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h-PURIFIABLE SUBMODULES AND ISOMORPHISM OF h-PURE HULLS

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Abstract. Let N be a h-purifiable submodule of a $QTAG$-module M and K be a h-pure hull of N in M. Then K is a direct summand of M if and only if $Soc(M)/Soc(N)$ is h-purifiable in $M/Soc(N)$. Also, if K is a direct summand of M, then all h-pure hulls of N are direct summands of M, there exists the same complementary summand of M for every h-pure hull of N, and all h-pure hulls of N are isomorphic.

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1. Introduction and Preliminary Terminology

Let R be any ring. A module M_R is called $QTAG$-module if it satisfies the following condition:

(I) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

The study of $QTAG$-modules was initiated by Singh [8]. Several authors worked a lot on these modules. Many interesting results have been surfaced, but there is much to explore. In the theory of abelian groups many concepts may be generalized for modules. Out of these these concepts purifiability is very fascinating, which has been generalized earlier in [3]. Here we continue the similar study of h-purifiable submodules with the help of their h-pure hulls.

A submodule N of M is said to be h-purifiable in M if there exists a h-pure submodule K of M containing N which is minimal among the h-pure submodules of M that contain N. Such a submodule K is said to be a h-pure hull of N in M. In a direct sum of uniserial modules, every subsocle is h-purifiable.

First of all, we consider our problems on the assumptions which extend subsocles to h-purifiable submodules and h-pure submodules to h-purifiable submodules in M.

Then we obtain that a h-pure hull of a h-purifiable submodule is a direct summand of M, but M is not necessarily a direct sum of uniserial modules. We give such an example.

Next, we characterize a h-purifiable submodule N of M that a h-pure hull of N is a summand of M. It is well-known that all h-pure hulls of a subsocle in direct sum of uniserial modules are isomorphic, but all h-pure hulls of the same subsocle in a closed module are not necessarily isomorphic.

Finally, we show that if a h-pure hull K of a h-purifiable submodule N of M is a direct summand of M, then all h-pure hulls of N are direct summand of M and there exists the same complementary summand for every h-pure hull of N, and so all h-pure hulls are isomorphic.

All rings examined in the current paper contain unity $(1 \neq 0)$ and modules are unital $QTAG$-modules. A module in which the lattice of its submodule is totally ordered is called a serial module; in addition if it has finite composition length it is called a uniserial module. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique decomposition series, $d(M)$ denotes its decomposition length. For a uniform element $x \in M,\ e(x) = d(xR)$ and $H_M(x) = \sup\left\{ d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. A submodule N of M is h-pure in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$.

The submodules $H_k(M), k \geq 0$ form a neighborhood system of zero, thus a topology known as h-topology arises. Closed modules [5] are also closed with respect to this topology. Thus the closure of $N \subseteq M$ is defined as $\overline{N} = \bigcap_{k=0}^{\infty} (N + H_k(M))$. Therefore the submodule $N \subseteq M$ is closed with respect to h-topology if $\overline{N} = N$.

A module M is said to be quasi-complete, if the closure $\overline{N}$ of every h-pure submodule N of M, is h-pure in M.

A module M is called h-pure-complete, if for every subsocle S of M there exists a h-pure submodule N of M such that $S = Soc(N)$.

A module M is said to be bounded, if there exists an integer n such that $H(x) \leq n$ for every uniform element $x \in M$.

Mehran et. al. [7] proved that almost all the results which hold for TAG-modules also hold good for $QTAG$-modules. The terminologies and notations are well-known and followed by [1, 2].

2. Main concepts and results

We start here with a recollection of the following notions from [3] and [4], respectively.

Definition 2.1. *A submodule N of a $QTAG$-module M is called h-purifiable in M if there exists a submodule K of M minimal among the h-pure submodules of M containing N.*

Such K is called h-pure hull of N in M.

Notation 2.1. *For any non-negative integer t and for a submodule N of a $QTAG$-module M, we denote by $N^t(M)$ the submodule $(N + H_{t+1}(M)) \cap Soc(H_t(M))$, by $N_t(M)$ the submodule $(N \cap Soc(H_t(M))) + Soc(H_{t+1}(M))$ and by $Q_t(M, N) = N^t(M)/N_t(M)$.*

It is trivial to see that

$$\begin{aligned} N^t(M) &= (N + H_{t+1}(M)) \cap Soc(H_t(M)) \\ &= Soc(N \cap H_t(M) + H_{t+1}(M)) \end{aligned}$$

and

$$\begin{aligned} N_t(M) &= (N \cap Soc(H_t(M))) + Soc(H_{t+1}(M)) \\ &= (Soc(N))^t(M) \end{aligned}$$

The following concept was defined in [6].

Definition 2.2. *A submodule N of a $QTAG$-module M is called a kV-submodule of M if there exists an integer k such that $N^t(M) \cong N_t(M)$, $\forall\ t \geq k$. If $k = 0$, then N is a V-submodule of M.*

Now we prove the following lemma:

Lemma 2.1. *Let N be a h-purifiable and V-submodule in a $QTAG$-module M. If K is a h-pure hull of N in M, then $\pi : M/N \to M/K$ is height-preserving on $(Soc(M) + N)/N$.*

Proof. Suppose that $x+N \in (Soc(M)+N)/N$ and $x+K = H(y')+K$, where $d\left(\frac{yR}{y'R}\right) = n$ for some $y \in M$. Let us assume that $H(y') \in Soc(M)$, where $d\left(\frac{yR}{y'R}\right) = n$. Since K is h-pure in M, we have $H(y') = H(z')$, where $d\left(\frac{yR}{y'R}\right) = d\left(\frac{zR}{z'R}\right) = n + 1$ for some $z \in K$. Note that, if N is a V-submodule in M, then $Soc(K) = Soc(N)$. Therefore

$H(y') - H(z') \in Soc(M)$, where $d\left(\dfrac{yR}{y'R}\right) = d\left(\dfrac{zR}{z'R}\right) = n$ and so $x + K \in H_n(w') + N$, where $w = y - z$ and $d\left(\dfrac{wR}{w'R}\right) = n$. Since $x - H_n(w') \in Soc(K) = Soc(N)$ such that $w = y - z$ and $d\left(\dfrac{wR}{w'R}\right) = n$, we have $x + N = H_n(w') + N$, where $w = y - z$ and $d\left(\dfrac{wR}{w'R}\right) = n$. Hence $\pi : M/N \to M/K$ is height preserving on $(Soc(M) + N)/N$. □

We continue with other statement, namely:

Lemma 2.2. *Let N be a h-purifiable and V-submodule in a $QTAG$-module M and K be a h-pure hull of N in M. If M/N is a direct sum of uniserial modules, then M/K is a direct sum of uniserial modules and K is a direct summand of M.*

Proof. Note that

$$(Soc(M) + N)/N \simeq Soc(M)/Soc(N)$$

and

$$Soc(M/K) \simeq Soc(M)/(K \cap Soc(M)) = Soc(M)/Soc(K) = Soc(M)/Soc(N).$$

Considering the map $\pi : M/N \to M/K$, $(Soc(M)+N)/N$ maps under π onto $Soc(M/K)$. Since M/N is a direct sum of uniserial modules and π is height-preserving on $(Soc(M)+N)/N$ by Lemma 2.1, M/K is a direct sum of uniserial modules. Hence K is a direct summand of M. □

And so, we prepare to prove the following:

Theorem 2.1. *Let N be a submodule of a $QTAG$-module M such that N is h-purifiable in M and K be a h-pure hull of N in M. If M/N is a direct sum of uniserial modules, then K is a direct summand of M.*

Proof. By [6, Lemma 4] and Lemma 2.2, we have N is a kV-submodule of M for some $k > 0$. Then $N \cap H_k(M)$ is a V-submodule in $H_k(M)$ and $H_k(K)$ is a h-pure hull of $N \cap H_k(M)$ in $H_k(M)$. Since M/N is a direct sum of uniserial modules and

$$H_k(M)/(H_k(M) \cap N) \simeq (H_k(M) + N)/N = H_k(M/N) < M/N,$$

therefore, $H_k(M)/(H_k(M) \cap N)$ is a direct sum of uniserial modules. By Lemma 2.2, $H_k(M)/H_k(K)$ is a direct sum of uniserial modules. Since

$$H_k(M)/H_k(K) = H_k(M)/(K \cap H_k(M)) \simeq (H_k(M) + K)/K = H_k(M/K)$$

and $(M/K)/H_k(M/K)$ is bounded, M/K is a direct sum of uniserial modules. Hence K is a direct summand of M. □

The following example demonstrates that N is h-purifiable in M and M/N is a direct sum of uniserial modules, but M is not a direct sum of uniserial modules.

Example: Let $N_1 = \bigoplus_{n=1}^{\infty}\langle a_n\rangle$ and $N_2 = \bigoplus_{n=2}^{\infty}\langle a_n\rangle$, where $e(a_n) = n$. Then $Soc(N_2) = Soc(H_1(N_2))$.

Let $M = \bar{N}_1$, then $M = \overline{\langle a_1\rangle} \oplus \bar{N}_2 = \langle a_1\rangle \oplus \bar{N}_2$, where N_1 and N_2 are h-pure in M. We have

$$\begin{aligned}
H_1(\bar{N}_2) &= \bigcap_n [H_1(N_2) + H_n(M)] \\
&= \bigcap_n [(N_2 \cap H_1(M)) + H_n(M)] \\
&= \bigcap_n [(N_2 + H_n(M)) \cap H_1(M)] \\
&= [\bigcap_n (N_2 + H_n(M))] \cap H_1(M) \\
&= \bar{N}_2 \cap H_1(M) \\
&= H_1(\bar{N}_2)
\end{aligned}$$

Since N_2 is h-pure in M, N_2 is a V-submodule in M. Therefore by [4, Theorem 4.6], $Soc(N_2 + H_n(M)) = Soc(N_2) + H_n(M)$ for all n.

We have

$$\begin{aligned}
Soc(\bar{N}_2) &= \bigcap_n [N_2 + H_n(M)] \cap Soc(M) \\
&= \bigcap_n [Soc(N_2 + H_n(M))] \\
&= \bigcap_n [Soc(N_2) + Soc(H_n(M))] \\
&= \bigcap_n [Soc(H_1(N_2)) + Soc(H_n(M))] \\
&\subset \bigcap_n Soc[(H_1(N_2) + H_n(M))] \\
&= \bigcap_n [H_1(N_2) + H_n(M)] \cap Soc(M) \\
&= Soc(H_1(\bar{N}_2))
\end{aligned}$$

Hence we have $Soc(\bar{N_2}) = Soc(H_1(\bar{N_2}))$.

Since $H_1(\bar{N_2})$ is essential in $\bar{N_2}$, $H_1(\bar{N_2})$ is a V-submodule in $\bar{N_2}$. Then $H_1(\bar{N_2})$ is h-purifiable in M, $\bar{N_2}$ is a h-pure hull of $H_1(\bar{N_2})$ in M, and $M/H_1(\bar{N_2})$ is a direct sum of uniserial modules, but $H_1(\bar{N_2})$ is not a direct sum of uniserial modules.

This completes the example.

Now we give a characterization of a h-purifiable submodule N of M that a h-pure hull of N is a direct summand of M

Theorem 2.2. *Let N be a submodule of a $QTAG$-module M such that N is h-purifiable in M and K be a h-pure hull of N of in M. Then K is a direct summand of M if and only if $Soc(M)/Soc(N)$ is h-purifiable in $M/Soc(N)$.*

Proof. Note that $K = N_1 \oplus N_2$, where N_1 and N_2 are submodules of K, then by [6, Lemma 4], $Soc(N_1) = Soc(N)$, and N_2 is bounded. If K is a direct summand of M, then we have $M = N_1 \oplus N_2 \oplus N_3$, for some submodule N_3 of M. Thus

$$\frac{M}{Soc(N)} = \frac{N_1}{Soc(N)} \oplus \frac{N_2 \oplus N_3 \oplus Soc(N)}{Soc(N)}$$

and

$$Soc\left(\frac{N_2 \oplus N_3 \oplus Soc(N)}{Soc(N)}\right) = \frac{Soc(N_2 \oplus N_3) \oplus Soc(N)}{Soc(N)} = \frac{Soc(M)}{Soc(N)}.$$

Hence $Soc(M)/Soc(N)$ is h-purifiable in $M/Soc(N)$.

Conversely, suppose that $Soc(M)/Soc(N)$ is h-purifiable in $M/Soc(N)$. Since N_1 is h-pure in M and $Soc(N_1) = Soc(N)$, then N_1 is a direct summand of M. Hence $M = N_1 \oplus L$, for some submodule L of M and so $K = N_1 \oplus (L \cap K)$. Since $H_n(K) = H_n(N_1)$ for some $n > 0$, $L \cap K$ is a bounded h-pure submodule of L. Therefore $M = N_1 \oplus (L \cap K) \oplus L' = K \oplus L'$ for some submodule L' of L.

□

Appealing to above theorem, the following immediately follows:

Corollary 2.1. *Let N be a submodule of a $QTAG$-module M such that N is h-purifiable in M and K be a h-pure hull of N in M. Then the following conditions hold:*

- (i) *If $M/Soc(N)$ is quasi-complete, then M is quasi-complete and K is a direct summand of M which is quasi-complete;*
- (ii) *If $M/Soc(N)$ is h-pure-complete, then M is h-pure-complete and K is a direct summand of M;*
- (iii) *If $M/Soc(N)$ is h-pure-complete and has an unbounded direct summand of M which is a direct sum of uniserial modules, then M has an unbounded direct*

summand of M which is a direct sum of uniserial modules and K is a direct summand of M;

(iv) *If $M/Soc(N)$ is a direct sum of closed modules, then M is a direct sum of closed modules and K is a direct summand of M which is a direct sum of closed modules.*

Proof. In every case, as an immediate consequence of Theorem 2.2, K is a direct summand of M. Hence, all of them are immediate. □

Now we are in the state to prove our main result which motivates this article:

Theorem 2.3. *Let N be a submodule of a $QTAG$-module M such that N is h-purifiable in M and K be a h-pure hull of N in M. If K is a direct summand of M, then the following conditions hold:*

(i) *All h-pure hulls of N are direct summands of M;*

(ii) *There exists the same complementary summand of M for every h-pure hull of N;*

(iii) *All h-pure hulls of N are isomorphic.*

Proof. Let K' be an another h-pure hull of N in M and $M = K \oplus T$ for some submodule T of M. If N is a V-submodule of M, then we have $Soc(K) = Soc(K') = Soc(N)$. Since

$$Soc(M) = Soc(K) \oplus Soc(T) = Soc(K') \oplus Soc(T),$$

we have $M = K' \oplus T$. Let us assume that N is a kV-submodule of M for some $k > 0$. Since $N \cap H_k(M)$ is a V-submodule in $H_k(M)$ and $H_k(K)$ is a h-pure hull of $N \cap H_k(M)$ in $H_k(M)$, we have $H_k(M) = H_k(K) \oplus H_k(T) = H_k(K') \oplus H_k(T)$.
Notice that,

$$(N+H_{n+1}(M))\cap Soc(H_n(M)) = [(N+H_{n+1}(K))\cap Soc(H_n(K))]+[(N\cap Soc(H_n(M)))+Soc(H_{n+1}(M))]$$

and $N + H_{n+1}(K) \supset Soc(H_n(K))$ for all $n \geq 0$. Hence we have

$$\begin{aligned}
Soc(H_{k-1}(M)) &= [(N + H_k(M)) \cap Soc(H_{k-1}(M))] \oplus S_{k-1} \\
&= [(N + H_k(K)) \cap Soc(H_{k-1}(K)] + [(N \cap Soc(H_{k-1}(M))) + Soc(H_k(M))] \oplus S_{k-1} \\
&= [Soc(H_{k-1}(K)) + Soc(H_k(M))] \oplus S_{k-1} \\
&= [Soc(H_{k-1}(K)) + Soc(H_k(K)) + Soc(H_k(T))] \oplus S_{k-1} \\
&= Soc(H_{k-1}(K)) \oplus Soc(H_k(T)) \oplus S_{k-1}
\end{aligned}$$

where S_{k-1} is a subsocle of M. By finitely many steps, we have

$$\begin{aligned} Soc(M) &= Soc(K) \oplus Soc(H_k(T)) \oplus S_{k-1} \oplus ... \oplus S_0 \\ &= Soc(K') \oplus Soc(H_k(T)) \oplus S_{k-1} \oplus ... \oplus S_0 \end{aligned}$$

where S_i is a subsocle of M, $0 \leq i \leq k-1$. Put $S = \bigoplus_{i=0}^{k-1} S_i$.

Since

$$(S \oplus Soc(H_k(T))) \cap H_k(M) = (S \cap H_k(M)) \oplus Soc(H_k(T)) = Soc(H_k(T))$$

and $Soc(H_k(T))$ is h-purifiable in $H_k(M)$, then there exists a h-pure hull L of $S \oplus Soc(H_k(T))$.

Since we have

$$H_M(x+y) = \min\{H_M(x), H_M(y)\}$$

and

$$H_M(x'+y') = \min\{H_M(x'), H_M(y')\}$$

for all $x \in Soc(K), x' \in Soc(K')$, and $y, y' \in Soc(L)$, we have $M = K \oplus L = K' \oplus L$. Hence (1) and (2) are proved. (3) is immediate by (2). □

With the last statement in hand, we establish the following corollaries about isomorphism of h-pure hulls.

Corollary 2.2. *Let N be a submodule of a $QTAG$-module M such that N is h-purifiable in M and M/N is a direct sum of uniserial modules, then all h-pure hulls of N in M are isomorphic.*

Corollary 2.3. *Let N be a submodule of a $QTAG$-module M such that N is h-purifiable in M. If $M/Soc(N)$ is h-pure-complete or a direct sum of closed modules, then all h-pure hulls of N in M are isomorphic.*

Corollary 2.4. *Let N be a submodule of a $QTAG$-module M such that N is h-purifiable in M and $Soc(M)/Soc(N)$ is h-purifiable in $M/Soc(N)$, then all h-pure hulls of N in M are isomorphic.*

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