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CERTAIN THIRD ORDER DIFFERENTIAL SUBORDINATION

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Abstract. In this present investigation, we obtain some results for certain third order differential subordination. We also discuss some application of our results.

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1. Introduction and Definitions

Let $\mathcal{H}(\mathbb{U})$ denote the class of functions analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

For $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $a \in \mathbb{C}$ define the class of functions

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

Let f and g be members of $\mathcal{H}(\mathbb{U})$. The function f is said to be *subordinate* to g, or (equivalently) g is said to be *superordinate* to f, written symbolically as

$$f \prec g \quad \text{in} \quad \mathbb{U} \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a *Schwarz function* w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, then we have the equivalence (cf., [4, 5])

$$f \prec g \quad \Leftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Now, we introduce following few important definitions.

Definition 1.1. *Let* $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ *and let* $h(z)$ *be univalent in* $\mathbb{U}$. *If* $p(z)$ *is analytic in* $\mathbb{U}$ *and satisfies the third-order differential subordination:*

$$\phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec h(z), \tag{1.1}$$

then $p(z)$ *is called a solution of the differential subordination. The univalent function* $q(z)$ *is called a dominant of the solutions of the differential subordination or more simply a dominant if*

$$p(z) \prec q(z)$$

for all $p(z)$ *satisfying* (1.1). *A dominant* $\tilde{q}(z)$ *that satisfies*

$$\tilde{q}(z) \prec q(z)$$

for all dominants $q(z)$ *of* (1.1) *is said to be the best dominant.*

Definition 1.2. *Let* $\mathcal{Q}$ *denote the class of functions* q *that are analytic and univalent on the set* $\overline{\mathbb{U}} \setminus E(q)$, *where*

$$E(q) = \left\{ \xi \in \partial\mathbb{U} : \lim_{z\to\xi} q(z) = \infty \right\},$$

and are such that $q'(\xi) \neq 0$ *for* $\xi \in \partial\mathbb{U} \setminus E(q)$. *Further, let the subclass of* $\mathcal{Q}$ *for which* $q(0) = a$ *be denoted by* $\mathcal{Q}(a)$.

We first recall the following lemma due to Miller and Mocanu [4, 5].

Lemma 1.1. [5] *Let* $q \in \mathcal{Q}(a)$, *and let* $p \in \mathcal{H}[a, n]$ *with* $p(z) \not\equiv a$ *and* $n \geq 1$. *If* p *is not subordinate to* q, *then there exist points* $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}$ *and* $w_0 \in \partial\mathbb{U} \setminus E(q)$, *and* $m \geq n \geq 1$ *for which* $p(\mathbb{U}_{r_0}) \subset q(\mathbb{U})$,

(i) $p(z_0) = q(w_0)$,
(ii) $z_0 p'(z_0) = m w_0 q'(w_0)$,
(iii) $\operatorname{Re} \dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m \left[\operatorname{Re} \dfrac{w_0 q''(w_0)}{q'(w_0)} + 1 \right]$.

Ponnusamy and Juneja [6] investigated the third order differential inequalities in the complex plane. Recently, Antonino and Miller [1] considered the third order differential inequalities and differential subordination in $\mathbb{C}$ and proved the following lemma which contains the third order differential inequalities.

Lemma 1.2. [1] *Let* $p \in \mathcal{H}[a, n]$ *with* $p(z) \not\equiv a$ *and* $n \geq 2$, *and let* $q \in \mathcal{Q}(a)$. *If there exist points* $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}$ *and* $w_0 \in \partial\mathbb{U} \setminus E(q)$ *such that* $p(z_0) = q(w_0)$, $p(\overline{\mathbb{U}}_{r_0}) \subset q(\mathbb{U})$,

$$\operatorname{Re} \left\{ \frac{w_0 q''(w_0)}{q'(w_0)} \right\} \geq 0 \quad \textit{and} \quad \left| \frac{zp'(z)}{q'(w)} \right| \leq n,$$

when $z \in \overline{\mathbb{U}}_{r_0}$ *and* $w \in \partial\mathbb{U} \setminus E(q)$, *then there exists a real constant* $k \geq n \geq 2$ *such that*

$$z_0 p'(z_0) = n w_0 q'(w_0),$$

$$\operatorname{Re} \frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq n \left[\operatorname{Re} \frac{w_0 q''(w_0)}{q'(w_0)} + 1 \right] \quad \textit{and}$$

$$\operatorname{Re} \frac{z_0^2 p'''(z_0)}{p'(z_0)} \geq n^2 \frac{w_0^2 q'''(w_0)}{q'(w_0)}.$$

Now, we consider a variation of the above Lemma 1.2 which is related to the differential subordination of two analytic functions.

Lemma 1.3. *Let* $q \in \mathcal{Q}(a)$, *and let* $p \in \mathcal{H}[a, n]$ *with* $p(z) \not\equiv a$ *and* $n \geq 2$. *If* p *is not subordinate to* q, *then there exist points* $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}$ *and* $w_0 \in \partial\mathbb{U} \setminus E(q)$ *such that* $p(\overline{\mathbb{U}}_{r_0}) \subset q(\mathbb{U})$, *and*

$$\operatorname{Re} \left\{ \frac{w_0 q''(w_0)}{q'(w_0)} \right\} \geq 0 \quad \textit{and} \quad \left| \frac{z p'(z)}{q'(w)} \right| \leq n,$$

when $z \in \overline{\mathbb{U}}_{r_0}$ *and* $w \in \partial\mathbb{U} \setminus E(q)$,

(i) $p(z_0) = q(w_0)$,

(ii) $z_0 p'(z_0) = n w_0 q'(w_0)$,

(iii) $\operatorname{Re} \frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq n \left[\operatorname{Re} \frac{w_0 q''(w_0)}{q'(w_0)} + 1 \right]$, *and*

(iv) $\operatorname{Re} \frac{z_0^2 p'''(z_0)}{p'(z_0)} \geq n^2 \frac{w_0^2 q'''(w_0)}{q'(w_0)}$.

Lemma 1.4. [2] *Let* K, L, N, γ, δ *be nonnegative real, fixed numbers, and let*

$$f(z) \prec 1 + Kz, \quad g(z) \prec 1 + Lz, \quad h(z) \prec Nz, \quad (z \in \mathbb{U}).$$

Then

$$\gamma f(z) + \delta g(z) \prec \gamma + \delta + (\gamma K + \delta L)z, \quad (z \in \mathbb{U}),$$

and

$$\gamma f(z) + \delta h(z) \prec \gamma + (\gamma K + \delta N)z, \quad (z \in \mathbb{U}).$$

In this paper, by making use of Lemma 1.3 we find the connection between certain third order differential subordination and the subordination expression of $f(z)/z$ and $f'(z)$. The solution of third order differential equation is obtained. Our results extend the result of Kanas and Owa [2]. Results of similar type, but mainly second and first order obtained in [2, 3]. Certain result of Kanas and Owa [2], Ponnusamy and Juneja [6], and Kanas and Stankiewicz [3] are obtained as special cases.

2. Proof of the Lemma 1.3

The proof of the Lemma 1.3 is essentially similar to Lemma 2.2d of Miller and Mocanu [5, p. 24].

Proof. Since $p(0) = a = q(0)$, and p and q are analytic in $\mathbb{U}$, we can define

$$r_0 = \sup\{r : p(\mathbb{U}_r) \subset q(\mathbb{U})\}.$$

Since $p \not\prec q$ we have $p(\mathbb{U}) \not\subset q(\mathbb{U})$. Thus there exists $0 < r_0 < 1$ for which $p(\overline{\mathbb{U}}_{r_0}) \not\subset q(\mathbb{U})$. Since p is analytic in $\mathbb{U}$, the set $p(\overline{\mathbb{U}}_{r_0})$ is bounded and $p(\overline{\mathbb{U}}_{r_0}) \subset \overline{q(\mathbb{U})}$, there exists $z_0 \in \partial\mathbb{U}_{r_0}$ such that $p(z_0) \in \partial q(\mathbb{U})$. This implies that there exists $w_0 \in \partial\mathbb{U} \setminus E(q)$ such that $p(z_0) = q(w_0)$. The remaining three conclusions of this lemma follow by applying Lemma 1.2. □

3. Third order subordination

By making use of Lemma 1.3, we prove the following result.

Theorem 3.1. *Let α, β and γ be real numbers such that $\gamma, \beta \geq 0$ and $\gamma + 2\beta + \alpha \geq 0$. If $f \in \mathcal{H}(\mathbb{U})$, $\operatorname{Re}(\xi q''(\xi)/q'(\xi)) \geq 0$ $(\xi \in \partial\mathbb{U} \setminus E(q))$ and satisfies*

$$\gamma z^2 f'''(z) + \beta z f''(z) + (\gamma + \alpha) f'(z) + [1 - (\gamma + \alpha)] \frac{f(z)}{z} \prec 1 + Mz \quad (z \in \mathbb{U}), \tag{3.1}$$

for some $M > 0$, then

$$\frac{f(z)}{z} \prec 1 + \frac{Mz}{1 + \gamma + 2\beta + \alpha} := q(z) \quad (z \in \mathbb{U}), \tag{3.2}$$

and the result is the best possible.

Proof. Let $p(z) = \dfrac{f(z)}{z}$. Then $p(0) = 1 = q(0)$, using (3.1) and a straightforward calculation shows that

$$\gamma z^3 p'''(z) + (3\gamma + \beta) z^2 p''(z) + (\gamma + 2\beta + \alpha) z p'(z) + p(z) \prec 1 + Mz \quad (z \in \mathbb{U}). \tag{3.3}$$

If $p \not\prec q$, then by Lemma 1.3, there exists $z_0 \in \mathbb{U}$, $\xi_o \in \partial\mathbb{U} \setminus E(q)$, and $n \geq 2$ such that

$$p(z_0) = q(\xi_o), \quad z_0 p'(z_0) = n\xi_o q'(\xi_o), \quad \operatorname{Re}\left\{\frac{\xi_o q''(\xi_o)}{q'(\xi_o)}\right\} \geq 0,$$

$$\operatorname{Re}\frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq n\left[\operatorname{Re}\frac{\xi_o q''(\xi_o)}{q'(\xi_o)} + 1\right], \quad \text{and} \quad \operatorname{Re}\frac{z_0^2 p'''(z_0)}{p'(z_0)} \geq n^2 \frac{\xi_o^2 q'''(\xi_o)}{q'(\xi_o)}.$$

In this case we have,

$$q(\xi_o) = 1 + \frac{M\xi_o}{1 + \gamma + 2\beta + \alpha}, \quad \xi_o q'(\xi_o) = \frac{M\xi_o}{1 + \gamma + 2\beta + \alpha},$$

and $q''(\xi_o) = q'''(\xi_o) = 0$. Subsequently, for $\xi_o = e^{i\theta}$, we get

$$\text{Re}\,\left(e^{-i\theta} z_0^2 p''(z_0)\right) \geq n(n-1)\frac{M}{1+\gamma+2\beta+\alpha},$$

and

$$\text{Re}\,\left(e^{-i\theta} z_0^3 p'''(z_0)\right) \geq 0.$$

Thus,

$$\begin{aligned}
&\left|\gamma z_0^3 p'''(z_0) + (3\gamma+\beta) z_0^2 p''(z_0) + (\gamma+2\beta+\alpha) z_0 p'(z_0) + p(z_0) - 1\right| \\
&= \left|\gamma e^{-i\theta} z_0^3 p'''(z_0) + (3\gamma+\beta) e^{-i\theta} z_0^2 p''(z_0) + \frac{nM(\gamma+2\beta+\alpha)}{1+\gamma+2\beta+\alpha} + \frac{M}{1+\gamma+2\beta+\alpha}\right| \\
&\geq \text{Re}\,\left(\gamma e^{-i\theta} z_0^3 p'''(z_0)\right) + (3\gamma+\beta)\text{Re}\,\left(e^{-i\theta} z_0^2 p''(z_0)\right) + \frac{M[n(\gamma+2\beta+\alpha)+1]}{1+\gamma+2\beta+\alpha} \\
&\geq \frac{M}{1+\gamma+2\beta+\alpha}\left[(3\gamma+\beta)n^2 + (\alpha+\beta-2\gamma)n + 1\right] \\
&\geq M,
\end{aligned}$$

for $n \geq 2$, $\gamma, \beta \geq 0$, and $\gamma + 2\beta + \alpha \geq 0$, which contradicts the assumption (3.1). It follows that $p \prec q$, which completes the proof of Theorem 3.1. □

Remark 1. *Note that the function*

$$q(z) = 1 + \frac{Mz}{\gamma+2\beta+\alpha+1} \quad (z \in \mathbb{U}),$$

realizes equality in the differential subordination (3.3)*, thus* $q(z)$ *is the best dominant of* (3.3)*, and obtained the result is the best possible.*

Remark 2. *From Theorem 3.1, it is easy to check the function*

$$f(z) = z + \frac{Mz^2}{1+\gamma+2\beta+\alpha},$$

is a solution of the differential equation

$$\gamma z^2 f'''(z) + \beta z f''(z) + (\gamma+\alpha) f'(z) + [1-(\gamma+\alpha)]\,\frac{f(z)}{z} = 1 + Mz \quad (z \in \mathbb{U}).$$

By taking $\gamma = 0$ in Theorem 3.1, we obtain the following result, which is essentially due to Kanas and Owa [2, Theorem 2.1, p. 27].

Theorem 3.2. *Let* α *and* β *be real numbers, such that* $\beta \geq 0, \alpha + 2\beta \geq 0$. *If* $f \in \mathcal{H}[0,1]$, *and*

$$\beta z f''(z) + \alpha f'(z) + (1-\alpha)\,\frac{f(z)}{z} \prec 1 + Mz, \quad (z \in \mathbb{U}),$$

for some $M > 0$, *then*

$$\frac{f(z)}{z} \prec 1 + \frac{Mz}{\alpha+2\beta+1} \quad (z \in \mathbb{U}),$$

and the result is the best as possible.

Remark 3. *Putting $\gamma = \beta = 0$ in Theorem 3.1, we get the result of Ponnusamy* [7, Example 6, p. 408].

Putting $\gamma = \alpha = 0$ in Theorem 3.1, we obtain the following result.

Corollary 3.3. *Let $\beta \geq 0$. If $f \in \mathcal{H}(\mathbb{U})$, $\mathrm{Re}\,(\xi q''(\xi)/q'(\xi)) \geq 0$ ($\xi \in \partial\mathbb{U} \setminus E(q)$) and satisfies*

$$\beta z f''(z) + \frac{f(z)}{z} \prec 1 + Mz \quad (z \in \mathbb{U}), \tag{3.4}$$

for some $M > 0$, then

$$\frac{f(z)}{z} \prec 1 + \frac{Mz}{1+2\beta} := q(z) \quad (z \in \mathbb{U}), \tag{3.5}$$

and the result is the best possible.

Theorem 3.4. *Let α, β and γ be real numbers such that $\gamma, \beta \geq 0$, $\gamma + \alpha \geq 1$. If $f \in \mathcal{H}(\mathbb{U})$, $\mathrm{Re}\,(\xi q_1''(\xi)/q_1'(\xi)) \geq 0$ ($\xi \in \partial\mathbb{U} \setminus E(q_1)$) and satisfies*

$$\gamma z^2 f'''(z) + \beta z f''(z) + (\gamma+\alpha) f'(z) + [1 - (\gamma+\alpha)] \frac{f(z)}{z} \prec 1 + Mz \quad (z \in \mathbb{U}), \tag{3.6}$$

for some $M > 0$, then

$$f'(z) \prec 1 + \frac{2Mz}{1+\gamma+2\beta+\alpha} := q_1(z) \quad (z \in \mathbb{U}). \tag{3.7}$$

Proof. Let $g(z) = f'(z)$, then $g(0) = 1$. A simple computations using (3.6) yields that

$$\gamma z^2 g''(z) + \beta z g'(z) + (\gamma+\alpha) g(z) + [1 - (\gamma+\alpha)] \frac{f(z)}{z} \prec 1 + Mz. \tag{3.8}$$

By using Theorem 3.1, we get

$$\frac{f(z)}{z} \prec 1 + \frac{Mz}{1+\gamma+2\beta+\alpha} \quad (z \in \mathbb{U}). \tag{3.9}$$

Applying Lemma 1.4 together with (3.8) and (3.9), we obtain

$$\gamma z^2 g''(z) + \beta z g'(z) + (\gamma+\alpha) g(z) \prec \gamma + \alpha + \frac{2(\gamma+\beta+\alpha)Mz}{1+\gamma+2\beta+\alpha} \tag{3.10}$$

If $g \not\prec q_1$, then by Lemma 1.1, there exists $z_0 \in \mathbb{U}$, $\xi_o \in \partial\mathbb{U} \setminus E(q_1)$, and $n \geq 1$ such that

$$g(z_0) = q_1(\xi_o), \quad z_0 g'(z_0) = n \xi_o q_1'(\xi_o),$$
$$\mathrm{Re}\, \frac{z_0 g''(z_0)}{g'(z_0)} + 1 \geq n \left[\mathrm{Re}\, \frac{\xi_o q_1''(\xi_o)}{q_1'(\xi_o)} + 1 \right].$$

In this case we have,

$$q_1(\xi_o) = 1 + \frac{2M\xi_o}{1+\gamma+2\beta+\alpha}, \quad \xi_o q_1'(\xi_o) = \frac{2M\xi_o}{1+\gamma+2\beta+\alpha},$$

and $q_1''(\xi_o) = 0$. Subsequently, for $\xi_o = e^{i\theta}$, we get

$$\text{Re}\,\left(e^{-i\theta} z_0^2 g''(z_0)\right) \geq n(n-1)\frac{2M}{1+\gamma+2\beta+\alpha}.$$

Thus

$$\begin{aligned}
&\left|\gamma z_0^2 g''(z_0) + \beta z_0 g'(z_0) + (\gamma+\alpha) g(z_0) - (\gamma+\alpha)\right| \\
&= \left|\gamma e^{-i\theta} z_0^2 g''(z_0) + \beta \frac{2nM}{1+\gamma+2\beta+\alpha} + \frac{2M(\gamma+\alpha)}{1+\gamma+2\beta+\alpha}\right| \\
&\geq \text{Re}\,\left(\gamma e^{-i\theta} z_0^2 g''(z_0)\right) + \frac{2nM\beta}{1+\gamma+2\beta+\alpha} + \frac{2nM(\gamma+\alpha)}{1+\gamma+2\beta+\alpha} \\
&\geq \frac{2M}{1+\gamma+2\beta+\alpha}\left[\gamma n^2 + (\beta-\gamma)n + (\gamma+\alpha)\right] \\
&\geq \frac{2(\gamma+\beta+\alpha)M}{1+\gamma+2\beta+\alpha},
\end{aligned}$$

for $n \geq 1, \gamma, \beta \geq 0$, and $\gamma + \alpha \geq 1$, which contradicts to (3.10). It follows that $g \prec q_1$, which completes the proof of Theorem 3.4.

□

By taking $\gamma = 0$ in Theorem 3.4, we obtain the following result by Kanas and Owa [2, Theorem 2.2, p. 29].

Theorem 3.5. *Let α and β be real numbers, such that $\beta \geq 0, \alpha \geq 1$. If $f \in \mathcal{H}[0,1]$, and the differential subordination*

$$\beta z f''(z) + \alpha f'(z) + (1-\alpha)\frac{f(z)}{z} \prec 1 + Mz, \quad (z \in \mathbb{U}),$$

holds true for some $M > 0$, then

$$f'(z) \prec 1 + \frac{2Mz}{\alpha + 2\beta + 1} \quad (z \in \mathbb{U}).$$

By taking $\gamma = 0$ and $\alpha = 1$ in Theorem 3.1 and Theorem 3.4, we obtain the results of Kanas and Stankiewicz [3].

Theorem 3.6. *Let $\beta \geq 0$. If $f \in \mathcal{H}[0,1]$, and the differential subordination*

$$\beta z f''(z) + f'(z) \prec 1 + Mz, \quad (z \in \mathbb{U}),$$

holds true for some $M > 0$, then

(i) $\dfrac{f(z)}{z} \prec 1 + \dfrac{Mz}{2(1+\beta)} \quad (z \in \mathbb{U}).$

(ii) $f'(z) \prec 1 + \dfrac{Mz}{1+\beta} \quad (z \in \mathbb{U}).$

Remark 4. *Putting $\gamma = \beta = 0$ in Theorem 3.4, we get the result of Ponnusamy* [7, Example 6, p. 408].

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