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REDUCING GRAPH COLORING TO CLIQUE SEARCH

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ABSTRACT. Coloring the nodes of a graph is a widely used technique to speed up practical clique search algorithms. This motivates our interest in various graph coloring schemes. Because of computational costs mainly simple greedy graph coloring procedures are considered. In this paper we will show that certain graph coloring schemes can be reduced to finding cliques in an appropriately constructed auxiliary graph. Once again because of computational costs involved one has to resort on not exhaustive clique search procedures. These lead to new greedy graph coloring algorithms which can be used as preconditioning tools before embarking on large scale clique searches.

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1. Introduction

Throughout this paper under graph we mean finite simple graph, that is, graphs in this paper have finitely many vertices, finitely many edges without any loop or double edges. Let $G = (V, E)$ be a finite simple graph. Here V is the set of vertices of G and E is the set of edges of G. Let D be a subset of V and let Δ be the subgraph of G induced by D. We say that Δ is a clique in G if any two distinct elements of D are adjacent in G. We call Δ a k-clique if the set D has k elements.

Finding cliques in a given graph is an important problem in discrete applied mathematics with many applications inside and outside of mathematics. For further details see [1], [2], [4], [7], [9], [17], [18].

We formally state the following clique search problem.

Problem 1. *Given a finite simple graph G and given a positive integer k. Decide if G contains a k-clique.*

Coloring the nodes of a graph is intimately related to finding cliques in the graph. Namely, many practical clique search algorithms employ coloring to speed up the computation by reducing the search space. Finding optimal or nearly optimal colorings is itself a computationally demanding problem. From this reason in the above computations computationally more feasible greedy algorithms are used to construct suboptimal colorings.

In this paper we propose to explore the opposite direction. We describe practical means to reduce coloring to clique search. From a given graph we construct a larger new auxiliary graph. The cliques in this auxiliary graph are associated with with colorings of the original graph.

In [6] an algorithm is presented to locate suboptimal cliques in relatively large graphs. This algorithm can be applied to the auxiliary graph to construct coloring for the original graph. Numerical experiments indicate that the program provides impressive results. We recommend this approach as a preconditioning tool before submitting the original graph to any of the standard clique search algorithms.

All the evidences available at the time of writing this paper indicate that the proposed coloring procedure translates to better running times in practical clique search algorithms.

2. Legal coloring of the nodes

We color the nodes of a graph G satisfying the following conditions.

(1) Each node of G receives exactly one color.
(2) Adjacent nodes in G cannot receive the same color.

This is the most commonly encountered coloring of the nodes of a graph and it is referred as legal coloring of the nodes. It is well known that coloring can be used for estimating clique size.

Let us suppose that Δ is an l-clique in G and let us suppose that the nodes of G have a legal coloring with k colors. Then $l \leq k$ holds.

Indeed, a legal coloring of the nodes of G gives a legal coloring of the nodes of Δ. Note that in a legal coloring of the nodes of Δ at least l colors must occur. This gives $l \leq k$, as required.

Problem 2. *Given a finite simple graph G and given a positive integer k. Decide if the nodes of G have a legal coloring using k colors.*

Both Problems 2 and 1 are decision problems. From the complexity theory of computations we know that these problems belong to the NP-complete complexity class.

Table 1. The adjacency matrix of the graph G in Example 1

	1	2	3	4	5	6
1	×	•		•	•	
2	•	×	•			
3		•	×	•		•
4	•		•	×	•	
5	•			•	×	
6			•			×

Problems 2 and 1 are polynomially reducible to each other. The point we would like to make here is that reducing Problem 2 to Problem 1 can be utilized in practical computations.

Here is a way how Problem 2 can be reduced to Problem 1.

Using the graph $G = (V, E)$ and using the positive integer k one constructs an auxiliary graph $\Gamma = (W, F)$. The nodes of Γ are the ordered pairs

$$(v, a), \text{ where } v \in V, \ 1 \le a \le k.$$

The intended meaning of the pair (v, a) is that node v of G receives color a.

Let us pick two distinct nodes

$$w_1 = (v_1, a_1) \text{ and } w_2 = (v_2, a_2)$$

of Γ. If the unordered pair $\{v_1, v_2\}$ is an edge of G, then in a legal coloring of the nodes of G the colors a_1, a_2 cannot be identical. When we construct Γ we do not connect the nodes w_1, w_2 if the unordered pair $\{v_1, v_2\}$ is an edge of G and if in addition $a_1 = a_2$ holds. In a coloring of the nodes of G a node cannot receive two distinct colors. Thus when we construct Γ we do not connect w_1, w_2 by an edge in Γ if $v_1 = v_2$.

Let n be the number of vertices of G, that is, let $n = |V|$. The graph Γ has nk vertices.

Observation 1. *If the nodes of the graph G have a legal coloring using k colors, then the graph Γ contains a n-clique.*

Proof. Let us assume that the nodes of G can be colored legally using k colors. Let $f : V \to \{1, \ldots, k\}$ be a function which describes this coloring. Let

$$D = \{(v, f(v)) : \ v \in V\}$$

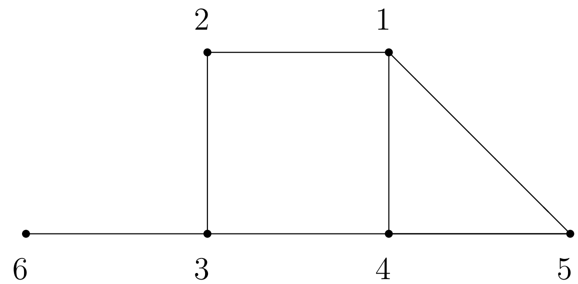

FIGURE 1. A graphical representation of the graph G in Example 1.

TABLE 2. The partitioned form of the adjacency matrix A of the auxiliary graph Γ in Example 1.

B	C	D	C	C	D
C	B	C	D	D	D
D	C	B	C	D	C
C	D	C	B	C	D
C	D	D	C	B	D
D	D	C	D	D	B

and let Δ be the subgraph of Γ induced by D. Clearly, D has n elements. We claim that Δ is an n-clique in Γ.

In order to verify this claim we pick two distinct nodes

$$w_1 = (v_1, f(v_1)) \text{ and } w_2 = (v_2, f(v_2))$$

from D.

If $v_1 = v_2$, then $f(v_1) = f(v_2)$ must hold as the node v_1 receives exactly one color. This means that $w_1 = w_2$. But we know that $w_1 \neq w_2$.

If $v_1 \neq v_2$ and the unordered pair $\{v_1, v_2\}$ is an edge of G, then $f(v_1) \neq f(v_2)$ holds since the coloring defined by f is legal. This means that we have connected the nodes w_1, w_2 by an edge in Γ when we constructed Γ.

If $v_1 \neq v_2$ and the unordered pair $\{v_1, v_2\}$ is not an edge of G, then we have connected the nodes w_1, w_2 by an edge in Γ when we have constructed Γ. $\square$

Observation 2. *If the graph Γ contains an n-clique, then the nodes of the graph G can be colored legally using k colors.*

TABLE 3. Monotonic matrices.

n	$\lvert V\rvert$	$\lvert E\rvert$	**L**	**D**	**Q**	**K**
3	27	189	6	6	5	**5**
4	64	1 296	12	10	9	**8**
5	125	5 500	20	17	16	**14**
6	216	17 550	30	25	22	**20**
7	343	46 305	42	36	32	**26**
8	512	106 624	56	45	41	**33**
9	729	221 616	72	60	52	**43**
10	1 000	425 250	90	74	67	**54**
11	1 331	765 325	110	93	82	
12	1 728	1 306 800	132	109	102	
13	2 197	2 135 484	156	129	121	
14	2 744	3 362 086	182	153	136	
15	3 375	5 126 625	210	177	161	

TABLE 4. Deletion error correcting codes.

n	$\lvert V\rvert$	$\lvert E\rvert$	**L**	**D**	**Q**	**K**
3	8	9	2	2	2	**2**
4	16	57	4	4	5	**4**
5	32	305	8	6	7	**6**
6	64	1 473	14	13	14	**12**
7	128	6 657	26	23	25	**20**
8	256	28 801	50	42	44	**37**
9	512	121 089	101	84	88	**68**
10	1 024	499 713	199	155	160	
11	2 048	2 037 761	395	306	301	

Proof. Let us suppose that Γ has an n-clique Δ and D is the set of nodes of Δ. Let

$$I_v = \{(v, a) :\ 1 \le a \le k\}$$

TABLE 5. Johnson codes.

n	$\|V\|$	$\|E\|$	L	D	Q	K
8	70	1 855	20	17	17	14
9	126	6 615	35	30	30	25
10	210	19 425	56	48	46	40
11	330	49 665	84	71	67	56
12	495	114 345	120	99	91	77
13	715	242 385	165	133	128	
14	1 001	480 480	220	171	164	
15	1 365	900 900	286	222	209	
16	1 820	1 611 610	364	281	265	
17	2 380	2 769 130	455	376	325	

for each $v \in V$. Obviously, I_v has k elements. Note that the sets I_v, $v \in V$ are pair-wise disjoint independent sets in Γ. Further note that the union of these sets is equal to W.

The nodes of Γ can be colored legally using n colors. The sets I_v, $v \in V$ can play the roles of the color classes of the nodes of Γ. Since Δ is a clique in Γ it follows that each I_v contains at most one element from D. Using the fact that $|D| = n$ we can conclude that D is a complete set of representatives of the sets I_v, $v \in V$.

Set

$$T = \{v : \ (v, a) \in D\}$$

We can see that $T = V$. Therefore each $v \in V$ receives exactly one color. We may express this result such that the map $f : V \to \{1, \ldots, k\}$ defined by $f(v) = a$ is a function. It remains to show that the function f describes a legal coloring of the nodes of G.

Suppose that the unordered pair $\{v_1, v_2\}$ is an edge of G. and consider two distinct nodes

$$w_1 = (v_1, f(v_1)) \text{ and } w_2 = (v_2, f(v_2))$$

of Δ. When we constructed the graph Γ we have connected the nodes w_1, w_2 by an edge in Γ because $f(v_1) \neq f(v_2)$. □

We will draw conclusions from the proof of Observation 2. This is the reason we have included a proof.

Example 1. *Let the finite simple graph $G = (V, E)$ be given by its adjacency matrix in Table 1. The graph has 6 nodes and 7 edges. A graphical representation of G is depicted in Figure 1.*

We would like to decide if the nodes of G have a legal coloring using $k = 3$ colors. The auxiliary graph Γ has $|V| \cdot k = 18$ nodes. Let A be the adjacency matrix of Γ. The matrix A can be partitioned into 3 by 3 blocks. The partitioned form of A is given in Table 2. Note that the pattern of the adjacency matrix of G and the pattern of the blocks in the adjacency matrix of Γ are similar. The block B is a 3 by 3 matrix in which all cells are empty. The block D is a 3 by 3 matrix in which all cells are filled with bullets. Finally, the block C is a 3 by 3 matrix, where the cells in the main diagonal are empty and all the other cells are filled with bullets.

Finding a legal coloring of a graph with 1000 nodes using 60 colors can be reduced to locate a 1000-clique in the auxiliary graph which has 60000 nodes. The size of the auxiliary graph makes us wonder if this proposal is reasonable. We carried out a large scale numerical experiment. The graphs we used are coming from coding theory. They are related to the existence of certain error correcting codes. (Details from the monotonic matrices can be found in [15].) The results are summarized in Tables 3, 4, and 3. The columns labeled by L contain the number of colors one gets using the simplest sequential greedy coloring procedure. The columns labeled by D holds the number of colors we get using the dsatur algorithm described in [3]. The columns labeled by Q and K contain the number of colors provided by the proposed clique approach.

One can locate a suboptimal size clique in the auxiliary graph using the following procedure. We check if the graph itself is a clique. If not, then let us pick a node with a maximum degree. Next we delete all the nodes that are not adjacent to this node. Repeating this procedure for the remaining graph eventually we end up with a clique. The results provided by this greedy clique search algorithm are listed in the columns with heading Q.

Locating independent sets in a graph is equivalent to locating cliques in the complement of the graph. In [6] an algorithm is advanced for locating independent sets is sparse graph. (The program is downloadable from http://algo2.iti.kit.edu/kamis/.) The algorithm of course is applicablee for locating cliques in dense graphs. We used this algorithm. The running time of the program was limited to 600 seconds by default. This limited the sizes of the auxiliary graphs we could complete the search. The results are listed in columns labeled by K. Cells remained empty when the auxiliary graph for too big to complete the search during alloted 600 second time. In spite of their limited nature the results are impressive in comparison with the other coloring algorithms.

In summary we can say that the numerical experiments show that the clique approach for coloring is a feasible proposal. Further, there are many local search based methods to locate cliques in a graph. It means there is a large number of potential greedy coloring methods.

3. 2-fold coloring of the nodes

We color the nodes of a given finite simple graph G with k colors satisfying the following conditions.

(1) Each node receives exactly two distinct colors.
(2) Adjacent nodes never receive the same color.

This coloring of the nodes of the graph G is referred as a 2-fold legal coloring of the nodes of G. Coloring can be used for bounding clique size.

Let us suppose that Δ is an l-clique in G and let us suppose that the nodes of G have a 2-fold legal coloring using k colors. Then $l \leq k/2$ holds.

Indeed, a 2-fold legal coloring of the nodes of G provides a 2-fold legal coloring of the nodes of Δ. Note that in a 2-fold legal coloring of the nodes of Δ at least $2l$ colors must occur. This gives $2l \leq k$, as required.

Problem 3. *Given a finite simple graph G and given a positive integer k. Decide if the nodes of G have a 2-fold legal coloring using k colors.*

Problem 3 can be reduced to Problem 1. Using the graph $G = (V, E)$ and using the integer k we construct an auxiliary graph $\Gamma = (W, F)$. The nodes of Γ are the ordered triples

$$(v, a, b), \text{ where } v \in V, \ 1 \leq a < b \leq k.$$

The intuitive meaning of the triple (v, a, b) is that node v receives the two distinct colors a, b. Let n be the number of vertices of G, that is, let $n = |V|$. The number of nodes of Γ is equal to $nk(k-1)/2$.

Let us consider two distinct nodes

$$w_1 = (v_1, a_1, b_1) \text{ and } w_2 = (v_2, a_2, b_2)$$

of Γ. If the unordered pair $\{v_1, v_2\}$ is an edge of G, then $\{a_1, b_1\} \cap \{a_2, b_2\} = \emptyset$ must hold in a 2-fold legal coloring of the nodes of G. Thus when we construct the graph Γ we do not connect w_1, w_2 by an edge in Γ if $\{v_1, v_2\}$ is an edge of G and $\{a_1, b_1\} \cap \{a_2, b_2\} \neq \emptyset$.

TABLE 6. The adjacency matrix of the graph G in Example 2

	1	2	3	4	5
1	×	•			•
2	•	×	•		
3		•	×	•	
4			•	×	•
5	•			•	×

TABLE 7. The partitioned form of the adjacency matrix A of the auxiliary graph Γ in Example 2.

B	C	D	D	C
C	B	C	D	D
D	C	B	C	D
D	D	C	B	C
C	D	D	C	B

In a 2-fold legal coloring of the nodes of G each node receives exactly two distinct colors. Thus when we construct the graph Γ we do not connect w_1, w_2 by an edge in Γ if $v_1 = v_2$.

Observation 3. *If the nodes of G have a 2-fold legal coloring with k colors, then Γ contains an n-clique.*

Proof. Suppose that the nodes of G have a 2-fold legal coloring with k-colors and $f : V \to P$ is a function which describes this coloring. Here

$$P = \{\{a, b\} : \ 1 \leq a < b \leq k\}.$$

Set $D = \{(v, f(v)) : \ v \in V\}$. It is clear that $|D| = n$. Let Δ be the subgraph of Γ induced by D. We will verify that Δ is a clique in Γ. In order to do so let us pick two distinct nodes

$$w_1 = (v_1, f(v_1)) \text{ and } w_2 = (v_2, f(v_2))$$

from Δ.

If $v_1 = v_2$, then $f(v_1) = f(v_2)$ must hold as f is a function. In this situation $w_1 = w_2$. But we know that $w_1 \neq w_2$. We are left with the case when $v_1 \neq v_2$.

TABLE 8. The 10 by 10 size block C in Example 2.

	1	1	1	1	2	2	2	3	3	4
	2	3	4	5	3	4	5	4	5	5
1,2	×							•	•	•
1,3		×				•	•			•
1,4			×		•		•		•	
1,5				×	•	•		•		
2,3			•	•	×					•
2,4		•		•		×			•	
2,5		•	•				×	•		
3,4	•			•			•	×		
3,5	•		•			•			×	
4,5	•	•			•					×

If the unordered pair $\{v_1, v_2\}$ is an edge of G, then $[f(v_1)] \cap [f(v_2)] = \emptyset$ since the function f codes a 2-fold legal coloring of the nodes of G. When we constructed Γ we have connected the nodes w_1, w_2 by an edge in Γ on the base that the unordered pair $\{v_1, v_2\}$ is an edge of G and $[f(v_1)] \cap [f(v_2)] = \emptyset$. We are left with the case when the unordered pair $\{v_1, v_2\}$ is not an edge of G. When we constructed Γ we have connected the nodes w_1, w_2 by an edge in Γ on the base that the unordered pair $\{v_1, v_2\}$ is not an edge of G. □

Observation 4. *If Γ contains an n-clique, then the nodes of G have a 2-fold legal coloring using k colors.*

Proof. Suppose that Γ contains an n-clique Δ whose set of nodes is equal to D. Let

$$I_v = \{(v, a, b) : \ 1 \leq a < b \leq k\}$$

for each $v \in V$. Clearly, $|I_v| = k(k-1)/2$. The reader will note that the sets I_v, $v \in V$ are pair-wise disjoint independent sets of Γ. We a can see that the union of these independent sets is equal to W.

The nodes of Γ have a legal coloring using n colors. The reason is that the sets I_v, $v \in V$ can be identified with with n color classes of the nodes of Γ. Each node of the n-clique Δ falls into exactly one color class. Set

$$T = \{v : \ (v, a, b) \in U\}.$$

It follows that $T = V$. We define a function $f : V \to P$ by setting $f(v) = \{a, b\}$ whenever $(v, a, b) \in U$. Here $P = \{\{a, b\} : 1 \leq a < b \leq k\}$. The function f describes a 2-fold coloring of the nodes of G. It remains to show that this coloring is a 2-fold legal coloring of the nodes of G.

Let us consider an unordered pair $\{v_1, v_2\}$ which is an edge of G and let us focus our attention to the distinct nodes

$$w_1 = (v_1, f(v_1)) \text{ and } w_2 = (v_2, f(v_2))$$

of Δ. When we constructed the graph Γ we connected the nodes w_1, w_2 by an edge in Γ exactly on the base that $[f(v_1)] \cap [f(v_2)] = \emptyset$. □

Example 2. *Let the finite simple graph $G = (V, E)$ be given by its adjacency matrix in Table 6. The graph is a circle which has 5 nodes and 5 edges.*

We would like to decide if the nodes of the graph G have a 2-fold legal coloring with 5 colors. In order to reduce the coloring problem to a clique problem we construct the auxiliary graph $\Gamma = (W, F)$. In this case $n = |V| = 5$ and $k = 5$. The graph Γ has $nk(k-1)/2 = 50$ nodes. Let A be the adjacency matrix of Γ. We partition A into 10 by 10 size blocks. Table 7 exhibits the adjacency matrix A in block form. Note that the pattern of A follows the pattern of the adjacency matrix of G.

The block B is a 10 by 10 matrix in which all the 100 cells are empty. The block D is a 10 by 10 matrix in which all the cells are filled with bullets. Finally, the block C is a 10 by 10 matrix given in Table 8. The rows and the columns of the block C are labeled by the 2-element subsets of the set $\{1, \ldots, k\}$. The cell at the intersection of the row labeled by $\{a_1, b_1\}$ and the column labeled by $\{a_2, b_2\}$ holds a bullet if $\{a_1, b_1\} \cap \{a_2, b_2\} = \emptyset$.

Problem 3 can be reduced to Problem 2 too. This means that instead of finding a a 2-fold coloring of the nodes of a given graph we may look for a (1-fold) legal coloring of the nodes of a larger auxiliary graph.

Using the graph $G = (V, E)$ and the positive integer k we construct an auxiliary graph $\Gamma = (W, F)$. The nodes of Γ are the ordered pairs

$$(v, \alpha), \text{ where } v \in V, \ 1 \leq \alpha \leq 2.$$

The number of the nodes of Γ is equal to $2n$.

Let us consider two distinct nodes

$$w_1 = (v_1, \alpha_1) \text{ and } w_2 = (v_2, \alpha_2)$$

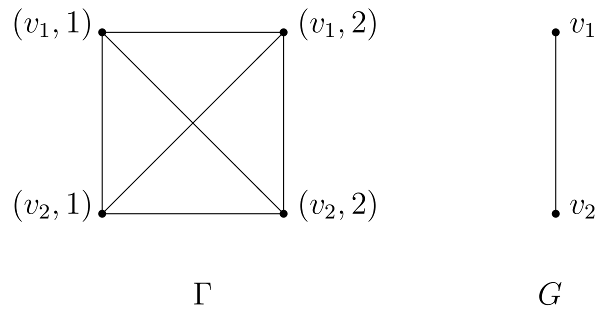

FIGURE 2. The correspondence between the graphs G and Γ.

of Γ. We connect the nodes w_1, w_2 by an edge in the graph Γ whenever $v_1 = v_2$. (In this situation $\alpha_1 \neq \alpha_2$ must hold as $w_1 \neq w_2$.) If the unordered pair $\{v_1, v_2\}$ is an edge of G, then we add the unordered pairs

$$\{(v_1, 1), (v_2, 1)\},\ \{(v_1, 1), (v_2, 2)\},$$

$$\{(v_1, 2), (v_2, 1)\},\ \{(v_1, 2), (v_2, 2)\},$$

as edges to Γ. Note that if the nodes w_1 and w_2 are adjacent in Γ and $v_1 \neq v_2$, then the nodes v_1 and v_2 must be adjacent in G.

Observation 5. *If the nodes of G have a 2-fold legal coloring using k colors, then the nodes of the auxiliary graph Γ have a legal coloring with k colors.*

Proof. Let us suppose that the nodes of G have a 2-fold legal coloring using k colors. We define a coloring of the nodes of Γ. If the node v of G receives the distinct colors a, b, then the nodes $(v, 1)$, $(v, 2)$ of Γ receive colors a, b, respectively.

It is clear that each node of the graph Γ receives exactly one color. It remains to verify that if $w_1 = (v_1, \alpha_1)$, $w_2 = (v_2, \alpha_2)$ are distinct adjacent nodes of Γ, then they do not receive the same color. We distinguish two cases depending on $v_1 = v_2$ or $v_1 \neq v_2$.

Case 1: $v_1 = v_2$. Now $\alpha_1 \neq \alpha_2$ must hold since otherwise the nodes w_1, w_2 are identical. We may assume that $\alpha_1 = 1$ and $\alpha_2 = 2$ since this is only a matter of exchanging the nodes. As $a \neq b$ then nodes w_1, w_2 of Γ receive distinct colors.

Case 2: $v_1 \neq v_2$. Now the unordered pair $\{v_1, v_2\}$ must be an edge of G. Let us suppose that the node v_1 of G receives the colors a_1, b_1 and the node v_2 of G receives the colors a_2, b_2. The colors a_1, b_1, a_2, b_2 are pair-wise distinct. Consequently the nodes w_1, w_2 of Γ receive distinct colors. □

Observation 6. *If the nodes of the auxiliary graph Γ have a legal coloring with k colors, then the nodes of the graph G have a 2-fold legal coloring with k colors.*

TABLE 9. The adjacency matrix of the auxiliary graph Γ in Example 3.

	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
	1	1	1	1	1	2	2	2	2	2	3	3	3	3	3
1,1	×	•			•	•	•			•	•	•			•
2,1	•	×	•			•	•	•			•	•	•		
3,1		•	×	•			•	•	•			•	•	•	
4,1			•	×	•			•	•	•			•	•	•
5,1	•			•	×	•			•	•	•			•	•
1,2	•	•			•	×	•			•	•	•			•
2,2	•	•	•			•	×	•			•	•	•		
3,2		•	•	•			•	×	•			•	•	•	
4,2			•	•	•			•	×	•			•	•	•
5,2	•			•	•	•			•	×	•			•	•
1,3	•	•			•	•	•			•	×	•			•
2,3	•	•	•			•	•	•			•	×	•		
3,3		•	•	•			•	•	•			•	×	•	
4,3			•	•	•			•	•	•			•	×	•
5,3	•			•	•	•			•	•	•			•	×

Proof. Let us assume that the nodes of the graph Γ have a legal coloring with k colors. We define a 2-fold coloring of the nodes of the graph G. If v is a node of G, then $w_1 = (v, 1)$ and $w_2 = (v, 2)$ are nodes of the graph Γ. Suppose that the node w_1 receives the colors a_1 and the node w_2 receives the color a_2. We assign the colors a_1, a_2 to the node v of G. As the nodes w_1 and w_2 are adjacent in Γ, the colors a_1 and a_2 must be distinct. In other words each node of G receives exactly two distinct colors.

It remains to show that if the unordered pair $\{v_1, v_2\}$ is an edge of G, then the nodes v_1 and v_2 cannot have the same color. The nodes $(v_1, 1)$, $(v_2, 1)$, $(v_1, 2)$, $(v_2, 2)$ of Γ are the nodes of a 4-clique in Γ. Since the nodes of Γ are legally colored, these nodes receive pair-wise distinct colors. Therefore the nodes v_1 and v_2 of G cannot receive the same color. □

Example 3. *Let the finite simple graph $G = (V, E)$ be given by its adjacency matrix in Table 6. The graph is a circle which has 5 nodes and 5 edges.*

TABLE 10. The simple greedy sequential coloring procedure applied to the auxiliary graph in Example 3.

$(1,1)$	[1]	1	1	[1]	[1]	[1]	1	1	[1]	[1]	[1]	1	1	[1]	1
$(2,1)$	$\to$	[2]	2	2	[2]	[2]	[2]	2	2	[2]	[2]	[2]	2	2	2
$(3,1)$		$\to$	[1]	1	1	[1]	[1]	[1]	1	1	[1]	[1]	[1]	1	1
$(4,1)$			$\to$	[2]	2	2	[2]	[2]	[2]	2	2	[2]	[2]	[2]	2
$(5,1)$				$\to$	[3]	3	3	[3]	[3]	[3]	3	3	[3]	[3]	3
$(1,2)$					$\to$	[4]	4	4	[4]	[4]	[4]	4	4	[4]	4
$(2,2)$						$\to$	[3]	3	3	[3]	[3]	[3]	3	3	3
$(3,2)$							$\to$	[4]	4	4	[4]	[4]	[4]	4	4
$(4,2)$								$\to$	[5]	5	5	[5]	[5]	[5]	5
$(5,2)$									$\to$	[6]	6	6	[6]	[6]	6
$(1,3)$										$\to$	[5]	5	5	[5]	5
$(2,3)$											$\to$	[6]	6	6	6
$(3,3)$												$\to$	[7]	7	7
$(4,3)$													$\to$	[8]	8
$(5,3)$														$\to$	7

TABLE 11. The nodes of the graph G and their colors in Example 3.

node	1	2	3	4	5
color	1	2	1	2	3
color	4	3	4	5	6
color	5	6	7	8	7

We would like to decide if the nodes of the graph G have a 3-fold legal coloring with 8 or less colors. In order to reduce this to a 1-fold legal coloring problem we construct the auxiliary graph $\Gamma = (W, F)$. The graph Γ has $3 \cdot |V| = 15$ nodes. The set of nodes of Γ is

$$W = \{(v, a) : \ v \in V, \ 1 \le a \le 3\}.$$

Table 9 exhibits the adjacency matrix of Γ. The greedy sequential coloring of the nodes is recorded in Table 10. The nodes of G and their colors can be seen in Table 11.

4. Coloring the edges of a graph

Instead of the nodes we are coloring the edges of a graph. For example we may color the edges of a graph G with k colors in the following way.

(1) Each edge of G receives exactly one color.
(2) If x, y, z are distinct nodes of a 3-clique in G, then the edges $\{x, y\}$, $\{y, z\}$, $\{x, z\}$ must receive three distinct colors.
(3) If x, y, u, v are distinct nodes of a 4-clique in G, then the edges $\{x, y\}$, $\{x, u\}$, $\{x, v\}$, $\{y, u\}$, $\{y, v\}$, $\{u, v\}$ must receive six distinct colors.

We call this type of coloring of the edges of G a legal or well or proper edge coloring. Edge coloring can be used for bounding clique size.

Let us suppose that Δ is an l-clique in G and let us suppose that the edges of G have a legal coloring using k colors. Then $l(l-1)/2 \leq k$ holds.

Indeed, a legal coloring of the edges of G provides a legal coloring of the edges of Δ. Note that in a legal coloring of the edges of Δ at least $l(l-1)/2$ colors must occur. This gives $l(l-1)/2 \leq k$, as required.

Problem 4. *Given a finite simple graph G and given a positive integer k. Decide if the edges of G have a legal coloring using k colors.*

Problem 4 can be reduced to Problem 2. Using the graph G and the positive number k we construct an auxiliary graph Γ. The edges of G will play the role of the vertices of Γ. Let

$$w_1 = \{u_1, v_1\} \text{ and } w_2 = \{u_2, v_2\}$$

be two distinct nodes of Γ. Of course w_1, w_2 are distinct edges of G.

Let us consider the set

$$X = \{u_1, v_1\} \cup \{u_2, v_2\} = \{u_1, v_1, u_2, v_2\}.$$

Plainly, $|X| \leq 4$. Since $w_1 \neq w_2$, it follows that $|X| \geq 3$. Let H_X be the subgraph of G induced by X.

When we construct the graph Γ we connect the nodes w_1, w_2 by an edge in Γ if H_X is a clique in G. As $3 \leq |X| \leq 4$, the graph H_X can only be a 3-clique or a 4-clique in G.

Observation 7. *If the edges of G have a legal coloring with k colors, then the vertices of Γ have a legal coloring with k colors.*

Proof. Suppose that the edges of G are legally colored using k colors. Let $f : E \rightarrow \{1, \dots, k\}$ be a function describing this coloring. As $W = E$, the function f describes a coloring of the nodes of Γ. The only thing we should prove is that adjacent nodes in Γ do not receive the same color.

Let

$$w_1 = \{u_1, v_1\} \text{ and } w_2 = \{u_2, v_2\}$$

be distinct adjacent nodes of Γ. This means that the subgraph H_X of G is a clique in G.

If $|X| = 4$, then H_X is a 4-clique. The six edges of H_X receive six distinct colors because f is a legal coloring of the edges of G. In particular the edges $\{u_1, v_1\}$, $\{u_2, v_2\}$ cannot receive the same color. Therefore the nodes w_1, w_2 of Γ cannot receive the same color.

If $|X| = 3$, then H_X is a 3-clique. The three edges of H_X receive three distinct colors because f is a legal coloring of the edges of G. In particular the edges $\{u_1, v_1\}$, $\{u_2, v_2\}$ cannot receive the same color. Therefore the nodes w_1, w_2 of Γ cannot receive the same color. □

Observation 8. *If the nodes of* Γ *have a legal coloring using* k *colors, then the edges of* G *have a legal coloring with* k *colors.*

Proof. Suppose that the nodes of Γ have a legal coloring with k colors. Let $f : W \to \{1, \dots, k\}$ be a function that represents this coloring. Since $W = E$, the function f records a coloring of the edges of G.

Let us consider the subgraph H_X of G. We should show that if H_X is a 3-clique in G, then the three edges of H_X are colored with three distinct colors. Similarly, if H_X is a 4-clique in G, then the six edges of H_X are colored with six distinct colors.

The three edges of the 3-clique H_X in G are nodes of a 3-clique in Γ. As f is legal coloring of the nodes of Γ it follows that the three edges of H_X receive three distinct colors.

Similarly, the six edges of the 4-clique H_X in G are nodes of a 6-clique in Γ. As f is legal coloring of the nodes of Γ it follows that the six edges of H_X receive six distinct colors. □

5. 3-CLIQUE FREE COLORING

We color the nodes of a graph G satisfying the following conditions.

(1) Each node of G receives exactly one color.
(2) The three nodes of a 3-clique in G cannot receive the same color.

We call this type of coloring of the nodes of G a 3-clique free coloring. Coloring can be used for estimating clique size.

Let us suppose that Δ is an l-clique in G and let us suppose that the nodes of G have a 3-clique free coloring with k colors. Then $l \leq 2k$ holds.

We indicate the proof in the case when l is an even number. A 3-clique free coloring of the nodes of G gives a 3-clique free coloring of the nodes of Δ. Note that in a 3-clique free coloring of the nodes of Δ at least $l/2$ colors must occur. This gives $l/2 \leq k$, as required.

Problem 5. *Given a finite simple graph G and given a positive integer k. Decide if the nodes of G have a 3-clique free coloring using k colors.*

Problem 5 can be reduced to Problem 1. Starting with the the graph $G = (V, E)$ and the positive integer k we construct an auxiliary graph $\Gamma = (W, F)$. The nodes of Γ are the triples

$$(\{u, v\}, a, b), \text{ where } \{u, v\} \in E, \ 1 \leq a, b, \leq k.$$

Let m be the number of edges of G, that is, let $m = |E|$. The number of the triples is equal to mk^2.

The triple $(\{u, v\}, a, b)$ intends to code the information that the end points u, v of the edge $\{u, v\}$ are colored with the colors a, b respectively. In this section we assume that each node of the graph G is end point of some edge of G. In other words we assume that the graph G does not contain isolated nodes.

Let us consider two distinct nodes

$$w_1 = (\{u_1, v_1\}, a_1, b_1) \text{ and } w_2 = (\{u_2, v_2\}, a_2, b_2)$$

of Γ. Set

$$X = \{u_1, v_1\} \cup \{u_2, v_2\} = \{u_1, v_1, u_2, v_2\}.$$

It is clear that $|X| \leq 4$ and since $u_1 \neq v_1$ we get that $|X| \geq 2$. Thus $2 \leq |X| \leq 4$. Let H_X be the subgraph of G induced by X. The nodes u_1, v_1, u_2, v_2 receive the colors a_1, b_1, a_2, b_2, respectively in the graph H_X.

When $|X| \leq 3$, then these nodes are not pair-wise distinct and it may happen that two distinct colors are assigned to a node in H_X. In this case we call the graph H_X a non-qualifying graph.

It also may happen that there is a 3-clique in H_X and all the three nodes of this 3-clique receive the same color. In this situation again we call the graph H_X a non-qualifying graph. In all the other cases H_X is called a qualifying graph.

When we construct the graph Γ we connect the nodes w_1, w_2 by an edge in Γ if H_X is a qualifying graph.

TABLE 12. The adjacency matrix of the graph G in Example 4

	1	2	3	4
1	×		•	•
2		×	•	
3	•	•	×	•
4	•		•	×

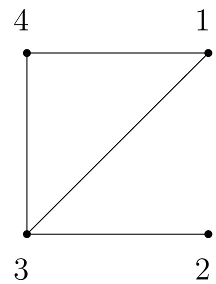

FIGURE 3. A graphical representation of the graph G in Example 4.

Observation 9. *If the nodes of G have a 3-clique free coloring with k colors, then the graph Γ contains an m-clique.*

Proof. Suppose that the nodes of the graph G have a 3-clique free coloring using k colors. Let $f : V \to \{1, \ldots, k\}$ be a function that codes this coloring. Set

$$D = \{(\{u, v\}, f(u), f(v)) : \ \{u, v\} \in E\}$$

and let Δ be the subgraph of Γ induced by D. It is clear that $|D| = m$. We claim that Δ is a clique in Γ.

In order to verify the claim let us choose two distinct nodes w_1, w_2 from D. Let us consider the subgraph H_X associated with w_1, w_2. Since f is a function, each node of H_X receives exactly one color. As f describes a 3-clique free coloring of the nodes of G, it follows that the restriction of f to the nodes of H_X is a 3-clique free coloring of the nodes of H_X. Thus H_X is a qualifying graph. Consequently, we connected w_1, w_2 by an edge in Γ when we constructed Γ. □

Observation 10. *If the auxiliary graph Γ contains an m-clique, then the nodes of the graph G have a 3-clique free coloring with k colors.*

TABLE 13. The nodes of the auxiliary graph Γ in Example 4.

1	$(\{1,3\},1,1)$	9	$(\{2,3\},1,1)$
2	$(\{1,3\},1,2)$	10	$(\{2,3\},1,2)$
3	$(\{1,3\},2,1)$	11	$(\{2,3\},2,1)$
4	$(\{1,3\},2,2)$	12	$(\{2,3\},2,2)$
5	$(\{1,4\},1,1)$	13	$(\{3,4\},1,1)$
6	$(\{1,4\},1,2)$	14	$(\{3,4\},1,2)$
7	$(\{1,4\},2,1)$	15	$(\{3,4\},2,1)$
8	$(\{1,4\},2,2)$	16	$(\{3,4\},2,2)$

Proof. Suppose that Γ contains an m-clique Δ and D is the set of nodes of Δ. Now $|D| = m$.

Set

$$I_{\{u,v\}} = \{(\{u,v\},a,b) : \ 1 \leq a,b, \leq k\}$$

for each $\{u,v\} \in E$. Obviously, $|I_{\{u,v\}}| = k^2$. Note that the sets $I_{\{u,v\}}$, $\{u,v\} \in E$ are pair-wise disjoint independent sets in Γ.

Indeed, if

$$w_1 = (\{u,v\},a_1,b_1) \text{ and } w_2 = (\{u,v\},a_2,b_2)$$

are distinct elements of $I_{\{u,v\}}$, then the graph H_X associated with w_1, w_2 has two nodes. From $w_1 \neq w_2$ it follows that $a_1 = a_2$, $b_1 = b_2$ cannot hold. Thus H_X is not qualifying. This means when we constructed Γ we did not connect w_1, w_2 by an edge in Γ.

The nodes of Γ have a legal coloring using m colors. The independent sets $I_{\{u,v\}}$, $\{u,v\} \in E$ can play the roles of the color classes.

As Δ is a clique in Γ each color class contains at most one element from D. Using the cardinality of D we can conclude that D is a complete set of representatives of the color classes.

Set

$$T = \{\{u,v\} : \ (\{u,v\},a,b) \in D\}.$$

It follows that $E = T$. Consequently, each node of G which is an end point of at least one edge of G receives at least one color. We claim that each node receives exactly one color.

In order to prove the claim assume on the contrary that more than one colors are assigned to a node of G. In this case there are distinct nodes w_1, w_2 of Δ such that a node

TABLE 14. The adjacency matrix of the auxiliary graph Γ in Example 4.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	×					•			•	•				•		
2		×			•	•			•	•					•	•
3			×				•	•			•	•	•	•		
4				×			•				•	•			•	
5		•			×					•		•			•	
6	•	•				×			•	•	•	•		•		•
7			•	•			×		•	•	•	•	•		•	
8			•					×	•		•			•		
9	•	•				•	•	•	×				•	•		
10	•	•			•	•	•			×					•	•
11			•	•		•	•	•			×				•	•
12			•	•	•	•	•					×	•	•		
13			•				•		•			•	×			
14	•		•			•		•	•			•		×		
15		•		•	•		•			•	•				×	
16		•				•				•	•					×

receives more than one color in the subgraph H_X associated with w_1, w_2. This means that H_X is not qualifying. On the other hand when we constructed Γ we connected w_1, w_2 by an edge on the base that the subgraph H_X was qualifying.

We may summarize our consideration by saying that we can define a function $f : V \to \{1, \ldots, k\}$ by setting $f(u) = b$ whenever $(\{u, v\}, a, b)$ is a node of Δ. It remains to show that the coloring of the nodes of G described by the function f is a 3-clique free coloring.

Suppose there is a 3-clique Ω in G whose nodes receive the same color. There are distinct nodes w_1, w_2 of Δ such that Ω is a 3-clique in the subgraph H_X associated with w_1, w_2. This means that H_X is not qualifying. On the other hand when we constructed Γ we connected w_1, w_2 by an edge in Γ because the subgraph H_X was qualifying. □

Example 4. *Let the finite simple graph $G = (V, E)$ be given by its adjacency matrix in Table 12. The graph has 4 nodes and 4 edges. Figure 3 depicts a possible geometric version of G .*

We wish to decide if the nodes of the graph G have a 3-clique free legal coloring with 2 colors. By constructing the auxiliary graph $\Gamma = (W, F)$ the question is reduced to a clique search. The graph Γ has $|V| \cdot k^2 = (4)(2^2) = 16$ nodes. The nodes of Γ are listed in Table 13.

References

[1] E. Balas, J. Xue, Weighted and unweighted maximum clique algorithms with upper bounds from fractional coloring, *Algorithmica* **15** (1996), 397–412.

[2] I. M. Bomze, M. Budinich, P. M. Pardalos, M. Pelillo, The Maximum Clique Problem, Handbook of Combinatorial Optimization Vol. 4, Kluwer Academic Publisher, 1999.

[3] D. Brelaz, New methods to color the vertices of a graph, *Communications of the ACM* **22** (1979), 251–256.

[4] R. Carraghan, P. M. Pardalos, An exact algorithm for the maximum clique problem, *Operation Research Letters* **9** (1990), 375–382.

[5] J. Hasselberg, P. M. Pardalos, and G. Vairaktarakis, Test case generators and computational results for the maximum clique problem, *Journal of Global Optimization* **3** (1993), 463–482. http://www.springerlink.com/content/p2m65n57u657605n

[6] S. Lamm, P. Sanders, C. Schulz, D. Strash, R. F. Werneck, Finding Near-Optimial Independent Sets at Scale. *Proceedings of the 16th Meeting on Algorithm Engineering and Experimentation (ALENEX'16)*. 2016.

[7] D. Kumlander, *Some Practical Algorithms to Solve the Maximum Clique problem* PhD. Thesis, Tallin University of Technology, 2005.

[8] C. Morgan, *A Combinatorial Search with Dancing Links*, PhD. Thesis, Univ. of Warwick, 1999–2000.

[9] P. R. J. Östergård, A fast algorithm for the maximum clique problem, *Discrete Applied Mathematics* **120** (2002), 197–207.

[10] P. Erdős, Graph theory and probability, *Canad. J. Math.* **11** (1959), 34–38.

[11] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-completeness*, Freeman, New York, 2003.

[12] J. Mycielski, Sur le coloriage des graphes, *Colloq. Math.* **3** (1955), 161–162.

[13] C. H. Papadimitriou, *Computational Complexity*, Addison-Wesley Publishing Company, Inc., 1994.

[14] S. Szabó, Parallel algorithms for finding cliques in a graph, *Journal of Physics: Conference Series* **268** (2011) 012030 DOI:10.1088/1742-6596/268/1/012030

[15] S. Szabó, Monotonic matrices and clique search in graphs, *Annales Univ. Sci. Budapest., Sect. Computatorica* **41** (2013), 307–322.

[16] S. Szabó and B. Zaválnij, Greedy algorithms for triangle free coloring, *AKCE International Journal of Graphs and Combinatorics* **9** No. 2 (2012), 169–186.

[17] E. Tomita and T. Seki, An efficient branch-and-bound algorithm for finding a maximum clique, *Lecture Notes in Computer Science* **2631** (2003), 278–289.

[18] D. R. Wood, An algorithm for finding a maximum clique in a graph, *Oper. Res. Lett.* **21** (1997), 211–217.