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UNIQUE POSITIVE SOLUTION OF SEMILINEAR ELLIPTIC EQUATIONS INVOLVING CONCAVE AND CONVEX NONLINEARITIES IN $\mathbb{R}^{\mathbb{N}}$

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ABSTRACT. In this article, we investigate the effect of the coefficient $a(z)$ on the existence of positive solution of the subcritical semilinear elliptic problem. We prove for sufficiently large $\lambda, \mu > 0$, there exists at least one positive solution for the problem

$$-\Delta v + \mu b(z)v = a(z)v^{p-1} + \lambda h(z)v^{q-1}$$

where $v \in H^1(\mathbb{R}^{\mathbb{N}}), 1 \leq q < 2 < p < 2^* = \dfrac{2N}{(N-2)}$ for $N \geq 3$.

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1. Introduction

For $N \geq 3, 1 \leqslant q < 2 < p < 2^* = \dfrac{2N}{(N-2)}$, we consider the semilinear elliptic equation

$$\begin{cases} -\Delta v + \mu b(z)v = a(z)v^{p-1} + \lambda h(z)v^{q-1} & \text{in } \mathbb{R}^N; \\ \mathrm{v} \in \mathrm{H}^1(\mathbb{R}^N), \end{cases} \qquad (\mathbf{E}_{\lambda,\mu})$$

where $\lambda, \mu > 0$ and

(a_1) a is a positive continuous function in $\mathbb{R}^N$ and $\lim_{|z|\to\infty} a(z) = a_\infty > 0$,

(a_2) there exists a point a_1 in $\mathbb{R}^N$ such that $a(a_1) = a_{max} = \max_{z\in\mathbb{R}^N} a(z)$, $a_\infty < a_{max}$,

(h_1) $h \in L^{\frac{p}{p-q}}(\mathbb{R}^{\mathbb{N}}) \cap L^\infty(\mathbb{R}^{\mathbb{N}})$ and $h \gneqq 0$.

Let $\| u \|_H^2 = \int_{\mathbb{R}^{\mathbb{N}}} (| \nabla u |^2 + | u |^2) dz$ is the norm in $H^1(\mathbb{R}^N)$ and $u_+ = \max\{u, 0\} \geqslant 0$.

Semilinear elliptic problems involving concave - convex nonlinearities in $\mathbb{R}^{\mathbb{N}}$

$$\begin{cases} -\Delta u + u = a(z)u^{p-1} + \lambda h(z)u^{q-1} & \text{in } \mathbb{R}^{\mathbb{N}}; \\ \mathrm{u} \in H^1(\mathbb{R}^{\mathbb{N}}), \end{cases}$$

have been studied by Huei-li Lin [4] ($b(z)=1, \mu=1$ and for $N \geq 3, 1 \leqslant q < 2 < p < 2^* = \dfrac{2N}{(N-2)}$) and Ambrosetti [1] ($a \equiv 1$ and $1 < q < 2 < p \leq 2^* = \dfrac{2N}{(N-2)}$). They proved that this equation has at least two positive solutions for small enough $\lambda > 0$. In [3], existence of at least four positive solutions for the general case $-\Delta v + v = a(z)v^{p-1} + \lambda h(z)v^{q-1}$ in $\mathbb{R}^N$, for small enough $\lambda > 0$ has been investigated.
In this paper, we study the existence of at least positive solution for equation $(E_{\lambda,\mu})$ in $\mathbb{R}^{\mathbb{N}}$.

In the special case where $\lambda = \varepsilon^2,\ \mu = \frac{1}{\varepsilon^2}$ by the change of variable $u(z) = \varepsilon^{\frac{2}{p-2}} v(\varepsilon z)$, equation $(E_{\lambda,\mu})$ is transformed to

$$\begin{cases} -\Delta u + b(\varepsilon z)u = a(\varepsilon z)u^{p-1} + \varepsilon^{\frac{-2(q+2-p)}{p-2}} h(z)u^{q-1} & \text{in } \mathbb{R}^{\mathbb{N}}; \\ u \in H^1(\mathbb{R}^N). \end{cases} \qquad (E_\varepsilon)$$

Associated with equation (E_ε), we consider the C^1-functional

$$J_\varepsilon(u) = \frac{1}{2} \parallel u \parallel_b^2 - \frac{1}{p}\int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z)u_+^p dz - \frac{1}{q}\int_{\mathbb{R}^{\mathbb{N}}} \varepsilon^{\frac{-2(q+2-p)}{p-2}} h(\varepsilon z)u_+^q dz;$$

where $\parallel u \parallel_b^2 = \int_{\mathbb{R}^{\mathbb{N}}} (|\nabla u|^2 + b(\varepsilon z)u^2)dz$ is an equivalent norm in $H^1(\mathbb{R}^N)$. Precisely choosing $d = \max\{1, b(\varepsilon z)\}$, we have $\parallel u \parallel_H \leq \parallel u \parallel_b \leq d \parallel u \parallel_H$.
We know that the nonnegative weak solutions of equation (E_ε) are corresponding to the critical points of J_ε.
This article is organized as follows. In section 2 we use the argument of Tarantello [5] to divide the Nehari manifold M_ε into the two parts M_ε^+ and M_ε^-. In section 3, we prove that the existence of a positive ground state solution $u_0 \in M_\varepsilon^+$ of equation (E_ε).
Let

$$S = \sup_{\substack{u \in H^1(\mathbb{R}^{\mathbb{N}}) \\ \parallel u \parallel_H = 1}} \parallel u \parallel_{L^p}$$

then

$$\parallel u \parallel_{L^p} \leq S \parallel u \parallel_H \text{ for any } u \in H^1(\mathbb{R}^{\mathbb{N}}) \setminus \{0\} . \qquad (1.1)$$

For the semilinear elliptic equations

$$\begin{cases} \text{-}\Delta u + u = a(\varepsilon z)u^{p-1} & \text{in } \mathbb{R}^{\mathbb{N}}; \\ \mathrm{u} \in H^1(\mathbb{R}^{\mathbb{N}}), \end{cases}$$

we define the energy functional $I_\varepsilon(u) = \frac{1}{2} \parallel u \parallel_H^2 - \frac{1}{p}\int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z)u_+^p dz$, and

$$\gamma_\varepsilon = \inf_{u \in N_\varepsilon} I_\varepsilon(u);$$

where $N_\varepsilon = \{u \in H^1(\mathbb{R}^{\mathbb{N}}) \setminus \{0\} | u_+ \not\equiv 0$ and $\langle I'_\varepsilon(u), u\rangle = 0\}$.
If $a = a_{max}$, we define $I_{max}(u) = \frac{1}{2} \parallel u \parallel_H^2 - \frac{1}{p}\int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z)u_+^p dz$, and

$$\gamma_{max} = \inf_{u \in \Omega} I_{max}(u);$$

where $\Omega = \{u \in H^1(\mathbb{R}^{\mathbb{N}}) \setminus \{0\} | u_+ \not\equiv 0$ and $\langle I'_{max}(u), u\rangle = 0\}$.
Lemma 1.1

$$\gamma_{max} = \frac{p-2}{2p}(a_{max}S^p)^{\frac{-2}{(p-2)}} > 0.$$

proof: if $I_{max} = \frac{1}{2} \parallel u \parallel_H^2 - \frac{1}{p}\int_{\mathbb{R}^{\mathbb{N}}} a_{max}u_+^p dz$, then

$$\gamma_{max} = \gamma_{max}(\Omega) = (\frac{1}{2} - \frac{1}{p})\gamma(\Omega)^{\frac{2p}{2-p}},$$

where $\gamma(\Omega) = \sup\left\{ \int_{\mathbb{R}^{\mathbb{N}}} a_{max}u^p \middle| u \in H^1(\mathbb{R}^N) \text{ and } \parallel u \parallel_H = 1\right\} = a_{max}^{\frac{1}{p}}$.
Moreover

$$\gamma_{max} = (\frac{1}{2} - \frac{1}{p})(a_{max}^{\frac{1}{p}}S)^{\frac{2p}{p-2}} > 0. \quad \square$$

2. The Nehari manifold

We define the Palais - Smale (denoted by (PS))$-$ sequences , $(PS)-$ value, and $(PS)-$ conditions in $H^1(\mathbb{R}^N)$ for J as follows.

Definition

(i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(PS)_\beta$-sequence in $H^1(\mathbb{R}^N)$ for J if $J(u_n) = \beta + o(1)$ and $J'(u_n) = o(1)$ strongly in $H^{-1}(\mathbb{R}^N)$ as $n \longrightarrow \infty$, where $H^{-1}(\mathbb{R}^N)$ is the dual space of $H^1(\mathbb{R}^N)$;
(ii) $\beta \in \mathbb{R}$ is a$(PS)-$ value in $H^1(\mathbb{R}^{\mathbb{N}})$for J if there is a $(PS)_\beta$- sequence in $H^1(\mathbb{R}^N)$ for J;
(iii) J satisfies the $(PS)_\beta$- condition in $H^1(\mathbb{R}^N)$ if every $(PS)_\beta$- sequence in $H^1(\mathbb{R}^N)$ for

J contains a convergent subsequence.

Next, since J_ε is not bounded form below in $H^1(\mathbb{R}^N)$, we consider the Nehari manifold

$$M_\varepsilon = \{u \in H^1(\mathbb{R}^N) \setminus 0 | u_+ \not\equiv 0 \text{ and } \langle J'_\varepsilon(u), u\rangle = 0\}, \qquad \textbf{(2.1)}$$

where

$$\langle J'_\varepsilon(u), u\rangle = \|u\|_H^2 - \int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z) u_+^p dz - \varepsilon^{\frac{-2(q+2-p)}{p-2}} \int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z) u_+^q dz.$$

Note that M_ε contains all nonnegative solutions of equation $(E_{\lambda,\mu})$.

Lemma 2.1 The energy functional J_ε is coercive and bounded from below on M_ε.
Proof. For $u \in M_\varepsilon$, by (3.1), the Holder inequality $(p_1 = \dfrac{p}{p-q}, p_2 = \dfrac{p}{q})$ and the Sobolev embedding theorem (1.1), we get

$$\begin{aligned} J_\varepsilon(u) &= (\frac{1}{2} - \frac{1}{p}) \parallel u \parallel_H^2 - (\frac{1}{q} - \frac{1}{p}) \int_{\mathbb{R}^{\mathbb{N}}} \varepsilon^{\frac{-2(q+2-p)}{p-2}} h(\varepsilon z) u_+^q dz \\ &\geq (\frac{1}{2} - \frac{1}{p}) \parallel u \parallel_H^2 - (\frac{1}{q} - \frac{1}{p}) \varepsilon^{\frac{-2(q+2-p)}{p-2}} \parallel h \parallel_\# S^q \parallel u \parallel_H^q \\ &\geq \frac{\parallel u \parallel_H^q}{p} \left[\frac{p-2}{2} \parallel u \parallel_H^{2-q} - (\frac{p-q}{q}) \varepsilon^{\frac{-2(q+2-p)}{p-2}} \parallel h \parallel_\# S^q \right] \geq 0. \end{aligned}$$

Since $\dfrac{p-2}{2} > 0$ and $(\dfrac{p-q}{q}) \varepsilon^{\frac{-2(q+2-p)}{p-2}} \parallel h \parallel_\# S^q > 0$, we have that J_ε is coercive and bounded from below on M_ε.

Define

$$\psi_\varepsilon(u) = \langle J'_\varepsilon(u), u\rangle.$$

Then for $u \in M_\varepsilon$, we get

$$\begin{aligned} \langle \psi'_\varepsilon(u), u\rangle &= 2\|u\|_H^2 - p \int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z) u_+^p dz - \Lambda q \int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z) u_+^q dz \\ &= (2-p) \parallel u \parallel_H^2 + \Lambda(p-q) \int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z) u_+^q dz \qquad (2.2) \\ &= (2-q) \parallel u \parallel_H^2 + (q-p) \int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z) u_+^p dz. \qquad (2.3) \end{aligned}$$

Now we divide the Nehari manifold into three disjoint subsets

$M_\varepsilon^+ = \{u \in M_\varepsilon \mid \langle \psi'_\varepsilon(u), u\rangle > 0\};$
$M_\varepsilon^0 = \{u \in M_\varepsilon \mid \langle \psi'_\varepsilon(u), u\rangle = 0\};$
$M_\varepsilon^- = \{u \in M_\varepsilon | \langle \psi'_\varepsilon(u), u\rangle < 0\}.$

Lemma 2.2 Under some assumptions $(a_1), (a_2)$ and (h_1), if $0 < \Lambda < \Lambda_0$, then $M_\varepsilon^0 = \emptyset$.

Proof. Assuming the contrary, there is $\lambda_0 \in \mathbb{R}$ and $0 < \lambda_0 < \Lambda_0$ such that $M_{\lambda_0}^0 = \emptyset$. Then for $u \in M_{\lambda_0}^0$,

$\| u \|_H^2 = \frac{p-q}{p-2}\lambda_0 \int_{\mathbb{R}^\mathbb{N}} h(\varepsilon z) u_+^q dz = \frac{p-q}{2-q}\int_{\mathbb{R}^\mathbb{N}} a(\varepsilon z) u_+^p dz.$

By the Holder and the Soblev embedding theorem, we get

$$\| u \|_H \geq \left[\frac{(2-q)}{(p-q)a_{max}} S^{-p}\right]^{\frac{1}{(p-2)}},$$

and

$$\| u \|_H \leq (\frac{p-q}{p-2}\lambda_0 \| h \|_\# S^q)^{\frac{1}{2-q}}.$$

Thus

$$\lambda_0 \geq (p-2)(\frac{2-q}{a_{max}})^{\frac{2-q}{p-2}} \left[(p-q)S^2\right]^{\frac{q-p}{p-2}} \| h \|_\#^{-1} = \Lambda_0,$$

which is a contradiction. $\square$

Lemma 2.3 Suppose that u is a local minimizer for J_ε on M_ε and $u \in M_\varepsilon^0$. Then $J'_\varepsilon(u) = 0$ in $H^{-1}(\mathbb{R}^\mathbb{N})$.

Proof. See Brown and Zhang [2,Theorem 2.3]. $\square$

Lemma 2.4 For each $u \in M_\varepsilon^+$ we have

$$\int_{\mathbb{R}^\mathbb{N}} h(\varepsilon z) u_+^q dz > 0 \text{ and } \| u \|_H < \left(\frac{p-q}{p-2}\Lambda \| h \|_{\neq} S^q\right)^{\frac{1}{(2-q)}}.$$

Proof. For $u \in M_\varepsilon^+$ we get

$$(2-p) \| u \|_H^2 + (p-q)\Lambda \int_{\mathbb{R}^\mathbb{N}} h(\varepsilon z) u_+^q dz > 0$$
$$(p-q)\Lambda \int_{\mathbb{R}^\mathbb{N}} h(\varepsilon z) u_+^q dz > (2-p) \| u \|_H^2$$
$$\int_{\mathbb{R}^\mathbb{N}} h(\varepsilon z) u_+^q dz > \frac{(2-p)}{\Lambda(p-q)} \| u \|_H^2 > 0.$$

For every $u \in M_\varepsilon^+ \subset M_\varepsilon$, by (2.2) and the Holder inequality ($p_1 = \frac{p}{p-q}, p_2 = \frac{p}{q}$), we have

$$\begin{aligned} 0 &< (p-q)\int_{\mathbb{R}^\mathbb{N}} \Lambda h(\varepsilon z) \| u_+^q dz - (p-2) \| u \|_H^2 \\ &\leq (p-q)\Lambda \| h \|_\# S^q \| u \|_H^q - (p-2) \| u \|_H^2 \\ \|u\|_H &\leq (\frac{p-q}{p-2}\Lambda \| h \|_\# S^q)^{\frac{1}{2-q}}. \quad \square \end{aligned}$$

Lemma 2.5 For each $u \in M_\varepsilon^-$ we have $\| u \|_H > \left[\frac{2-q}{(p-q)a_{max}} S^p\right]^{\frac{1}{p-2}}$.

Proof. For every $u \in M_\varepsilon^-$, by (2.3), we have that

$$\begin{aligned} \| u \|_H^2 &< \frac{p-q}{2-q} \int_{\mathbb{R}^\mathbb{N}} a(\varepsilon z) u_+^p dz \\ &\le \frac{p-q}{2-q} S^p \| u \|_H^p \, a_{max}. \end{aligned}$$

$$\| u \|_H \ge \left[\frac{(2-q)}{(p-q)a_{max}} S^{-p}\right]^{\frac{1}{(p-2)}}. \quad \square$$

Lemma 2.6 If $0 < \Lambda < \frac{q\Lambda_0}{2}$ and $u \in M_\varepsilon^-$ then $J_\varepsilon(u) > 0$.

Proof. For $u \in M_\varepsilon^-$ we have

$$\begin{aligned} J_\varepsilon(u) &= (\frac{1}{2} - \frac{1}{p}) \| u \|_H^2 - (\frac{1}{q} - \frac{1}{p}) \int_{\mathbb{R}^\mathbb{N}} \Lambda h(\varepsilon z) u_+^q dz. \\ &\ge \frac{\| u \|_H^q}{p} (\frac{P-2}{2} \| u \|_H^{2-q} - \frac{p-q}{q} \Lambda \| h \|_\# S^q). \\ &> \frac{1}{p} (\frac{2-p}{(p-q)a_{max} S^p})^{\frac{q}{p-2}} \\ &\times (\frac{p-2}{2} (\frac{2-q}{(p-q)a_{max} S^p})^{\frac{2-q}{p-2}} - \frac{p-q}{q} \Lambda \| h \|_\# S^q). \end{aligned}$$

So $J_\varepsilon(u) \geqslant d_0 > 0$ for some $d_0 = d_0(\varepsilon, p, q, S, \| h \|_\#, a_{max})$. $\square$

For $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ and $u_+ \not\equiv 0$, let

$$\bar{l} = \bar{l}(u) = \left[\frac{(2-q) \| u \|_H^2}{(p-q) \int_{\mathbb{R}^\mathbb{N}} a(\varepsilon z) u_+^p dz}\right]^{\frac{1}{p-2}} > 0 .$$

Lemma 2.7 For every $u \in H^1(\mathbb{R}^\mathbb{N}) \setminus \{0\}$ and $u_+ \not\equiv 0$, we have that if $\int_{\mathbb{R}^\mathbb{N}} \Lambda h(z) u_+^q dz = 0$, then there is a unique positive number $l^- = l^-(u) > \bar{l}$ such that $l^- u \in M_\varepsilon^-$ and $J_\varepsilon(l^- u) = \sup_{l \geqslant 0} J_\varepsilon(lu)$.

Proof. For every $u \in H^1(\mathbb{R}^\mathbb{N}) \setminus \{0\}$ and $u_+ \not\equiv 0$, define

$$k(l) = k_u(l) = l^{2-q} \| u \|_H^2 - l^{p-q} \int_{\mathbb{R}^\mathbb{N}} a(\varepsilon z) u_+^p dz \text{ for } l \geqslant 0.$$

Clearly, $k(0) = 0$ and $k(l) \to -\infty$ as $l \to \infty$. Since

$$k'(l) = \frac{1}{l^{q+1}} \left[(2-q) \| lu \|_H^2 - (p-q) \int_{\mathbb{R}^\mathbb{N}} a(\varepsilon z)(lu_+)^p dz\right] \text{ for } l \geqslant 0$$

then $k'(\bar{l}) = 0, k'(l) > 0$ for $0 < l < \bar{l}$, and $k'(l) < 0$ for $l > \bar{l}$. Thus , $k(l)$ gets its maximum at $\bar{l}$. Furthermore, by the Sobolev embedding theorem, we have that

$$k(\bar{l}) = \left[\frac{(2-q) \parallel u \parallel_H^2}{(p-q)\int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z)u_+^p dz}\right]^{\frac{(2-q)}{(p-2)}} \parallel u \parallel_H^2$$

$$- \left[\frac{(2-q) \parallel u_H^2}{(p-q)\int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z)u_+^p dz}\right]^{\frac{(p-q)}{(p-2)}} \int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z)u_+^p dz$$

$$\geq (p-2)(2-q)^{\frac{2-q}{p-2}}(p-q)^{\frac{q-p}{p-2}} S^{\frac{p(q-2)}{p-2}} \parallel u \parallel_H^q . \qquad (2.4)$$

Since $\int_{\mathbb{R}^{\mathbb{N}}} \Lambda h(z)u_+^q dz = 0$, there exists a unique positive number $l^- = l^-(u) > \bar{l}$ such that $k(l^-) = \int_{\mathbb{R}^{\mathbb{N}}} \Lambda h(z)u_+^q dz = 0$ and $k'(l^-) > 0$. Then

$$\frac{d}{dl}J_\varepsilon(lu)\Big|_{l=l^-} = \frac{1}{l}(\parallel lu \parallel_H^2 - \int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z)(lu_+)^p dz - \int_{\mathbb{R}^{\mathbb{N}}} \Lambda h(\varepsilon z)(lu_+)^q dz)\Big|_{l=l^-} = 0$$

$$\frac{d^2}{dl^2}J_\varepsilon(lu)\Big|_{l=l^-} = \frac{1}{l^2}(\parallel lu \parallel_H^2 -(p-1)\int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z)(lu_+)^p dz$$

$$-(q-1)\int_{\mathbb{R}^{\mathbb{N}}} \Lambda h(\varepsilon z)(lu_+)^q dz)\Big|_{l=l^-} < 0.$$

Furthermore $J_\varepsilon(lu) \to -\infty$ as $l \to \infty$, so it is not difficult to find that $l^-u \in M_\varepsilon^-$ and $J_\varepsilon(l^-u) = \sup_{l\geq 0} J_\varepsilon(lu)$. □

Lemma 2.8 if $0 < \Lambda < \Lambda_0$ and $\int_{\mathbb{R}^{\mathbb{N}}} \Lambda h(\varepsilon z)u_+^q dz > 0$, then there is unique positive number $l^+ = l^+(u) < \bar{l} < l^- = l^-(u)$ such that $l^+u \in M_\varepsilon^-$, and

$$J_\varepsilon(l^+u) = \inf_{0\leq l\leq \bar{l}} J_\varepsilon(lu), \quad J_\varepsilon(l^-u) = \sup_{l\geq \bar{l}} J_\varepsilon(lu).$$

Proof. Since $0 < \Lambda < \Lambda_0$ and $\int_{\mathbb{R}^{\mathbb{N}}} \Lambda h(\varepsilon z)u_+^q dz > 0$, by (2.4), then

$$k(0) = 0 < \Lambda\int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z)u_+^q dz \leq \Lambda \parallel h \parallel_\# S^q \parallel u \parallel_H^q$$

$$< (p-2)(2-q)^{\frac{2-q}{p-2}}(p-q)^{\frac{q-p}{p-2}} S^{\frac{p(q-2)}{p-2}} \parallel u \parallel_H^q \leq k(\bar{l}).$$

It follows that there are unique positive number $l^+ = l^+(u)$ and $l^- = l^-(u)$ such that $l^+ < \bar{l} < l^-$, $k(l^+) = \int_{\mathbb{R}^{\mathbb{N}}} \Lambda h(\varepsilon z)u_+^q dz = k(l^-)$ and $k'(l^-) < 0 < k'(l^+)$. We also have that $l^+u \in M_\varepsilon^+, l^-u \in M_\varepsilon^-, J_\varepsilon(l^+u) \leq J_\varepsilon(lu) \leq J_\varepsilon(l^-u)$ for every $l \in [l^+, l^-]$, and $J_\varepsilon(l^+u) \leq J_\varepsilon(lu)$ for every $l \in [0, \bar{l}]$. Hence, $J_\varepsilon(l^+u) = \inf_{0\leq l\leq \bar{l}} J_\varepsilon(lu),\ J_\varepsilon(l^-u) = \sup_{l\geq \bar{l}} J_\varepsilon(lu)$.
□

Applying lemma 2.2, we have

$$M_\varepsilon = M_\varepsilon^+ \cup M_\varepsilon^-,$$

where

$$M_\varepsilon^+ = \left\{ u \in M_\varepsilon \middle| (2-q) \parallel u \parallel_H^2 -(p-q) \int_{\mathbb{R}^\mathbb{N}} a(\varepsilon z) u_+^p dz > 0 \right\},$$

$$M_\varepsilon^- = \left\{ u \in M_\varepsilon \middle| (2-q) \parallel u \parallel_H^2 -(p-q) \int_{\mathbb{R}^\mathbb{N}} a(\varepsilon z) u_+^p dz < 0 \right\}.$$

Define

$$\alpha_\varepsilon = \inf_{u \in M_\varepsilon} J_\varepsilon(u); \quad \alpha_\varepsilon^+ = \inf_{u \in M_\varepsilon^+} J_\varepsilon(u); \quad \alpha_\varepsilon^- = \inf_{u \in M_\varepsilon^-} J_\varepsilon(u).$$

Lemma 2.9 If $0 < \Lambda < \Lambda_0$, then $\alpha_\varepsilon \leq \alpha_\varepsilon^+ < 0$.

Proof. Suppose $u \in M_\varepsilon^+$, by (2.2) we get that

$$(p-2) \parallel u \parallel_H^2 < (p-q)\Lambda \int_{\mathbb{R}^\mathbb{N}} h(z) u_+^q dz.$$

Then

$$\begin{aligned} J_\varepsilon(u) &= (\frac{1}{2} - \frac{1}{p}) \parallel u \parallel_H^2 -(\frac{1}{q} - \frac{1}{p})\Lambda \int h(\varepsilon z) u_+^q dz \\ &< \left[(\frac{1}{2} - \frac{1}{p}) - (\frac{1}{q} - \frac{1}{p}) \frac{p-2}{p-q} \right] \parallel u \parallel_H^2 \\ &= -\frac{(2-q)(p-2)}{2pq} \parallel u \parallel_H^2 < 0. \end{aligned}$$

By the definition α_ε and α_ε^+, we conclude that $\alpha_\varepsilon \leq \alpha_\varepsilon^+ < 0$. □

Lemma 2.10 If $0 < \Lambda < \dfrac{q\Lambda_0}{2}$, then $\alpha_\varepsilon^- \geq d_0 > 0$ for some $d_0 = d_0(\varepsilon, p, q, S, \parallel h \parallel_\#)$.

Proof. See [3, Lemma 2.5]. □

Corollary 2.11

(a) There exists a $(PS)_{\alpha_\varepsilon}-$sequence $\{u_n\}$ in M_ε for J_ε;

(b) There exists a $(PS)_{\alpha_\varepsilon^+}-$sequence $\{u_n\}$ in M_ε^+ for J_ε;

(c) There exists a $(PS)_{\alpha_\varepsilon^-}-$sequence $\{u_n\}$ in M_ε^- for J_ε.

3. Existence of a ground state solution

At first, we show that J_ε satisfies the $(PS)_\beta-$ condition in $H^1(\mathbb{R}^N)$ for $\beta \in (-\infty, \gamma_{max} - C_0 \Lambda^{\frac{2}{2-q}})$, where

$$C_0 = (2-q) \left[(p-q) \parallel h \parallel_\# S^q \right]^{\frac{2}{2-q}} \Big/ \left[2pq(p-2)^{\frac{q}{2-q}} \right].$$

Lemma 3.1 Under some assumptions a_1, a_2, h_1 and $0 < \Lambda < \Lambda_0$. If $\{u_n\}$ is a $(PS)_\beta-$sequence in $H^1(\mathbb{R}^N)$ for J_ε with $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^\mathbb{N})$, then $J'_\varepsilon(u) = 0$ in $H^{-1}(\mathbb{R}^\mathbb{N})$.

Proof. Suppose $\{u_n\}$ be a $(PS)_\beta-$ sequence in $H^1(\mathbb{R}^N)$ for J_ε such that $J_\varepsilon(u_n) = \beta + o_n(1)$ and $J'_\varepsilon(u_n) = o_n(1)$ in $H^{-1}(\mathbb{R}^N)$. Then

$$\begin{aligned} \mid \beta \mid + o_n(1) + \frac{d_n \parallel u_n \parallel_H}{p} &\geq J_\varepsilon(u_n) - \frac{1}{p}\langle J'_\varepsilon(u_n), u_n\rangle \\ &= (\frac{1}{2} - \frac{1}{p}) \parallel u_n \parallel_H^2 - (\frac{1}{q} - \frac{1}{p}) \int_{\mathbb{R}^\mathbb{N}} \Lambda h(\varepsilon z)(u_n)_+^q dz \\ &\geq \frac{p-2}{2p} \parallel u_n \parallel_H^2 - \frac{p-q}{pq}\Lambda \parallel h \parallel_\# S^q \parallel u_n \parallel_H^q \\ &\geq \frac{p-2}{2p} \parallel u_n \parallel_H^2. \end{aligned}$$

Then

$$\parallel u_n \parallel \geq 2p(\mid \beta \mid + o_n(1))/(2d_n - (p-2)),$$

where $d_n = o_n(1)$ as $n \to \infty$. It follows that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^\mathbb{N})$, so there exist a subsequence $\{u_n\}$ and $u \in H^1(\mathbb{R}^N)$ such that $J'_\varepsilon(u) = 0$ in $H^{-1}(\mathbb{R}^\mathbb{N})$. □

Lemma 3.2 For $0 < \Lambda < \Lambda_0$, if $\{u_n\}$ is a $(PS)_\beta-$sequence in $H^1(\mathbb{R}^N)$ for J_ε with $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^\mathbb{N})$, we have $J_\varepsilon(u) \geqslant -C_0\Lambda^{\frac{2}{2-q}} \geqslant -C'_0$, where

$$C'_0 = (p-2)(2-q)^{\frac{p}{p-2}}/(2pq(a_{max}(p-q))^{\frac{2}{p-2}} S^{\frac{2p}{p-2}}).$$

Proof. We have $\langle J'_\varepsilon(u), u\rangle = 0$, that is , $\int_{\mathbb{R}^\mathbb{N}} a(\varepsilon z)u_+^p dz = \parallel u \parallel_H^2 - \int_{\mathbb{R}^\mathbb{N}} \Lambda h(\varepsilon z)u_+^q dz$. Hence, by the Young inequality ($p_1 = \dfrac{2}{q}$ and $p_2 = \dfrac{2}{2-q}$)

$$\begin{aligned} J_\varepsilon(u) &= \left(\frac{1}{2} - \frac{1}{p}\right) \parallel u \parallel_H^2 - \left(\frac{1}{q} - \frac{1}{p}\right) \int_{\mathbb{R}^\mathbb{N}} \Lambda h(\varepsilon z)u^q dz \\ &\geq \frac{p-2}{2p} \parallel u \parallel_H^2 - \frac{p-q}{pq}\Lambda \parallel h \parallel_\# S^q \parallel u \parallel_H^q \\ &\geq \frac{p-2}{2p} \parallel u \parallel_H^2 \\ &- \frac{p-2}{pq}\left[\frac{q \parallel u \parallel_H^2}{2} + (\frac{p-q}{p-2}\Lambda \parallel h \parallel_\# S^q)^{\frac{2}{2-q}} \frac{2-q}{2}\right] \\ &= -\Lambda a^{\frac{2}{2-q}} (2-q) \left[(p-q) \parallel h \parallel_\# S^q\right]^{\frac{2}{2-q}} \Big/ \left[2pq(p-2)^{\frac{q}{2-q}}\right] \\ &\geq -\frac{(p-2)(2-q)^{\frac{p}{p-2}}}{2pq\left[a_{max}(p-q)\right]^{\frac{2}{p-2}} S^{\frac{2p}{p-2}}} = -C'_0. \quad \square \end{aligned}$$

Lemma 3.3 For $0 < \Lambda < \Lambda_0$ the functional J_ε satisfies the $(PS)_\beta$- condition in $H^1(\mathbb{R}^\mathbb{N})$ for $\beta \in (-\infty, \gamma_{max} - C_0\Lambda^{\frac{2}{2-q}})$.

Proof. Suppose $\{u_n\}$ be a $(PS)_\beta-$ sequence in $H^1(\mathbb{R}^N)$ for J_ε such that $J_\varepsilon(u_n) = \beta + o_n(1)$ and $J'_\varepsilon(u_n) = o_n(1)$ in $H^{-1}(\mathbb{R}^N)$. Then it follows that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^\mathbb{N})$ and so there exist a subsequence $\{u_n\}$ and $u \in H^1(\mathbb{R}^\mathbb{N})$ such that $J'_\varepsilon(u) = 0$ in $H^{-1}(\mathbb{R}^\mathbb{N}), u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N), u_n \to u$ a.e. in $\mathbb{R}^N, u_n \to u$ in $L^s_{loc}(\mathbb{R}^\mathbb{N})$ for every $1 \leq s < 2^*$.

Next, we claim that

$$\int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z)|u_n - u|^q dz \to 0 \text{ as } n \to \infty. \tag{3.1}$$

Using the Brezis-Lieb lemma to get

$$\int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z)(u_n - u)_+^q dz = \int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z)(u_n)_+^q dz - \int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z)u^q dz + o_n(1).$$

For every $\sigma > 0$, there is $r > 0$ such that $\int_{[B^N(0;r)]^c} h(\varepsilon z)^{\frac{p}{p-q}} dz < \sigma$. By the Holder inequality and the Sobolev embedding theorem, we get

$$\begin{aligned}
\left| \int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z) \mid u_n - u \mid^q dz \right| &\leq \int_{B^N(0;r)} h(\varepsilon z) \mid u_n - u \mid^q dz \\
&\quad + \int_{[B^N(0;r)]^c} h(\varepsilon z) \mid u_n - u \mid^q dz \\
&\leq \| h \|_{\#} \left(\int_{\mathbb{R}^{\mathbb{N}}} \mid u_n - u \mid^p dz \right)^{\frac{q}{p}} \\
&\quad + S^q \left(\int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z)^{\frac{p}{p-q}} dz \right)^{\frac{p-q}{p}} \| u_n - u \|_H^q \\
&\leq o_n(1) + \sigma C'.
\end{aligned}$$

Using this fact, we get

$$\int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z)(u_n - u)_+^p dz = \int_{\mathbb{R}^{\mathbb{N}}} a_{max}(u_n - u)_+^p dz + o_n(1). \tag{3.2}$$

Let $p_n = u_n - u$. Suppose $p_n \nrightarrow 0$ strongly in $H^1(\mathbb{R}^N)$. By (3.1) and (3.2) we conclude that

$$\begin{aligned}
\| p_n \|_H^2 &= \| u_n \|_H^2 - \| u \|_H^2 + o_n(1) \\
&= \int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z)(u_n)_+^p dz - \int_{\mathbb{R}^{\mathbb{N}}} \lambda h(\varepsilon z)(u_n)_+^q dz \\
&\quad - \int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z)u^p dz + \int_{\mathbb{R}^{\mathbb{N}}} \lambda h(\varepsilon z)u^q dz + o_n(1) \\
&= \int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z)(u_n - u)_+^p dz + o_n(1) = \int_{\mathbb{R}^{\mathbb{N}}} a_{max}(p_n)_+^p dz + o_n(1).
\end{aligned}$$

Then

$$\begin{aligned}
I_{max}(p_n) &= \frac{1}{2} \| p_n \|_H^2 - \frac{1}{p} \int_{\mathbb{R}^{\mathbb{N}}} a_{max}(p_n)_+^p dz \\
&= (\frac{1}{2} - \frac{1}{p}) \| p_n \|_H^2 + o_n(1) > 0.
\end{aligned}$$

By Theorem 4.3 in Wang [6] , there exists a sequence $\{s_n\} \subset \mathbb{R}^+$ such that $s_n = 1 + o_n(1), \{s_n p_n\} \subset \Omega$ and $I_{max}(s_n p_n) = I_{max}(p_n) + o_n(1)$. It follows that

$$\begin{aligned}
\gamma_{max} &\leq I_{max}(s_n p_n) = I_{max}(p_n) + o_n(1) \\
&= J_\varepsilon(u_n) - J_\varepsilon(u) + o_n(1)
\end{aligned}$$

$$= \beta - J_\varepsilon(u) + o_n(1)$$
$$= J_\varepsilon(u_n) - J_\varepsilon(u)$$
$$= J_\varepsilon(p_n) \to o_n(1) < \gamma_{max},$$

which is a contradiction. Hence , $u_n \to u$ strongly in $H^1(\mathbb{R}^\mathbb{N})$. □

Theorem 3.4 Let $\Lambda = \varepsilon^{\frac{-2(q+2-p)}{p-2}}$. Then for

$$0 < \Lambda < \Lambda_0 = (p-2)(\frac{2-q}{a_{max}})^{\frac{2-q}{p-2}} [(p-q)S^2]^{\frac{q-p}{p-2}} \parallel h \parallel_{\#}^{-1},$$

where $\parallel h \parallel_\#$ is the norm in $L^{\frac{P}{p-q}}(\mathbb{R}^N)$, The problem (E_ε) admits at least one positive ground state solution u_0 of the problem (E_ε) in $\mathbb{R}^N$. Moreover, $u_0 \in M_\varepsilon^+$ and

$$J_\varepsilon(u_0) = \alpha_\varepsilon = \alpha_\varepsilon^+ \geq -C_0\Lambda^{\frac{2}{2-q}}.$$

Proof. Consider minimizing sequence $\{u_n\} \subset M_\varepsilon$ for J_ε such that $J_\varepsilon(u_n) = \alpha_\varepsilon + o_n(1)$ and $J'_\varepsilon(u_n) = o_n(1)$ in $H^{-1}(\mathbb{R}^\mathbb{N})$.

By Lemma 3.2 (i), there is a subsequence $\{u_n\}$ and $u_0 \in H^1(\mathbb{R}^\mathbb{N})$. We claim that $u_0 \in M_\varepsilon^+(M_\varepsilon^0 = \emptyset$ for $0 < \Lambda < \Lambda_0)$ and $J_\varepsilon(u_0) = \alpha_\varepsilon$. On the contrary, if $u_0 \in M_\varepsilon^-$ we get that

$$\int_{\mathbb{R}^\mathbb{N}} \Lambda h(\varepsilon z)(u_0)_+^q dz > 0.$$

Otherwise,

$$\begin{aligned} \parallel u_n \parallel_H^2 - \int_{\mathbb{R}^\mathbb{N}} a(\varepsilon z)(u_n)_+^p dz &= \int_{\mathbb{R}^\mathbb{N}} \Lambda h(\varepsilon z)(u_n)_+^q dz \\ &= \int_{\mathbb{R}^\mathbb{N}} \Lambda h(\varepsilon z)(u_0)_+^q dz + o_n(1) \\ &= o_n(1). \end{aligned}$$

It follows that $\lim_{n\to\infty}(\frac{1}{2} - \frac{1}{p}) \parallel u_n \parallel_H^2 = \alpha_\varepsilon$, which contradicts to $\alpha_\varepsilon < 0$. By Lemma 2.11 (b), then there are positive numbers $l^+ < \bar{l} < l^- = 1$ such that $l^+u_0 \in M_\varepsilon^+$,$l^-u_0 \in M_\varepsilon^-$ which is a contradiction. Hence, $u_0 \in M_\varepsilon^+$

$$-C_0\Lambda^{\frac{2}{2-q}} \leq J_\varepsilon(u_0) = \alpha_\varepsilon = \alpha_\varepsilon^+.$$

By Lemma 2.4 and the maximum principle, then u_0 is a positive solution of the problem ($E_{\lambda,\mu}$). □

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