Asia Pacific Journal of Mathematics, Vol. 3, No. 2 (2016), 114-135 ISSN 2357-2205

RANDOM FIXED POINT THEOREMS FOR RANDOM MAPPINGS

R. A. RASHWAN[1], H. A. HAMMAD[2,*]

[1]Department of Mathematics, Faculty of Science, Assuit University, Assuit 71516, Egypt

[2]Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt

*Corresponding author

Abstract. The aim of this paper is to prove some random common fixed point theorems for weakly compatible mappings in Hilbert spaces. Our results extend and generalize several known results in the literature.

2010 Mathematics Subject Classification. 47H10.

Key words and phrases.random operators; weakly and semi-compatible mappings; Hilbert spaces.

1. Introduction

After the Banach's fixed point theorem many researchers worked on Hilbert spaces for generalizing this principle, they worked for Random operators. Random fixed point theorems are stochastic generalization of classical fixed point theorems. Bharucha-Reid [11] attracted the attention of several mathematicians in his survey article and gave wings to this theory. the results of Spacek and Hans in multivalued contractive mappings was extended by Itoh [15]. Random fixed point theorem has become the full fledged research area and various ideas associated with random fixed point theory are used to obtain the solution of nonlinear random system (see [11, 16, 18, 20]). Some of its recent literature noted in [7,8,12,17,19,21]. Beg [2, 3], and Beg and Shahzad [7, 10] studied the structure of common random fixed points and random coincidence points of a pair of compatible random operators and proved fixed point theorems for contractive random operators in Polish spaces. Recently Beg and Shahzad [7, 9] had used different iteration processes to obtain common random fixed points. Recently Badshah and Shrivastava [1] introduced the concept of semi-compatibility in Polish spaces. We first review the following concepts which are essentials for our study in this paper.

2. Preliminaries Notes

Throughout this paper, (Ω, Σ) denotes a measurable space consisting of a set Ω and sigma algebra Σ of subsets of Σ, X stands for a separable Banach space and C is a nonempty subset of X.

Definition 2.1 [12] A function $f : \Omega \longrightarrow X$ is said to be measurable if $f^{-1}(B) \in \Sigma$ for every Borel subset B of X. Let C is a nonempty subset of X. A mapping $f : \Omega \longrightarrow C$ is measurable if $f^{-1}(B \cap C) \in \Sigma$ for every Borel subset B of X.

Definition 2.2 [13] A mapping $T : \Omega \times C \longrightarrow C$ is random operator, if for each fixed $t \in C$, the mapping $T(., t) : \Omega \longrightarrow C$ is measurable.

Definition 2.3 [4] A random operator $T : \Omega \times C \longrightarrow C$ is continuous if $T(., t) : \Omega \longrightarrow C$ is continuous, for each $t \in \Omega$.

Definition 2.4 [14] A measurable mapping $\xi : \Omega \longrightarrow C$ is random fixed point of a random operator $T : \Omega \times C \longrightarrow C$ if $T(t, \xi(t)) = \xi(t)$ for each $t \in \Omega$.

Definition 2.5 [5] Let X be a Polish space that is separable complete metric space and $T, S : \Omega \times X \longrightarrow X$ then T and S are:
(i) compatible if $S(t, .)$ and $T(t, .)$ are compatible for each $t \in \Omega$. That is,

$$\lim_{n\to\infty} \|S(t, T(t, \xi_n(t))) - T(t, S(t, \xi_n(t)))\| = 0,$$

provided that $\lim_{n\to\infty} S(t, \xi_n(t))$ and $\lim_{n\to\infty} T(t, \xi_n(t))$ exists in X and $\lim_{n\to\infty} S(t, \xi_n(t)) = \lim_{n\to\infty} T(t, \xi_n(t))$ for each $t \in \Omega$, where ξ_n is a sequence of measurable mappings.

(ii) weakly compatible if

$$S(t, T(t, \xi(t))) = T(t, S(t, \xi(t))),$$

for every $t \in \Omega$ whenever $T(t, \xi(t)) = S(t, \xi(t))$, where ξ is a measurable mapping.

Definition 2.6 [13] A random operators $T, S : \Omega \times X \longrightarrow X$ are semi-compatible if

$$\lim_{n\to\infty} \|S(t, T(t, \xi_n(t))) - T(t, \xi_n(t))\| = 0,$$

whenever $\xi_n : \Omega \to X$, $n > 0$ is a measurable mapping such that

$$T(t, \xi_n(t)), S(t, \xi_n(t) \longrightarrow \xi(t) \text{ as } n \longrightarrow \infty,$$

for some measurable mapping $\xi : \Omega \to X$.

3. Results

We start with the following result:

Theorem 3.1 Let C be a non-empty closed subset of separable Hilbert space X. Let T be a self random operator defined on C such that $T(t,.) : C \longrightarrow C$ (for all $t \in \Omega$) satisfy the following condition:

$$\begin{aligned}\|T(t,x(t)) - T(t,y(t))\|^2 \leq\ & \alpha\ \frac{\|y(t) - T(t,y(t))\|^2\,[1 + \|x(t) - T(t,x(t))\|^2]}{1 + \|x(t) - y(t)\|^2} \\ & +\beta\ \frac{\|y(t) - T(t,y(t))\|^2 + \|y(t) - T(t,x(t))\|^2}{1 + \|y(t) - T(t,y(t))\|^2 . \|y(t) - T(t,x(t))\|^2} \\ & +\gamma\ \|x(t) - y(t)\|^2\end{aligned}$$

$$+\delta\ [\|x(t) - T(t,y(t))\|^2 + \|y(t) - T(t,x(t))\|^2], \tag{3.1}$$

for all $x(t), y(t)$ in $T(t,.)$, where $\alpha, \beta, \gamma, \delta \in [0,1)$ with $(\alpha + \beta + \gamma + 4\delta < 1)$.
Then T has a unique random fixed point.

Proof: We constract a sequence $\{x_n\}$ as $x_0 : \Omega \longrightarrow C$ is arbitrary measurable function, for $t \in \Omega$ and $n = 1, 2, 3, ...$

$$\mathbf{x}_{n+1=}T(t, \mathbf{x}_n(t)).$$

For fixed $t \in \Omega$ and $n = 1, 2, 3, ...$

$$
\begin{aligned}
\|x_n(t)-\mathbf{x}_{n+1}(t)\|^2 &= \|T(t,x_{n-1}(t))-T(t,x_n(t))\|^2 \\
&\le \alpha\ \frac{\|\mathbf{x}_n(t)-T(t,x_n(t))\|^2\,[1+\|x_{n-1}(t)-T(t,x_{n-1}(t))\|^2]}{1+\|x_{n-1}(t)-x_n(t)\|^2} \\
&\quad +\beta\ \frac{\|x_n(t)-T(t,x_n(t))\|^2+\|x_n(t)-T(t,x_{n-1}(t))\|^2}{1+\|x_n(t)-T(t,x_n(t))\|^2\,.\,\|x_n(t)-T(t,x_{n-1}(t))\|^2} \\
&\quad +\gamma\ \|x_{n-1}(t)-x_n(t)\|^2 \\
&\quad +\delta\ [\|x_{n-1}(t)-T(t,x_n(t))\|^2+\|x_n(t)-T(t,x_{n-1}(t))\|^2] \\
&\le \alpha\ \frac{\|x_n(t)-x_{n+1}(t)\|^2\,[1+\|x_{n-1}(t)-x_n(t)\|^2]}{1+\|x_{n-1}(t)-x_n(t)\|^2} \\
&\quad +\beta\ \frac{\|x_n(t)-x_{n+1}(t)\|^2+\|x_n(t)-x_n(t)\|^2}{\|x_n(t)-x_{n+1}(t)\|^2\,.\,\|x_n(t)-x_n(t)\|^2} \\
&\quad +\gamma\ \|x_{n-1}(t)-x_n(t)\|^2+\delta\ [\|x_{n-1}(t)-x_{n+1}(t)\|^2+\|x_n(t)-x_n(t)\|^2]
\end{aligned}
$$

$$= \alpha\ \|x_n(t)-x_{n+1}(t)\|^2+\beta\ \|x_n(t)-x_{n+1}(t)\|^2+\gamma\ \|x_{n-1}(t)-x_n(t)\|^2$$

(3.2) $$+\delta\ [\|x_{n-1}(t)-x_{n+1}(t)\|^2+\|x_n(t)-x_n(t)\|^2].$$

By a Parallelogram law, $\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+2\|y\|^2$, which implies that $\|x+y\|^2 = 2\|x\|^2+2\|y\|^2-\|x-y\|^2 \le 2\|x\|^2+2\|y\|^2$, we can write

$$
\begin{aligned}
\|x_{n-1}(t)-x_{n+1}(t)\|^2 &= \|x_{n-1}(t)-x_n(t)+x_n(t)-x_{n+1}(t)\|^2 \\
&= 2\|x_{n-1}(t)-x_n(t)\|^2+2\|x_n(t)-x_{n+1}(t)\|^2 \\
&\quad -\|x_{n-1}(t)-x_n(t)-[x_n(t)-x_{n+1}(t)]\|^2
\end{aligned}
$$

(3.3) $$\le 2\|x_{n-1}(t)-x_n(t)\|^2+2\|x_n(t)-x_{n+1}(t)\|^2.$$

From (3.3) in (3.2), we get

$$
\begin{aligned}
\|x_n(t)-x_{n+1}(t)\|^2 &\le \alpha\ \|x_n(t)-x_{n+1}(t)\|^2+\beta\ \|x_n(t)-x_{n+1}(t)\|^2+\gamma\ \|x_{n-1}(t)-x_n(t)\|^2 \\
&\quad +2\delta\ [\|x_{n-1}(t)-x_n(t)\|^2+\|x_n(t)-x_{n+1}(t)\|^2]
\end{aligned}
$$

$$\Longrightarrow \|x_n(t)-x_{n+1}(t)\|^2 \le \frac{\gamma+2\delta}{1-\alpha-\beta-2\delta}\ \|x_{n-1}(t)-x_n(t)\|^2,$$

let $S=\frac{\gamma+2\delta}{1-\alpha-\beta-2\delta}<1$, then

$$\|x_n(t)-x_{n+1}(t)\|^2 \le S\ \|x_{n-1}(t)-x_n(t)\|^2.$$

Proceeding in this way we can get

$$\|x_n(t)-x_{n+1}(t)\|^2 \le S^n\ \|x_\circ(t)-x_1(t)\|^2, \text{ where } n=1,2,3,..$$

For any integer p,

$$\begin{aligned} \|x_n(t) - x_{n+p}(t)\| &\leq \|x_n(t) - x_{n+1}(t)\| + \|x_{n+1}(t) - x_{n+2}(t)\| + \ldots\ldots + \|x_{n+p-1}(t) - x_{n+p}(t)\| \\ &\leq (S^n + S^{n+1} + \ldots\ldots + S^{n+p-1}) \|x_\circ(t) - x_1(t)\| \\ &\leq \frac{S^n}{1-S} \|x_\circ(t) - x_1(t)\| \text{ since } 0 < S < 1. \end{aligned}$$

So, as $n \longrightarrow \infty$, $\frac{S^n}{1-S} \|x_\circ(t) - x_1(t)\| \longrightarrow 0$. Then $\|x_n(t) - x_{n+p}(t)\| \to 0$ as $n \longrightarrow \infty$. Therefore $\{x_n\}$ is a Cauchy sequence in C. Since C is closed subset of a Hilbert space X, there exists an element $\nu(t) \in C(\Omega)$ such that

$$\lim_{n\to\infty} x_n(t) = \nu(t), \text{ for all } t \in \Omega.$$

Now, further we have

$$\begin{aligned} &\Rightarrow \|\nu(t) - T(t,\nu(t))\|^2 = \|\nu(t) - x_n(t) - (T(t,\nu(t)) - x_n(t))\|^2 \\ &\leq \|\nu(t) - x_n(t)\|^2 + \|T(t,\nu(t)) - x_n(t)\|^2 \\ &\quad +2\,\mathrm{Re}\,\langle \nu(t) - x_n(t), T(t,\nu(t)) - x_n(t)\rangle \\ &= \|\nu(t) - x_n(t)\|^2 + \|T(t,x_{n-1}(t)) - T(t,\nu(t))\|^2 + 2\,\mathrm{Re}\,\langle \nu(t) - x_n(t), T(t,\nu(t)) - x_n(t)\rangle \\ &\Rightarrow \|\nu(t) - T(t,\nu(t))\|^2 \leq \|\nu(t) - x_n(t)\|^2 \\ &\quad +\alpha\, \frac{\|\nu(t) - T(t,\nu(t))\|^2 [1 + \|x_{n-1}(t) - T(t,x_{n-1}(t))\|^2]}{1 + \|x_{n-1}(t) - \nu(t)\|^2} \\ &\quad +\beta\, \frac{\|\nu(t) - T(t,\nu(t))\|^2 + \|\nu(t) - T(t,x_{n-1}(t))\|^2}{1 + \|\nu(t) - T(t,\nu)\|^2 . \|\nu(t) - T(t,x_{n-1}(t))\|^2} + \gamma\; \|x_{n-1}(t) - \nu(t)\|^2 \end{aligned}$$

(3.4)

$$+\delta\, [\|x_{n-1}(t) - T(t,\nu(t))\|^2 + \|\nu(t) - T(t,x_{n-1}(t))\|^2] + 2\,\mathrm{Re}\,\langle \nu(t) - x_n(t), T(t,\nu) - x_n(t)\rangle$$

Taking $n \longrightarrow \infty$ in (3.4), we obtain that

$$x_n(t) \longrightarrow \nu(t),\; x_{n-1}(t) \longrightarrow \nu(t),$$

$$\begin{aligned} &\Rightarrow \|\nu(t) - T(t,\nu(t))\|^2 \leq \alpha\; \|\nu(t) - T(t,\nu(t))\|^2 + \beta\; \|\nu(t) - T(t,\nu(t))\|^2 \\ &\quad +\delta\; \|\nu(t) - T(t,\nu(t))\|^2 \\ &\Rightarrow (1 - \alpha - \beta - \delta)\, \|\nu(t) - T(t,\nu(t))\|^2 \leq 0, \end{aligned}$$

(since $\alpha + \beta + \delta < 1$), therefore $\|\nu(t) - T(t,\nu(t))\|^2 \leq 0$, implies $\nu(t) = T(t,\nu(t))$. So ν is a fixed point of T.

Uniqueness: Let $\omega \in C$ is another fixed point of T, where $\omega \neq \nu$. Then

$$\begin{aligned}
\|\nu(t)-\omega(t)\|^2 &= \|T(t,\nu(t))-T(t,\omega(t))\|^2 \\
&\leq \alpha \, \frac{\|\omega(t)-T(t,\omega(t))\|^2\,[1+\|\nu(t)-T(t,\nu(t))\|^2]}{1+\|\nu(t)-\omega(t)\|^2} \\
&\quad +\beta \, \frac{\|\omega(t)-T(t,\omega(t))\|^2+\|\omega(t)-T(t,\nu(t))\|^2}{1+\|\omega(t)-T(t,\omega(t))\|^2+\|\omega(t)-T(t,\nu(t))\|^2} \\
&\quad +\gamma \, \|\nu(t)-\omega(t)\|^2+\delta[\|\nu(t)-T(t,\omega(t))\|^2+\|\omega(t)-T(t,\nu(t))\|^2] \\
&= \beta \, \|\nu(t)-\omega(t)\|^2+\gamma \, \|\nu(t)-\omega(t)\|^2+2\delta \, \|\nu(t)-\omega(t)\|^2 \\
&\Rightarrow (1-\gamma-\beta-2\delta)\,\|\nu(t)-\omega(t)\|^2 \leq 0,
\end{aligned}$$

(since $\gamma+\beta+2\delta<1$), then $\|\nu(t)-\omega(t)\|^2=0$. So $\nu(t)=\omega(t)$.
Hence $\nu(t):\Omega\to X$ is a unique random fixed point of the random operator T.

Theorem 3.2 Let C be a non-empty closed subset of separable Hilbert space X. Let T,F be two self random operators defined on C such that $T(t,.),F(t,.):C\longrightarrow C$ (for all $t\in\Omega$) satisfy the following condition:

$$\begin{aligned}
\|T(t,x(t))-F(t,y(t))\|^2 &\leq \alpha \, \frac{\|y(t)-F(t,y(t))\|^2\,[1+\|x(t)-T(t,x(t))\|^2]}{1+\|x(t)-y(t)\|^2} \\
&\quad +\beta \, \frac{\|y(t)-F(t,y(t))\|^2+\|y(t)-T(t,x(t))\|^2}{1+\|y(t)-F(t,y(t))\|^2\,.\,\|y(t)-T(t,x(t))\|^2}+\gamma \, \|x(t)-y(t)\|^2
\end{aligned}$$

$$+\delta \, [\|x(t)-F(t,y(t))\|^2+\|y(t)-T(t,x(t))\|^2], \tag{3.5}$$

for all $x(t)$, $y(t)$ in $T(t,.)$, where $\alpha,\beta,\gamma,\delta\in[0,1)$ with $(\alpha+\beta+\gamma+4\delta<1)$.
Then T and F have a unique common random fixed point.

Proof: We constract a sequence $\{x_n\}$ as follows: Let $x_0:\Omega\longrightarrow C$ is arbitrary measurable function, for $t\in\Omega$ and $n=1,2,3,...$

$$x_{n+1=}T(t,x_n(t)),\ x_{n+2=}F(t,x_n(t)).$$

For fixed $t \in \Omega$ and $n = 1, 2, 3, ...$

$$\begin{aligned}
\|x_n(t) - x_{n+1}(t)\|^2 &= \|T(t, x_{n-1}(t)) - F(t, x_n(t))\|^2 \\
&\leq \alpha \frac{\|x_n(t) - F(t, x_n(t))\|^2 [1 + \|x_{n-1}(t) - T(t, x_{n-1}(t))\|^2]}{1 + \|x_{n-1}(t) - x_n(t)\|^2} \\
&\quad +\beta \frac{\|x_n(t) - F(t, x_n(t))\|^2 + \|x_n(t) - T(t, x_{n-1}(t))\|^2}{1 + \|x_n(t) - F(t, x_n(t))\|^2 . \|x_n(t) - T(t, x_{n-1}(t))\|^2} \\
&\quad +\gamma \|x_{n-1}(t) - x_n(t)\|^2 + \delta[\|x_{n-1}(t) - F(t, x_n(t))\|^2 + \|x_n(t) - T(t, x_{n-1}(t))\|^2] \\
&\leq \alpha \frac{\|x_n(t) - x_{n+1}(t)\|^2 [1 + \|x_{n-1}(t) - x_n(t)\|^2]}{1 + \|x_{n-1}(t) - x_n(t)\|^2} \\
&\quad +\beta \frac{\|x_n(t) - x_{n+1}(t)\|^2 + \|x_n(t) - x_n(t)\|^2}{\|x_n(t) - x_{n+1}(t)\|^2 . \|x_n(t) - x_n(t)\|^2} \\
&\quad +\gamma \|x_{n-1}(t) - x_n(t)\|^2 + \delta[\|x_{n-1}(t) - x_{n+1}(t)\|^2 + \|x_n(t) - x_n(t)\|^2].
\end{aligned}$$

$$\Rightarrow \|x_n(t) - x_{n+1}(t)\|^2 \leq (\frac{\gamma + 2\delta}{1 - \alpha - \beta - 2\delta}) \|x_{n-1}(t) - x_n(t)\|^2 ,$$

since $S = \frac{\gamma+2\delta}{1-\alpha-\beta-2\delta} < 1$, then

$$\|x_n(t) - x_{n+1}(t)\|^2 \leq S \ \|x_{n-1}(t) - x_n(t)\|^2 .$$

Proceeding in this way we can get

$$\|x_n(t) - x_{n+1}(t)\|^2 \leq S^n \ \|x_\circ(t) - x_1(t)\|^2 , \text{ where } n = 1, 2, 3, ..$$

For any integer p,

$$\begin{aligned}
\|x_n(t) - x_{n+p}(t)\| &\leq \|x_n(t) - x_{n+1}(t)\| + \|x_{n+1}(t) - x_{n+2}(t)\| + \\
&\quad ...... + \|x_{n+p-1}(t) - x_{n+p}(t)\| \\
&\leq (S^n + S^{n+1} + ... + S^{n+p-1}) \|x_\circ(t) - x_1(t)\| \\
&\leq (\frac{S^n}{1 - S}) \|x_\circ(t) - x_1(t)\| \text{ since } 0 < S < 1.
\end{aligned}$$

So, as $n \longrightarrow \infty$, $\frac{S^n}{1-S} \|x_\circ(t) - x_1(t)\| \longrightarrow 0$. Then $\|x_n(t) - x_{n+p}(t)\| \rightarrow 0$ as $n \longrightarrow \infty$. Therefore $\{x_n\}$ is a Cauchy sequence in C. Since C is closed subset of a Hilbert space X, there exists an element $\nu(t) \in C(\Omega)$ such that

$$\lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} x_{n+1}(t) = \lim_{n\to\infty} x_{n+2}(t) = \lim_{n\to\infty} x_{n-1}(t) = \nu(t), \text{ for all } t \in \Omega.$$

Now, further we have

$$
\begin{aligned}
\Rightarrow\ & \|\nu(t)-F(t,\nu(t))\|^2 = \|\nu(t)-x_n(t)-(F(t,\nu(t))-x_n(t))\|^2 \\
\leq\ & \|\nu(t)-x_n(t)\|^2 + \|F(t,\nu(t))-x_n(t)\|^2 \\
& +2\operatorname{Re}\langle \nu(t)-x_n(t), F(t,\nu(t))-x_n(t)\rangle \\
=\ & \|\nu(t)-x_n(t)\|^2 + \|T(t,x_{n-1}(t))-F(t,\nu(t))\|^2 + 2\operatorname{Re}\langle \nu(t)-x_n(t), F(t,\nu(t))-x_n(t)\rangle \\
\Rightarrow\ & \|\nu(t)-F(t,\nu(t))\|^2 \leq \|\nu(t)-x_n(t)\|^2 \\
& +\alpha\ \frac{\|\nu(t)-F(t,\nu(t))\|^2\,[1+\|x_{n-1}(t)-T(t,x_{n-1}(t))\|^2]}{1+\|x_{n-1}(t)-\nu(t)\|^2} \\
& +\beta\ \frac{\|\nu(t)-F(t,\nu(t))\|^2+\|\nu(t)-T(t,x_{n-1}(t))\|^2}{1+\|\nu(t)-F(t,\nu(t))\|^2.\,\|\nu(t)-T(t,x_{n-1}(t))\|^2} + \gamma\ \|x_{n-1}(t)-\nu(t)\|^2 \\
& +\delta\ [\|x_{n-1}(t)-F(t,\nu(t))\|^2 + \|\nu(t)-T(t,x_{n-1}(t))\|^2] \\
& +2\operatorname{Re}\langle \nu(t)-x_n(t), F(t,\nu(t))-x_n(t)\rangle\ .
\end{aligned}
$$

As $n \longrightarrow \infty$, we obtain that

$$x_n(t) \longrightarrow \nu(t),\ x_{n-1}(t) \longrightarrow \nu(t),$$

$$
\begin{aligned}
\Rightarrow\ & \|\nu(t)-T(t,\nu(t))\|^2 \leq \alpha\ \|\nu(t)-F(t,\nu(t))\|^2 + \beta\ \|\nu(t)-F(t,\nu(t))\|^2 \\
& +\delta\,\|\nu(t)-F(t,\nu(t))\|^2 \\
\Rightarrow\ & (1-\alpha-\beta-\delta)\,\|\nu(t)-F(t,\nu(t))\|^2 \leq 0,
\end{aligned}
$$

(since $\alpha+\beta+\delta<1$), therefore $\|\nu(t)-F(t,\nu(t))\|^2 \leq 0$, implies $\nu(t)=F(t,\nu(t))$. So ν is a fixed point of F. By the same way we can obtain that $\nu(t)=T(t,\nu(t))$.
Uniqueness: It is obvious, as in the proof Theorem 3.1.
Then T and F have a unique common random fixed point.

Theorem 3.3 Let C be a non-empty closed subset of separable Hilbert space X. Let E, J, S, H, A and $B : \Omega \times C \longrightarrow C$ be six random operators satisfy the following conditions:

(i) $A(t,X) \subset E(t,J(t,X))$ and $B(t,X) \subset S(t,H(t,X))$,

(ii)

$$\begin{aligned}
&\|A(t,x(t)) - B(t,y(t))\|^2 \\
\leq\ & \alpha\frac{\|E(t,J(t,y(t))) - B(t,y(t))\|^2\,[1+\|S(t,H(t,x(t))) - A(t,x(t))\|^2]}{1+\|S(t,H(t,x(t))) - E(t,J(t,y(t)))\|^2} \\
&+\beta\frac{\|E(t,J(t,y(t))) - B(t,y(t))\|^2 + \|E(t,J(t,y(t))) - A(t,x(t))\|^2}{1+\|E(t,J(t,y(t))) - B(t,y(t))\|^2 . \|E(t,J(t,y(t))) - A(t,x(t))\|^2} \\
&+\gamma\,\|S(t,H(t,x(t))) - E(t,J(t,y(t)))\|^2
\end{aligned}$$

(3.6) $$+\delta[\|S(t,H(t,x(t))) - B(t,y(t))\|^2 + \|E(t,J(t,y(t))) - A(t,x(t))\|^2],$$

(iii)

(3.7) $$AH = HA, SH = HS, JB = BJ \text{ and } EJ = JE.$$

For all $x(t), y(t) \in C$ and $t \in \Omega$, where $\alpha,\beta,\gamma,\delta \in [0,1)$ with $(\alpha+\beta+\gamma+4\delta < 1)$.

If either:
(iv) (A, SH) are semi-compatible, SH or A is continuous and (B, EJ) are weakly compatible,
Or:
(v) (B, EJ) are semi-compatible, EJ or B is continuous and(A, SH) are weakly compatible.
Then E, J, S, H, A and B have a unique common random fixed point.

Proof: Let $x_0 : \Omega \longrightarrow C$ is arbitrary measurable mapping. We define a sequence of measurable mapping $x_n : \Omega \longrightarrow C$ as follows:
(3.8)

$$A(t,x_{2n}(t)) = E(t,J(t,x_{2n+1}(t))) = f_{2n}(t) B(t,x_{2n+1}(t)) = S(t,H(t,x_{2n+2}(t))) = f_{2n+1}(t).$$

From condition (3.6) we have for $n = 1,2,3,...$ and $t \in \Omega$.

$$\begin{aligned}
&\|f_{2n}(t) - f_{2n+1}(t)\|^2 = \|A(t,x_{2n}(t)) - B(t,x_{2n+1}(t))\|^2 \\
&\leq \alpha\frac{\|E(t,J(t,x_{2n+1}(t))) - B(t,x_{2n+1}(t))\|^2\,[1+\|S(t,H(t,x_{2n}(t))) - A(t,x_{2n}(t))\|^2]}{1+\|S(t,H(t,x_{2n}(t))) - E(t,J(t,x_{2n+1}(t)))\|^2} \\
&\quad+\beta\frac{\|E(t,J(t,x_{2n+1}(t))) - B(t,x_{2n+1}(t))\|^2 + \|E(t,J(t,x_{2n+1}(t))) - A(t,x_{2n}(t))\|^2}{1+\|E(t,J(t,x_{2n+1}(t))) - B(t,x_{2n+1}(t))\|^2 . \|E(t,J(t,x_{2n+1}(t))) - A(t,x_{2n}(t))\|^2} \\
&\quad+\gamma\,\|S(t,H(t,x_{2n}(t))) - E(t,J(t,x_{2n+1}(t)))\|^2 \\
&\quad+\delta[\|S(t,H(t,x_{2n}(t))) - B(t,x_{2n+1}(t))\|^2 + \|E(t,J(t,x_{2n+1}(t))) - A(t,x_{2n}(t))\|^2].
\end{aligned}$$

It follows by (3.8) that

$$\begin{aligned}\|f_{2n}(t)-f_{2n+1}(t)\|^2 &= \alpha\frac{\|f_{2n}(t)-f_{2n+1}(t)\|^2\,[1+\|f_{2n-1}(t)-f_{2n}(t)\|^2]}{1+\|f_{2n-1}(t)-f_{2n}(t)\|^2}\\ &\quad +\beta\frac{\|f_{2n}(t)-f_{2n+1}(t)\|^2+\|f_{2n}(t)-f_{2n}(t)\|^2}{1+\|f_{2n}(t)-f_{2n+1}(t)\|^2\,.\,\|f_{2n}(t)-f_{2n}(t)\|^2}\\ &\quad +\gamma\,\|f_{2n-1}(t)-f_{2n}(t)\|^2\end{aligned}$$

$$+\delta[\|f_{2n-1}(t)-f_{2n+1}(t)\|^2+\|f_{2n}(t)-f_{2n}(t)\|^2]. \tag{3.9}$$

By a Parallelogram law, we can write as (3.3)

$$\begin{aligned}\|f_{n-1}(t)-f_{n+1}(t)\|^2 &= \|f_{n-1}(t)-f_n(t)+f_n(t)-f_{n+1}(t)\|^2\\ &= 2\,\|f_{n-1}(t)-f_n(t)\|^2+2\,\|f_n(t)-f_{n+1}(t)\|^2\\ &\quad -\|f_{n-1}(t)-f_n(t)-[f_n(t)-f_{n+1}(t)]\|^2\end{aligned}$$

$$\leq 2\,\|f_{n-1}(t)-f_n(t)\|^2+2\,\|f_n(t)-f_{n+1}(t)\|^2\,. \tag{3.10}$$

Applying (3.10) in (3.9) we get

$$\|f_{2n}(t)-f_{2n+1}(t)\|^2\leq(\frac{\gamma+2\delta}{1-\alpha-\beta-2\delta})\,\|f_{2n-1}(t)-f_{2n}(t)\|^2=S\,\|f_{2n-1}(t)-f_{2n}(t)\|^2$$

where $S=\frac{\gamma+2\delta}{1-\alpha-\beta-2\delta}<1$. Similarly proceeding the same way, by induction we get a measurable mapping $f_{2n}(t):\Omega\longrightarrow X$ such that $n=1,2,3,...$and $t\in\Omega$, we have

$$\|f_{2n}(t)-f_{2n+1}(t)\|^2\leq S^{2n}\,\|f_\circ(t)-f_1(t)\|^2\,,$$

since $S^{2n}\to 0$ as $n\longrightarrow\infty$ and by the argument of Theorem 3.1, we have $\{f_{2n}(t)\}$ is a Cauchy sequence and hence is convertgent in the separable Hilbert space X. For $t\in\Omega$, let $\{f_{2n}(t)\}$ and it subsequences converges to some measurable mapping $x:\Omega\longrightarrow C$, therefore

$$\begin{aligned}f_{2n}(t) &\longrightarrow x(t)\text{ as }n\longrightarrow\infty,\\ A(t,x_{2n}(t)) &= E(t,J(t,x_{2n+1}(t)))\longrightarrow x(t)\text{ as }n\longrightarrow\infty,\end{aligned}$$

$$B(t,x_{2n+1}(t))=S(t,H(t,x_{2n+2}(t)))\longrightarrow x(t)\text{ as }n\longrightarrow\infty. \tag{3.11}$$

Now, for every $t\in\Omega$. If SH is continuous, we have

(3.12)

$$S(t,H(t,A(t,x_{2n}(t))))\longrightarrow S(t,H(t,x(t))),\ S(t,H(t,S(t,H(t,x_{2n+2}(t)))))\longrightarrow S(t,H(t,x(t))),$$

since A and SH are semi-compatible, hence

$$A(t,S(t,H(t,x_{2n}(t))))\longrightarrow S(t,H(t,x(t))). \tag{3.13}$$

For every $t \in \Omega$, we get

$$\|A(t, S(t, H(t, x_{2n}(t)))) - B(t, x_{2n+1}(t))\|^2$$

$$\leq \alpha \frac{\|E(t,J(t,x_{2n+1}(t)))-B(t,x_{2n+1}(t))\|^2[1+\|S(t,H(t,S(t,H(t,x_{2n}(t)))))-A(t,S(t,H(t,x_{2n}(t))))\|^2]}{1+\|S(t,H(t,S(t,H(t,x_{2n}(t)))))-E(t,J(t,x_{2n+1}(t)))\|^2}$$

$$+\beta \frac{\|E(t,J(t,x_{2n+1}(t)))-B(t,x_{2n+1}(t))\|^2+\|E(t,J(t,x_{2n+1}(t)))-A(t,S(t,H(t,x_{2n}(t))))\|^2}{1+\|E(t,J(t,x_{2n+1}(t)))-B(t,x_{2n+1}(t))\|^2.\|E(t,J(t,x_{2n+1}(t)))-A(t,S(t,H(t,x_{2n}(t))))\|^2}$$

$$+\gamma \|S(t, H(t, S(t, H(t, x_{2n}(t))))) - E(t, J(t, x_{2n+1}(t)))\|^2$$

$$+\delta[\|S(t, H(t, S(t, H(t, x_{2n}(t))))) - B(t, x_{2n+1}(t))\|^2+\|E(t, J(t, x_{2n+1}(t))) - A(t, S(t, H(t, x_{2n}(t))))\|^2].$$

Letting $n \to \infty$ and using (3.11), (3.12) and (3.13) we get

$$\begin{aligned} \|S(t, H(t, x(t))) - x(t)\|^2 &\leq \beta \|x(t) - S(t, H(t, x(t)))\|^2 + \gamma \|S(t, H(t, x(t))) - x(t)\|^2 \\ &\quad +2\delta \|S(t, H(t, x(t))) - x(t)\|^2 \\ &\Rightarrow (1 - \beta - \gamma - 2\delta) \|S(t, H(t, x(t))) - x(t)\|^2 \leq 0, \end{aligned}$$

so that

$$S(t, H(t, x(t))) = x(t), \ t \in \Omega. \tag{3.14}$$

Again using condition (3.6),

$$\begin{aligned} &\|A(t, x(t)) - B(t, x_{2n+1}(t))\|^2 \\ &\leq \alpha \frac{\|E(t, J(t, x_{2n+1}(t))) - B(t, x_{2n+1}(t))\|^2 [1 + \|S(t, H(t, x(t)) - A(t, x(t))\|^2]}{1 + \|S(t, H(t, x(t))) - E(t, J(t, x_{2n+1}(t)))\|^2} \\ &\quad +\beta \frac{\|E(t, J(t, x_{2n+1}(t))) - B(t, x_{2n+1}(t))\|^2 + \|E(t, J(t, x_{2n+1}(t))) - A(t, x(t))\|^2}{1 + \|E(t, J(t, x_{2n+1}(t))) - B(t, x_{2n+1}(t))\|^2 . \|E(t, J(t, x_{2n+1}(t))) - A(t, x(t))\|^2} \\ &\quad +\gamma \|S(t, H(t, x(t))) - E(t, J(t, x_{2n+1}(t)))\|^2 \\ &\quad +\delta[\|S(t, H(t, x(t))) - B(t, x_{2n+1}(t))\|^2 + \|E(t, J(t, x_{2n+1}(t))) - A(t, x(t))\|^2]. \end{aligned}$$

Letting $n \to \infty$ and using (3.14), we get

$$\|A(t, x(t)) - x(t)\|^2 \leq \beta \|A(t, x(t)) - x(t)\|^2 + \delta \|x(t) - A(t, x(t))\|^2, \tag{3.15}$$

from (3.15), we get

$$(1 - \beta - \delta) \|A(t, x(t)) - x(t)\|^2 \leq 0, \ \Rightarrow x(t) = A(t, x(t)), \tag{3.16}$$

Both (3.14) and (3.16) give

$$x(t) = A(t, x(t)) = S(t, H(t, x(t))),\ t \in \Omega. \tag{3.17}$$

Since $A(t, X) \subset E(t, J(t, X))$ and hence there exists a measurable mapping $z : \Omega \longrightarrow C$ such that $A(t, \mathbf{x}(t)) = E(t, J(t, z(t)))$.
from (3.17), we have

$$x(t) = A(t, x(t)) = S(t, H(t, x(t))) = E(t, J(t, z(t))). \tag{3.18}$$

From condition (3.6), we obtain

$$\begin{aligned}
& \|A(t, x_{2n}(t)) - B(t, z(t))\|^2 \\
\leq\ & \alpha \frac{\|E(t, J(t, z(t))) - B(t, z(t))\|^2 [1 + \|S(t, H(t, x_{2n}(t))) - A(t, x_{2n}(t))\|^2]}{1 + \|S(t, H(t, x_{2n}(t))) - E(t, J(t, z(t)))\|^2} \\
& + \beta \frac{\|E(t, J(t, z(t))) - B(t, z(t))\|^2 + \|E(t, J(t, z(t))) - A(t, x_{2n}(t))\|^2}{1 + \|E(t, J(t, z(t))) - B(t, z(t))\|^2 . \|E(t, J(t, z(t))) - A(t, x_{2n}(t))\|^2} \\
& + \gamma \|S(t, H(t, x_{2n}(t))) - E(t, J(t, z(t)))\|^2 \\
& + \delta[\|S(t, H(t, x_{2n}(t))) - B(t, z(t))\|^2 + \|E(t, J(t, z(t))) - A(t, x_{2n}(t))\|^2].
\end{aligned}$$

Putting $n \longrightarrow \infty$, applying (3.18) and (3.11), we have

$$\|x(t) - B(t, z(t))\|^2 \leq \alpha \|x(t) - B(t, z(t))\|^2$$

$$+\beta \|x(t) - B(t, z(t))\|^2 + \delta[\|x(t) - B(t, z(t))\|^2, \tag{3.19}$$

from (3.19), we have

$$(1 - \alpha - \beta - \delta) \|x(t) - B(t, z(t))\|^2 \leq 0 \ \Rightarrow x(t) = B(t, z(t) \text{ for } t \in \Omega,$$

therefore,

$$B(t, z(t)) = E(t, J(t, z(t))) = x(t). \tag{3.20}$$

Since (B, EJ) are weakly compatible then,

$$B(t, E(t, J(t, z(t)))) = E(t, J(t, B(t, z(t)))), \tag{3.21}$$

from (3.20), we get

$$B(t, x(t)) = E(t, J(t, x(t))) \text{ for } t \in \Omega. \tag{3.22}$$

Again using condition (3.6), we get

$$\begin{aligned}
& \|A(t,x(t))-B(t,x(t))\|^2 \\
\leq\ & \alpha\frac{\|E(t,J(t,x(t)))-B(t,x(t))\|^2\,[1+\|S(t,H(t,x(t))-A(t,x(t))\|^2]}{1+\|S(t,H(t,x(t)))-E(t,J(t,x(t)))\|^2} \\
& +\beta\frac{\|E(t,J(t,x(t)))-B(t,x(t))\|^2+\|E(t,J(t,x(t)))-A(t,x(t))\|^2}{1+\|E(t,J(t,x(t)))-B(t,x(t))\|^2\,.\,\|E(t,J(t,x(t)))-A(t,x(t))\|^2} \\
& +\gamma\,\|S(t,H(t,x(t)))-E(t,J(t,x(t)))\|^2 \\
& +\delta[\|S(t,H(t,x(t)))-B(t,x(t))\|^2+\|E(t,J(t,x(t)))-A(t,x(t))\|^2].
\end{aligned}$$

By using (3.18) and (3.22), we have

$$\begin{aligned}
\|x(t)-E(t,J(t,x(t)))\|^2 \ \leq\ & \beta\,\|x(t)-E(t,J(t,x(t)))\|^2+\gamma\,\|x(t)-E(t,J(t,x(t)))\|^2 \\
& +2\delta\,\|x(t)-E(t,J(t,x(t)))\|^2,
\end{aligned}$$

so that

(3.23) $\quad (1-\gamma-\beta-2\delta)\,\|x(t)-E(t,J(t,x(t)))\|^2\leq 0\ \Rightarrow x(t)=E(t,J(t,x(t)))$ for $t\in\Omega$.

From (3.18), (3.22) and (3.23), we get

$$\begin{aligned}
x(t) &= B(t,x(t))=E(t,J(t,x(t))) \\
\text{i.e. } x(t) &= B(t,x(t))=E(t,J(t,x(t)))=A(t,x(t))=S(t,H(t,x(t))),\ t\in\Omega.
\end{aligned}$$

That is $x(t)$ is a common random fixed point of A, B, EJ and SH.
Now, we need to prove that

$$x(t)=E(t,x(t))=J(t,x(t))=S(t,x(t))=H(t,x(t))$$

By $HA=AH$ and condition (3.6) we have

$$\begin{aligned}
& \|H(t,x(t))-x(t)\|^2=\|H(t,A(t,x(t)))-B(t,x(t))\|^2=\|A(t,H(t,x(t)))-B(t,x(t))\|^2 \\
\leq\ & \alpha\frac{\|E(t,J(t,x(t)))-B(t,x(t))\|^2\,[1+\|S(t,H(t,H(t,x(t))))-A(t,H(t,x(t)))\|^2]}{1+\|S(t,H(t,H(t,x(t))))-E(t,J(t,x(t)))\|^2} \\
& +\beta\frac{\|E(t,J(t,x(t)))-B(t,x(t))\|^2+\|E(t,J(t,x(t)))-A(t,H(t,x(t)))\|^2}{1+\|E(t,J(t,x(t)))-B(t,x(t))\|^2\,.\,\|E(t,J(t,x(t)))-A(t,H(t,x(t)))\|^2} \\
& +\gamma\,\|S(t,H(t,H(t,x(t))))-E(t,J(t,x(t)))\|^2
\end{aligned}$$

(3.24) $\quad +\delta[\|S(t,H(t,H(t,x(t))))-B(t,x(t))\|^2+\|E(t,J(t,x(t)))-A(t,H(t,x(t)))\|^2].$

Since $HA = AH$ and $SH = HS$ we get $H(t, A(t, x(t))) = A(t, H(t, x(t))) = H(t, x(t))$ and $S(t, H(t, H(t, x(t)))) = H(t, S(t, H(t, x(t)))) = H(t, x(t))$. Applying this in (3.24) we obtain

$$\|H(t,x(t)) - x(t)\|^2 \le \beta \|H(t,x(t)) - x(t)\|^2 + \gamma \|H(t,x(t)) - x(t)\|^2$$

$$+2\delta \|H(t,x(t)) - x(t)\|^2 \tag{3.25}$$

From (3.25), we have

$$(1-\beta-\gamma-2\delta)\|H(t,x(t)) - x(t)\|^2 \le 0 \;\Rightarrow x(t) = H(t,x(t)). \tag{3.26}$$

Since $x(t) = S(t, H(t, x(t)))$ and by (3.26), we have $x(t) = S(t, x(t))$,

$$\text{i.e. } x(t) = H(t,x(t)) = S(t,x(t)) \text{ for } t \in \Omega. \tag{3.27}$$

Again since $JB = BJ$ and using condition (3.6) we have

$$\|x(t) - J(t,x(t)\|^2 = \|A(t,x(t)) - J(t,B(t,x(t)))\|^2 = \|A(t,x(t)) - B(t,J(t,x(t)))\|^2$$

$$\le \quad \alpha\frac{\|E(t,J(t,J(t,x(t)))) - B(t,J(t,x(t)))\|^2 [1 + \|S(t,H(t,x(t))) - A(t,x(t))\|^2]}{1 + \|S(t,H(t,x(t))) - E(t,J(t,J(t,x(t))))\|^2}$$

$$+\beta\frac{\|E(t,J(t,J(t,x(t)))) - B(t,J(t,x(t)))\|^2 + \|E(t,J(t,J(t,x(t)))) - A(t,x(t))\|^2}{1 + \|E(t,J(t,y)) - B(t,J(t,x(t)))\|^2 . \|E(t,J(t,J(t,x(t)))) - A(t,x(t))\|^2}$$

$$+\gamma \|S(t,H(t,x(t))) - E(t,J(t,J(t,x(t))))\|^2$$

$$+\delta[\|S(t,H(t,x(t))) - B(t,J(t,x(t)))\|^2 + \|E(t,J(t,J(t,x(t)))) - A(t,x(t))\|^2], \tag{3.28}$$

Since $JB = BJ$ and $EJ = JE$ we get $J(t, B(t, x(t))) = B(t, J(t, x(t))) = J(t, x(t))$ and $E(t, J(t, J(t, x(t)))) = J(t, E(t, J(t, x(t)))) = J(t, x(t))$. Applying this in (3.28) we obtain

$$\|x(t) - J(t,x(t)\|^2 \le \beta \|x(t) - J(t,x(t)\|^2 + \gamma \|x(t) - J(t,x(t))\|^2$$

$$+2\delta \|x(t) - J(t,x(t))\|^2 . \tag{3.29}$$

From (3.29), we have

$$(1-\beta-\gamma-2\delta)\|J(t,x(t)) - x(t)\|^2 \le 0 \;\Rightarrow x(t) = J(t,x(t)). \tag{3.30}$$

Since $x(t) = E(t, J(t, x(t)))$ and by (3.30), we have $x(t) = E(t, x(t))$,

$$\text{i.e. } x(t) = J(t,x(t)) = E(t,x(t)) \text{ for } t \in \Omega. \tag{3.31}$$

From (3.27) and (3.31), we have

$$x(t) = H(t,x(t)) = S(t,x(t)) = J(t,x(t)) = E(t,x(t)),\; t \in \Omega,$$

taht is $x(t)$ is a common random fixed point of E, J, S, H, A and B. For the uniqueness of the common random fixed point $x(t)$ of E, J, S, H, A and B, let $h : \Omega \to X$ be another common random fixed point of E, J, S, H, A and B, using condition (3.6) we have

$$\begin{aligned}
\|x(t) - h(t)\|^2 &= \|A(t, x(t)) - B(t, h(t))\|^2 \\
&\leq \alpha \frac{\|E(t, J(t, h(t))) - B(t, h(t))\|^2 \left[1 + \|S(t, H(t, x(t))) - A(t, x(t))\|^2\right]}{1 + \|S(t, H(t, x)) - E(t, J(t, h(t)))\|^2} \\
&\quad + \beta \frac{\|E(t, J(t, h(t))) - B(t, h(t))\|^2 + \|E(t, J(t, h(t))) - A(t, x(t))\|^2}{1 + \|E(t, J(t, h(t))) - B(t, h(t))\|^2 . \|E(t, J(t, h(t))) - A(t, x(t))\|^2} \\
&\quad + \gamma \|S(t, H(t, x(t))) - E(t, J(t, h(t)))\|^2 \\
&\quad + \delta[\|S(t, H(t, x(t))) - B(t, h(t))\|^2 + \|E(t, J(t, h(t))) - A(t, x(t))\|^2], \\
&\Rightarrow (1 - \gamma - \beta - 2\delta) \|x(t) - h(t)\|^2 \leq 0,
\end{aligned}$$

so that $\|x(t) - h(t)\|^2 = 0$. So

$$h(t) = x(t) \text{ for } t \in \Omega.$$

Now, suppose A is continuous, then

$$A(t, S(t, H(t, x_{2n}(t)))) \longrightarrow A(t, x(t)), \ A(t, A(t, x_{2n+2}(t))) \longrightarrow A(t, x(t)), \tag{3.32}$$

since pair (A, SH) are semi-compatible, hence

$$S(t, H(t, A(t, x_{2n}(t)))) \longrightarrow A(t, x(t)). \tag{3.33}$$

For every $t \in \Omega$, we get

$$\|A(t, A(t, x_{2n}(t))) - B(t, x_{2n+1}(t))\|^2$$

$$\leq \alpha \frac{\|E(t,J(t,x_{2n+1}(t)))-B(t,x_{2n+1}(t))\|^2[1+\|S(t,H(t,A(t,x_{2n}(t))))-A(t,A(t,x_{2n}(t)))\|^2]}{1+\|S(t,H(t,A(t,x_{2n}(t))))-E(t,J(t,x_{2n+1}(t)))\|^2}$$

$$+\beta \frac{\|E(t,J(t,x_{2n+1}(t)))-B(t,x_{2n+1}(t))\|^2+\|E(t,J(t,x_{2n+1}(t)))-A(t,A(t,x_{2n}(t)))\|^2}{1+\|E(t,J(t,x_{2n+1}(t)))-B(t,x_{2n+1}(t))\|^2.\|E(t,J(t,y))-A(t,A(t,x_{2n}(t)))\|^2}$$

$$+\gamma \|S(t, H(t, A(t, x_{2n}(t)))) - E(t, J(t, x_{2n+1}(t)))\|^2$$

$$+\delta[\|S(t, H(t, A(t, x_{2n}(t)))) - B(t, x_{2n+1}(t))\|^2 + \|E(t, J(t, x_{2n+1}(t))) - A(t, A(t, x_{2n}(t)))\|^2],$$

Letting $n \to \infty$ and using (3.11), (3.32) and (3.33) we get

$$\begin{aligned} \|A(t,x(t)) - x(t)\|^2 &\leq \beta \|x(t) - A(t,x(t))\|^2 + \gamma \|A(t,x(t)) - x(t)\|^2 \\ &\quad +2\delta \|A(t,x(t)) - x(t)\|^2 \\ &\Rightarrow (1-\beta-\gamma-2\delta) \|A(t,x(t)) - x(t)\|^2 \leq 0, \end{aligned}$$

which implies

$$A(t,x(t)) = x(t), \ t \in \Omega. \tag{3.34}$$

Since $A(t,X) \subset E(t,J(t,X))$ and hence there exists a measurable mapping $z : \Omega \longrightarrow C$ such that $A(t,\mathbf{x}(t)) = E(t,J(t,z(t)))$.
from (3.34), we have

$$x(t) = A(t,x(t)) = E(t,J(t,z(t))). \tag{3.35}$$

From condition (3.6), we obtain

$$\|A(t,A(t,x_{2n}(t))) - B(t,z(t))\|^2$$

$$\begin{aligned} \leq\ & \alpha \frac{\|E(t,J(t,z(t))) - B(t,z(t))\|^2 [1 + \|S(t,H(t,A(t,x_{2n}(t)))) - A(t,A(t,x_{2n}(t)))\|^2]}{1 + \|S(t,H(t,A(t,x_{2n}(t)))) - E(t,J(t,z(t)))\|^2} \\ & +\beta \frac{\|E(t,J(t,z(t))) - B(t,z(t))\|^2 + \|E(t,J(t,z(t))) - A(t,A(t,x_{2n}(t)))\|^2}{1 + \|E(t,J(t,z(t))) - B(t,z(t))\|^2 . \|E(t,J(t,z(t))) - A(t,A(t,x_{2n}(t)))\|^2} \\ & +\gamma \|S(t,H(t,A(t,x_{2n}(t)))) - E(t,J(t,z(t)))\|^2 \\ & +\delta[\|S(t,H(t,A(t,x_{2n}(t)))) - B(t,z(t))\|^2 + \|E(t,J(t,z(t))) - A(t,A(t,x_{2n}(t)))\|^2], \end{aligned}$$

putting $n \longrightarrow \infty$, applying (3.32), (3.33), (3.34), (3.35) and (3.11), we have

$$\|x(t) - B(t,z(t))\|^2 \leq \alpha \|x(t) - B(t,z(t))\|^2 + \beta \|x(t) - B(t,z(t))\|^2$$

$$+\delta[\|x(t) - B(t,z(t))\|^2 , \tag{3.36}$$

from (3.36), we have

$$(1-\alpha-\beta-\delta) \|x(t) - B(t,z(t))\|^2 \leq 0 \ \Rightarrow x(t) = B(t,z(t), \ t \in \Omega,$$

therefore,

$$B(t,z(t)) = E(t,J(t,z(t))) = x(t). \tag{3.37}$$

Since B and EJ are weakly compatible then, they commute at their coincidence point $z(t)$, i.e.

(3.38) $$B(t, E(t, J(t, z(t)))) = E(t, J(t, B(t, z(t)))) \Rightarrow B(t, x(t)) = E(t, J(t, x(t))), t \in \Omega,$$

using condition (3.6), we get

$$\begin{aligned}
& \|A(t, x_{2n}(t)) - B(t, x(t))\|^2 \\
\leq\ & \alpha \frac{\|E(t, J(t, x(t))) - B(t, x(t))\|^2 [1 + \|S(t, H(t, x_{2n}(t))) - A(t, x_{2n}(t))\|^2]}{1 + \|S(t, H(t, x_{2n}(t))) - E(t, J(t, x(t)))\|^2} \\
& + \beta \frac{\|E(t, J(t, x(t))) - B(t, x(t))\|^2 + \|E(t, J(t, x(t))) - A(t, x_{2n}(t))\|^2}{1 + \|E(t, J(t, x(t))) - B(t, x(t))\|^2 . \|E(t, J(t, x(t))) - A(t, x_{2n}(t))\|^2} \\
& + \gamma \|S(t, H(t, x_{2n}(t))) - E(t, J(t, x(t)))\|^2 \\
& + \delta [\|S(t, H(t, x_{2n}(t))) - B(t, x(t))\|^2 + \|E(t, J(t, x(t))) - A(t, x_{2n}(t))\|^2].
\end{aligned}$$

By using (3.38) and (3.37), we have

$$\begin{aligned}
\|x(t) - B(t, x(t))\|^2 & \leq \beta \|x(t) - B(t, x(t))\|^2 + \gamma \|x(t) - B(t, x(t))\|^2 \\
& \quad + 2\delta \|x(t) - B(t, x(t))\|^2, \\
& \Rightarrow (1 - \beta - \gamma - 2\delta) \|x(t) - B(t, x(t))\|^2 \leq 0,
\end{aligned}$$

which implies

(3.39) $$x(t) = B(t, x(t)) \text{ for } t \in \Omega.$$

From (3.34), (3.38) and (3.39), we get

(3.40) $$x(t) = A(t, x(t)) = B(t, x(t)) = E(t, J(t, x(t))), \text{ for } t \in \Omega.$$

Since $B(t, X) \subset S(t, H(t, X))$, and hence there exists a measurable mapping $k : \Omega \longrightarrow C$ such that $B(t, x(t)) = S(t, H(t, k(t)))$.
from (3.40), we have

(3.41) $$x(t) = A(t, x(t)) = B(t, x(t)) = S(t, H(t, k(t))).$$

Again using condition (3.6), we have

$$\begin{aligned}
& \|A(t,k(t)) - B(t,x(t))\|^2 \\
\leq\ & \alpha \frac{\|E(t,J(t,x(t))) - B(t,x(t))\|^2 [1 + \|S(t,H(t,k(t))) - A(t,k(t))\|^2]}{1 + \|S(t,H(t,k(t))) - E(t,J(t,x(t)))\|^2} \\
& + \beta \frac{\|E(t,J(t,x(t))) - B(t,x(t))\|^2 + \|E(t,J(t,x(t))) - A(t,k(t))\|^2}{1 + \|E(t,J(t,x(t))) - B(t,x(t))\|^2 . \|E(t,J(t,x(t))) - A(t,k(t))\|^2} \\
& + \gamma \|S(t,H(t,k(t))) - E(t,J(t,x(t)))\|^2 \\
& + \delta [\|S(t,H(t,k(t))) - B(t,x(t))\|^2 + \|E(t,J(t,x(t))) - A(t,k(t))\|^2],
\end{aligned}$$

using (3.38) and (3.41), we obtain

$$\begin{aligned}
\|A(t,k(t)) - x(t)\|^2 &\leq \beta \|A(t,k(t)) - x(t)\|^2 + \delta \|x(t) - A(t,k(t))\|^2 \\
&\Rightarrow (1 - \beta - \delta) \|A(t,k(t)) - x(t)\| \leq 0
\end{aligned}$$

(3.42) i.e. $x(t) = A(t,k(t)),\ t \in \Omega.$

Both (3.41) and (3.42) imply that

(3.43) $x(t) = A(t,k(t)) = S(t,H(t,k(t)))$ for $t \in \Omega$.

since A and SH are semi-compatible, hence

$$S(t,H(t,A(t,k(t)))) \longrightarrow A(t,k(t)) = x(t$$

(3.44) i.e. $S(t,H(t,x(t))) = A(t,k(t)) = x(t)$ for $t \in \Omega$.

From (3.40) and (3.44) we have

$$x(t) = A(t,x(t)) = B(t,x(t)) = S(t,H(t,x(t))) = E(t,J(t,x(t))),\ t \in \Omega,$$

that is $x(t)$ is a common random fixed point of A, B, SH and EJ.
Similarly, we can prove that $x(t)$ is a unique common random fixed point of E, J, S, H, A and B.
Now, if the second condition of the Theorem 3.3 is satisfied that is, B and EJ are semi-compatible, EJ or B is continuous and (A, SH) are weakly compatible. Then the proof is similar (iv)

If we put $J = H = I$ (the identity random mapping) in Theorem 3.3, we have the following corollary:

Corollary 3.1 Let C be a non-empty closed subset of separable Hilbert space X. Let

E, S, A and $B : \Omega \times C \longrightarrow C$ be six random operators satisfy the following conditions:

$$\begin{aligned}
& \|A(t,x(t)) - B(t,y(t))\|^2 \\
\leq \; & \alpha \frac{\|E(t,y(t)) - B(t,y(t))\|^2 \left[1 + \|S(t,x(t)) - A(t,x(t))\|^2\right]}{1 + \|S(t,x(t)) - E(t,y(t))\|^2} \\
& + \beta \frac{\|E(t,y(t)) - B(t,y(t))\|^2 + \|E(t,y(t)) - A(t,x(t))\|^2}{1 + \|E(t,y(t)) - B(t,y(t))\|^2 . \|E(t,y(t)) - A(t,x(t))\|^2} \\
& + \gamma \|S(t,x(t)) - E(t,y(t))\|^2 \\
& + \delta [\|S(t,x(t)) - B(t,y(t))\|^2 + \|E(t,y(t)) - A(t,x(t))\|^2],
\end{aligned}$$

for all $x(t)$, $y(t) \in C$ and $t \in \Omega$,

$$A(t,X) \subset E(t,X) \text{ and } B(t,X) \subset S(t,X),$$

where $\alpha, \beta, \gamma, \delta \in [0,1)$ with $(\alpha + \beta + \gamma + 4\delta < 1)$.
Then A, B, F and E have a unique common random fixed point if one of the following conditions is satisfied
(1) (A, S) are semi-compatible, S or A is continuous and (B, E) are weakly compatible,
(2) (B, E) are semi-compatible, E or B is continuous and (A, S) are weakly compatible.

Remark 3.1 Also, if we put $J = H = E = S = I$ (the identity random mapping) in Theorem 3.3 we obtain Theorem 3.2.

Now we give an example to justify our results

Example 3.1 Let $X = R$, $\Omega = [0,1]$ and Σ be the sigma algebra of lebesgue's measurable subset of $[0,1]$, let $C = [0,\infty)$, if we define the distance $d(x,y) = \|x(t) - y(t)\|$ on R, then X being a Hilbert space, define the random operators $T, F : \Omega \times C \longrightarrow C$ as $T(t,x) = \frac{1-t^2+x}{2}$ and $F(t,x) = \frac{1-t^2+5x}{6}$, also by given measurable sequence $g_n : \Omega \to C$ as $g_n(t) = (1-t^2)^{1+\frac{1}{n}}$ for every $t \in \Omega$ and $n \in N$. Taking the limit as $n \to \infty$, we have $g(t) = 1 - t^2$ also measurable mapping from Ω to C for every $t \in \Omega$.

Verification:
It is clearly

$$T(t, g(t)) = \frac{1 - t^2 + g(t)}{2} = \frac{1 - t^2 + 1 - t^2}{2} = 1 - t^2 = g(t),$$

hence, $g(t)$ is a random fixed point of T, for every $t \in \Omega$.
for fixed $x = t^2$ and $y = t$ for all $t = 1 \in \Omega$, then

$$\begin{aligned}
\|T(t,x(t)) - T(t,y(t))\|^2 &= 0 \\
\alpha \frac{\|y(t) - T(t,y(t))\|^2 \left[1 + \|x(t) - T(t,x(t))\|^2\right]}{1 + \|x(t) - y(t)\|^2} &= \frac{5}{16}\alpha, \\
\beta \frac{\|y(t) - T(t,y(t))\|^2 + \|y(t) - T(t,x(t))\|^2}{1 + \|y(t) - T(t,y(t))\|^2 . \|y(t) - T(t,x(t))\|^2} &= \frac{8}{17}\beta, \\
\gamma \|x(t) - y(t)\|^2 &= 0, \\
\delta[\|x(t) - T(t,y(t))\|^2 + \|y(t) - T(t,x(t))\|^2] &= \frac{1}{2}\delta.
\end{aligned}$$

The inequality (3.1) leads to

$$0 \leq \frac{5}{16}\alpha + \frac{8}{17}\beta + \frac{1}{2}\delta < 1,$$

satisfying for any value of α, β and δ in $[0, 1)$.
Therefore all requirements of Theorem 3.1 are satisfied.

Again for fixed $x = t^2$ and $y = \frac{3}{5}t$ for all $t = 1 \in \Omega$, then,

$$\begin{aligned}
\|T(t,x(t)) - F(t,y(t))\|^2 &= 0 \\
\alpha \frac{\|y(t) - F(t,y(t))\|^2 \left[1 + \|x(t) - T(t,x(t))\|^2\right]}{1 + \|x(t) - y(t)\|^2} &= \frac{1}{100}\alpha \\
\beta \frac{\|y(t) - F(t,y(t))\|^2 + \|y(t) - T(t,x(t))\|^2}{1 + \|y(t) - F(t,y(t))\|^2 . \|y(t) - T(t,x(t))\|^2} &= \frac{1}{50}\beta \\
\gamma \|x(t) - y(t)\|^2 &= \frac{4}{25}\gamma \\
\delta[\|x(t) - F(t,y(t))\|^2 + \|y(t) - T(t,x(t))\|^2] &= \frac{12}{50}\delta
\end{aligned}$$

The inequality (3.5) leads to

$$0 \leq \frac{1}{100}\alpha + \frac{1}{50}\beta + \frac{4}{25}\gamma + \frac{12}{50}\delta < 1,$$

satisfying for any value of α, β, γ and δ in $[0, 1)$ and

$$F(t, g(t)) = \frac{1 - t^2 + 5g(t)}{6} = \frac{1 - t^2 + 5 - 5t^2}{6} = 1 - t^2 = g(t),$$

hence, $g(t) = T(t, g(t)) = F(t, g(t))$ is a common random fixed point of T and F, for every $t \in \Omega$.
Therefore all conditions of Theorem 3.2 are obtained.

References

[1] V. H. Badshah and . N. Shrivastava, *Semi-compatibility and random fixed points on Polish spaces*, *Varahmihir Jour. Math. Sci.*, 6 (2006), 561-568.

[2] I. Beg, *Random fixed points of random operators satisfying semicontractivity conditions*, *Mathematica Japonica*, 46 (1997), 151-155.

[3] I. Beg, *Approximation of random fixed points in normed spaces*, *Nonlinear Anal.*, 51 (2002), 1363-1372.

[4] I. Beg and . M. Abbas, *Iterative procedures for solution of random equations in Banach spaces*, *J. Math. Anal. Appl.*, 315 (2006), 181-201.

[5] I. Beg and . M. Abbas, *Common random fixed point of compatible random operators, Hindawi Pub. Corp.*, 2006 (2006), 1-15.

[6] I. Beg and . N. Shahzad, *Fixed point theorems in probabilistic analysis, Bull. Amer. Math. Soc.*, 82 (1976), 641-657.

[7] I. Beg and . N. Shahzad, *Random fixed points of random multivalued operators on Polish spaces*, *Nonlinear Anal.*, 20 (1993), 835-847.

[8] I. Beg and . N. Shahzad, *Random approximations and random fixed point theorems*, *J. Appl. Math. Stoch. Anal*, 7 (1994), 145-150.

[9] I. Beg and . N. Shahzad, *Random fixed point theorems for nonexpansive and contractive-type random operators on Banach spaces*, *J. Appl. Math. Stoch. Anal.*, 7 (1994), 569-580.

[10] I. Beg and . N. Shahzad, *Common random fixed points of random multivalued operators on metric spaces*, *Bollettino della Unione Matem' atica Italiana, Serie* VII. A9 (1995), 493-503.

[11] A. T. Bharucha-Reid, *fixed point theorems in probabilistic analysis, Bull. Amer. Math. Soc.*, 82 (1976), 641-657.

[12] B. S. Choudhury, *Convergence of a random iteration scheme to a random fixed point*, *J. Appl. Math. Stoch. Anal.* 8 (1995), 139-142.

[13] B. S. Choudhury and . M. Ray, *Convergence of an iteration leading to a soulation of a random operator equation, J. Appl. Math. Stoch. Anal.*, 12 (1999), 161-168.

[14] B.S. Choudhury and . A. Upadhyay, *An iteration leading to random solutions and fixed points of operators*, *Soochow J. Math.*, 25 (1999), 395-400.

[15] S. Itoh, *Random fixed-point theorems with an application to random differential equations in Banach spaces*, *J. Math. Anal. Appl.*, 67 (1979), 261-273.

[16] D. Oregan, *Fixed points and random fixed points for weakly inward approximable maps*, *Proc. Amer. Math. Soc.*, 126 (1998), 3045-3053.

[17] N. S. Papageorgiou, *Random fixed point theorems for measurable multifunctions in Banach space*, *Proc. Amer. Math. Soc.*, 97 (1986), 507-514.

[18] V. M. Sehgal and . S. P. Singh, *On random approximations and a random fixed point theorem for set valued mappings, Proc. Amer. Math. Soc.*, 95 (1985), 91-94.

[19] V. M. Sehgal and . C. Waters, *Some random fixed point theorems for condensing operators*, *Proc. Amer. Math. Soc.*, 90 (1984), 425-429.

[20] K. K. Tan and . X. Z. Yuan, *On deterministic and random fixed points, Proc. Amer. Math. Soc.*, 119 (1993), 849-856.

[21] H. K. Xu, *Some random fixed point theorems for condensing and non expansive operators*, *Proc. Amer. Math. Soc.*, 110 (1990), 395-400.