

$$
(4.1)\qquad \begin{cases} -\sum_{i=1}^{N}\partial_{x_i}a_i(x,\partial_{x_i}u)=\lambda\, f_u(x,u,v)+\mu\, g_u(x,u,v) & \text{in } \Omega \\ \\ -\sum_{i=1}^{N}\partial_{x_i}b_i(x,\partial_{x_i}u)=\lambda\, f_v(x,u,v)+\mu\, g_v(x,u,v) & \text{in } \Omega \\ \\ \qquad\qquad u\in W_0^{1,p(\vec{x})},\ v\in W_0^{1,q(\vec{x})} \end{cases}
$$

where $2<p^-<p(x)<p^+<N, 2<q^-<q(x)<q^+<N$. $\lambda,\mu>0$ are parameters. The functions $a_i(x,s)$ are functions satisfying (A1),(A2),(A3) and (A4). The function $b_i(x,s)$ will satisfy: $B_i:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$, $i\in\{1,\ldots,N\}$, the antiderivatives of the Carathéodory functions $b_i:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$, respectively $f:\Omega\times\mathbb{R}\to\mathbb{R}$; that is,

$$
B_i(x,s)=\int_0^s b_i(x,t)\,dt,
$$

For every $i\in\{1,\ldots,N\}$, we work under the following hypotheses.

(B1) There exists a positive constant $\bar{c}_i$ such that b_i fulfills

$$
|b_i(x,s)|\le \bar{c}_i\left(1+|s|^{p_i(x)-1}\right),
$$

for all $x\in\Omega$ and all $s\in\mathbb{R}$.

(B2) There exists $k_i>0$ such that

$$
k_i|s|^{p_i(x)}\le b_i(x,s)s\le p_i(x)\ B_i(x,s),
$$

for all $x\in\Omega$ and all $s\in\mathbb{R}$.

(B3) The monotonicity condition

$$
[b_i(x,s)-b_i(x,t)](s-t)>0
$$

takes place for all $x\in\Omega$ and all $s,t\in\mathbb{R}$ with $s\neq t$.

(B4) $b_i(x,0)=0$ for all $x\in\partial\Omega$.

(F1) the functions $f,g\in C(\Omega\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ such that:

$$
(4.2)\qquad |f_t(x,t,t')+f_{t'}(x,t,t')|\le C(1+(|t|+|t'|)^{\rho(x)-1})\forall(x,t,t')\in\Omega\times\mathbb{R}\times\mathbb{R}
$$

$$
(4.3)\qquad |g_{t'}(x,t,t')+g_{t'}(x,t,t')|\le C(1+(|t|+|t'|)^{\rho(x)-1})\forall(x,t,t')\in\Omega\times\mathbb{R}\times\mathbb{R}
$$

where C is a positive constant, $\rho : \Omega \to (1, \infty)$ is a continuous function and

$$S_+^+ < \rho^- \leq \rho^+ < S_\infty^-$$

where $S_+^+ = max(P_+^+, Q_+^+)$ and $S_\infty^- = min(P_\infty^-, Q_\infty^-)$ and $S_-^- = min(P_-^-, Q_-^-)$

We set $E = W_0^{1,\vec{p}(x)}(\Omega)$,and $E' = W_0^{1,\vec{q}(x)}(\Omega)$, $U = (u, v)$, We shall look for a weak-solution of (4.1) in the space $E \times E'$ which is endowed with the Cartesian norm $\|(u, v)\| = \|u\|_E + \|v\|_{E'}$. We set

$$f(x, U) = f_u(x, u, v) + f_v(x, u, v), \quad g(x, U) = g_u(x, u, v) + g_v(x, u, v)$$
$$F(x, u, v) = \int_0^u f_u(x, s, v)\, ds + \int_0^v f_v(x, u, s)\, ds \text{ and } C_i(x, U) = A_i(x, u) + B_i(x, v)$$

Theorem 4.1. *Let us suppose* $2 < p_i(x),\ q_i(x) < N$ *and* $(A1), (A2), (A3), (B1), (B2), (B3)$*, the function* f *satisfy* (4.2). *Furthermore, suppose that:*

$$\max\{\limsup_{|t|+|t'|\to 0} \frac{\sup_{x\in\Omega} F(x, t, t')}{(|t| + |t'|)^{S_+^+}}, \limsup_{|U|\to\infty} \frac{\sup_{x\in\Omega} F(x, U)}{(U)^{S_-^-}}\} \leq 0 \tag{4.4}$$

and

$$\sup_{U\in E\times E'} \left(\int_{\mathbb{R}^N} F(x, U)dx \right) > 0$$

If we set:

$$\omega = inf\{ \frac{\sum_{i=1}^N \int_\Omega C_i(x, U)dx}{\int_\Omega F(x, U)dx} : \ U \in E \times E',\ \int_\Omega F(x, U)dx > 0\}$$

then for each compact interval $[a, b] \subset]\omega, +\infty[$*, there exists* $r_1 > 0$ *with the following property: for every* $\lambda \in [a, b]$*, and every function* $g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ *which is measurable in* Ω *and continuous in* $\mathbb{R}$ *satisfying* (4.3) *, there exists* $\delta > 0$ *such that for each* $\mu \in [0, \delta]$*, the two- equation system* (4.1) *has at least three nonzero weak solutions in* E *whose norms are less than* r_1.

To prove this result we need to define the following functions
$\Phi, J, \Psi : E \times E' \longrightarrow \mathbb{R}$

$$\Phi(U) = \Phi(u) + \Phi(v) = \sum_{i=1}^N \int_\Omega C_i(x, U)dx$$

and

$$J(U) = J(u) + J(v) = \int_{\mathbb{R}^n} F(x, U)\, dx$$

and

$$\Psi(U) = \Psi(u) + \Psi(v) = \int_{\mathbb{R}^n} G(x, U)\, dx$$

The functions $\Phi(U)$, $J(U)$, $\Psi(U)$ need to satisfy the conditions of the theorem 1.1, i.e

$\Phi : E\times E' \to \mathbb{R}$, a coercive, sequentially weakly lower semi continuous C^1 functional, belonging to $\mathcal{W}_{E\times E'}$, bounded on each bounded subset of $E\times E'$, and whose derivative admits a continuous inverse on X^*.

On one hand we have

$$\Phi(u) \geq \frac{\|u\|_E^{P_+^+}}{P_+^+ N^{P_+^+-1}}, \ \forall, \ \|u\|_E < 1.$$

and

$$\Phi(v) \geq \frac{\|v\|_{E'}^{Q_+^+}}{Q_+^+ N^{Q_+^+-1}}, \ \forall, \ \|v\|_{E'} < 1.$$

then

$$\Phi(U) = \Phi(u) + \Phi(v) \geq \frac{\|u\|_E^{P_+^+}}{P_+^+ N^{P_+^+-1}} + \frac{\|v\|_{E'}^{Q_+^+}}{Q_+^+ N^{Q_+^+-1}} \ \forall \ \|u\|_E < 1 \ and \ \|v\|_{E'} < 1.$$

$$\Phi(U) \geq K(\|u\|_E^{S_+^+} + \|v\|_{E'}^{S_+^+})$$

then for a very small value of $\|u\|_E, \ and, \ \|v\|_{E'}$ we have

$$\Phi(U) \geq K(\|u\|_E + \|v\|_{E'})^{S_+^+}$$

On the other hand we have for a very big value of $\|u\|_E, \ and, \ \|v\|_{E'}$ we have

$$\Phi(u) = \sum_{i=1}^{N} \int_\Omega A_i(x,u)dx \geq \frac{k^+}{P_+^+}\left(\frac{\|u\|_E^{P_-^-}}{N^{P_-^- -1}} - N\right)$$

and

$$\Phi(v) = \sum_{i=1}^{N} \int_\Omega A_i(x,v)dx \geq \frac{k^+}{Q_+^+}\left(\frac{\|v\|_{E'}^{Q_-^-}}{N^{Q_-^- -1}} - N\right)$$

then

$$\Phi(U) = \Phi(u) + \Phi(v) \geq \frac{k^+}{P_+^+}\left(\frac{\|u\|_E^{P_-^-}}{N^{P_-^- -1}} - N\right) + \frac{k^+}{Q_+^+}\left(\frac{\|v\|_{E'}^{Q_-^-}}{N^{Q_-^- -1}} - N\right)$$

then

$$\Phi(U) \geq C(\|u\|_E^{P_-^-} + \|v\|_{E'}^{Q_-^-} - 2N)$$

$$\Phi(U) \geq C((\|u\|_E + \|v\|_{E'})^{S_-^-} - 2N)$$

$$\alpha = \max\left\{0, \limsup_{\|U\|\to+\infty} \frac{J(U)}{\Phi(U)}, \limsup_{\|U\|\to 0} \frac{J(U)}{\Phi(U)}\right\},$$

$$\beta = \sup_{x\in\Phi^{-1}(]0,+\infty[)} \frac{J(U)}{\Phi(U)},$$

with $\alpha = 0 < \beta = \infty$ Then, for each compact interval $[a, b] \subset]0, \infty[$ there exists $r_1 > 0$ such that the problem (4.1) has at least three solutions in $E \times E'$ whose norms are less than r_1.

5. Example

Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded domain with smooth boundary,p_i continuous functions on Ω and $2 \leq p_i(x), q_i(x) < N$, for each $x \in \Omega$ and every $i \in \{1, 2, ..., N\}, \lambda, \mu > 0$ are real numbers, C_1, C_2 two positive constants, $m_1, m_2, m_3, m_4, m_5 : \Omega \to \mathbb{R}$ continuous functions such that

$$m_1(x) > 0 \ for \ \ every \ x \in \Omega, \tag{5.1}$$

$$S_+^+ < m_2^- \leq m_2^+ < m_4^- \leq m_4^+ < S_{-,\infty} \tag{5.2}$$

and

$$S_+^+ < m_3(x) < m_5(x) < S_{-,\infty} \tag{5.3}$$

for every $x \in \Omega$ and $S_+^+, S_{-,\infty}$ defined in section 3

$$\gamma = \inf\{\frac{\sum_1^N \int_\Omega (A_i(x, \partial_i u) + B_i(x, \partial_i v))dx}{\int_\Omega m_1(x)(\frac{C_1(|u|^{m_2(x)} + |v|^{m_2(x)})}{m_2(x)} - C_2(\frac{(|u|^{m_4(x)} + |v|^{m_4(x)})}{m_4(x)})dx} : u \in E\}$$

where $E = W_0^{1,\vec{p}(.)} \times W_0^{1,\vec{q}(.)}$, Then, for each compact interval $[a, b] \subset (\gamma, \infty)$, there exists $r > 0$ with the following property: for every $\lambda \in [a, b]$ and for function $g, f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by:

$$g(x, t, t') = (|t| + |t'|)^{m_3(x)-2}(t + t')$$

and

$$f(x, u) = m_1(x)(C_1|u|^{m_2(x)-2}u - C_2|u|^{m_4(x)-2}u)$$

then there exists $\delta > 0$ such that, for each $\nu \in [0, \delta]$ the anisotropic mean curvature problem with variable exponent :

$$\begin{cases} -\sum_{i=1}^N \partial_{x_i}((1 + |\partial_{x_i} u|^2)^{\frac{p_i(x)-2}{2}} \partial_{x_i} u) = \lambda f(x, u) + \mu g(x, u, v) & \text{in } \Omega \\ -\sum_{i=1}^N \partial_{x_i}((1 + |\partial_{x_i} u|^2)^{\frac{q_i(x)-2}{2}} \partial_{x_i} v) = \lambda f(x, v) + \mu g(x, u, v) & \text{in } \Omega \\ u \in W_0^{1,p(\vec{x})}, \ v \in W_0^{1,q(\vec{x})} \end{cases} \tag{5.4}$$

has at least three solutions whose norms are less than r.

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