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SUMMATION THEOREMS FOR ${}_pF_{q+1}[(\alpha_p); g \pm m, (\beta_q); z]$ VIA MELLIN-BARNES TYPE CONTOUR INTEGRAL AND ITS APPLICATIONS

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Abstract. In this paper, we find summation theorems for generalized hypergeometric functions ${}_pF_{q+1}[(\alpha_p); g\pm m, (\beta_q);\ z]$ by means of Mellin-Barnes type contour integral representation. Some closed forms (believed to be new) for Clausen hypergeometric functions ${}_3F_2[a, b, c; a-m, 1+b-c\pm p; -1]$, ${}_3F_2[a, b, c; a-m, b-c\pm p; -1]$, ${}_3F_2[a, b, c; a-m, (1+b+c\pm p)/2; 1/2]$, ${}_3F_2[a, b, c; a-m, (b+c\pm p)/2; 1/2]$, ${}_3F_2[a, b, 1-b\pm p; a-m, h; 1/2]$, ${}_3F_2[a, b, -b\pm p; a-m, h; 1/2]$, ${}_3F_2[a, b, c; a-m, c+1; 1]$ and Gauss hypergeometric functions ${}_2F_1[a, b; 1+a-b\pm p-m; -1]$, ${}_2F_1[a, b; a-b\pm p-m; -1]$, ${}_2F_1[a, b; (1+a+b\pm p-2m)/2; 1/2]$, ${}_2F_1[a, b; (a+b \pm p - 2m)/2; 1/2]$, ${}_2F_1[a, 1 - a \pm p; g - m; 1/2]$, ${}_2F_1[a, -a \pm p; g - m; 1/2]$ with suitable convergence conditions, are also obtained.

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1. Introduction, Definitions and Preliminaries

In the usual notation, let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. Also let

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\}\ , \qquad \mathbb{N} = \{1, 2, 3, \dots\} = \mathbb{N}_0 \backslash \{0\}\ ,$$

$$\mathbb{Z}_0^- = \{0, -1, -2, \dots\} = \mathbb{Z}^- \cup \{0\}\ , \qquad \mathbb{Z}^- = \{-1, -2, -3, \dots\}$$

and $\mathbb{Z} = \mathbb{Z}_0^- \cup \mathbb{N}$ being the sets of integers.

In terms of Gamma function $\Gamma(z)$, the widely-used Pochhammer symbol $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined, in general, by

$$(1.1) \qquad (\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1\,, & (\nu = 0; \lambda \in \mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1)\,, & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C})\,, \end{cases}$$

it is being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ quotient exists (see, for details, [15])

$$(1.2) \qquad \int_0^\infty e^{-st} t^{\alpha-1} \mathrm{d}t = \frac{\Gamma(\alpha)}{s^\alpha},$$

$$\Big(\Re(s) > 0,\ 0 < \Re(\alpha) < \infty\,,\ \text{or}\ \Re(s) = 0,\ 0 < \Re(\alpha) < 1\Big).$$

Here *generalized hypergeometric function* ${}_pF_q$ with p numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and q denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$ is defined by

$$(1.3) \qquad {}_pF_q\left[\begin{array}{c} \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \\ \beta_1, \beta_2, \ldots, \beta_q; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n} \frac{z^n}{n!}\,,$$

$\Big(p,\ q \in \mathbb{N}_0;\ p \leqq q+1\,;\ p \leqq q$ and $|z| < \infty,$ or $p = q+1$ and $|z| < 1,$ or

$p = q+1, |z| = 1$ and $\Re(\omega) > 0,$ or $p = q+1, |z| = 1, z \neq 1$ and $0 \geq \Re(\omega) > -1\Big),$

where

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j\,,$$

$$\Big(\alpha_j \in \mathbb{C}\ (j = 1, 2, \ldots, p);\ \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-(j = 1, 2, \ldots, q)\Big).$$

Of all the integrals which contain gamma functions in their integrands the most important ones are the so-called Mellin-Barnes integrals. Such integrals were first introduced by S. Pincherle in the year 1888; their theory has been developed by H. Mellin in the year 1910 and they were used for a complete integration of the hypergeometric differential equation by E.W. Barnes in the year 1908. The generalized Hypergeometric function [13, p. 100, Theorem 35 and p. 102, Theorem 36] is defined by means of Mellin-Barnes type contour integral in the following form; when $p \leq q+1$, then

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)}{\Gamma(\beta_1)\Gamma(\beta_2)\ldots\Gamma(\beta_q)}{}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{array} z\right]=$$

$$(1.4)\qquad = \frac{1}{2\pi\omega}\int_{L_1}\frac{(-z)^{\xi}\Gamma(-\xi)\Gamma(\alpha_1+\xi),\ldots,\Gamma(\alpha_p+\xi)}{\Gamma(\beta_1+\xi),\ldots,\Gamma(\beta_q+\xi)}d\xi\ ,$$

where L_1 is a suitable Mellin-Barnes path of integration [See 13, p. 95, figure (5), p. 98, figure(6)] and $\alpha_i, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\{i = 1,2,3,\ldots,p\,; j = 1,2,3,\ldots,q\}$ and $\omega = \sqrt{-1}$, $z \neq 0$.

When $p = q+1$ then $|arg(-z)| < \pi$ and suppose that $|z| < 1$.

When $p = q$ then $|arg(-z)| < \frac{\pi}{2}$ i.e. $\Re(z) < 0$.

When $p < q$ then equation (1.4) is also valid.

In an attempt to give a meaning to the symbol ${}_pF_q$, when $p > q+1$. Firstly the G-function was defined by Meijer [6] in the year 1936 by means of a finite series of generalized hypergeometric functions. Later on the Meijer's G-function of order (m,n,p,q) was defined by means of Mellin-Barnes type contour integral formula [3, p. 207, Entry(5.3.1); see also 4, p. 143, Entry(5.2.1); 5, p. 2, Entries(1.1.1, 1.1.3); 7, p. 1064, (21)], in the following form

$$G^{m,n}_{p,q}\left(z\,\middle|\,\begin{array}{c}a_1,a_2,\ldots,a_n;a_{n+1},\ldots,a_p\\ b_1,b_2,\ldots,b_m;b_{m+1},\ldots,b_q\end{array}\right)=$$

$$(1.5)\qquad = \frac{1}{2\pi\omega}{}_{L_2}\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}z^s\mathbf{d}s\ ,$$

$$(a_k - b_j \neq 1,2,3,\ldots;\ k = 1,2,3,\ldots,n \text{ and } j = 1,2,3,\ldots,m)\ ,$$

where $0 \leq m \leq q, 0 \leq n \leq p\,; z \neq 0$ and L_2 is a suitable contour (See three cases of contour in the monographs [3, p.207, Entries (2,3,4); 4, p.144, Entries (2,3,4); 9, p.617, Entries (1,2,3,4)]) and an empty product is interpreted as 1 and the parameters are such that no pole of $\Gamma(b_j-s)$, $j = 1,2,3,\ldots,m$ coincides with any pole of $\Gamma(1-a_k+s)$, $k = 1,2,3,\ldots,n$. Without any loss of generality, we are assuming that $p \leq q$.

Translation property [7, p. 1066, Entry(24)]

$$z^{\sigma} G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, a_2, \dots, a_n, a_{n+1}, \dots, a_p \\ b_1, b_2, \dots, b_m, b_{m+1}, \dots, b_q \end{matrix}\right) =$$

$$= G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1+\sigma, a_2+\sigma, \dots, a_n+\sigma, a_{n+1}+\sigma, \dots, a_p+\sigma \\ b_1+\sigma, b_2+\sigma, \dots, b_m+\sigma, b_{m+1}+\sigma, \dots, b_q+\sigma \end{matrix}\right). \tag{1.6}$$

Symmetric property (Transformation formula)

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, a_2, \dots, a_n, a_{n+1}, \dots, a_p \\ b_1, b_2, \dots, b_m, b_{m+1}, \dots, b_q \end{matrix}\right) =$$

$$= G_{q,p}^{n,m}\left(\frac{1}{z} \,\middle|\, \begin{matrix} 1-b_1, 1-b_2, \dots, 1-b_m, 1-b_{m+1}, \dots, 1-b_q \\ 1-a_1, 1-a_2, \dots, 1-a_n, 1-a_{n+1}, \dots, 1-a_p \end{matrix}\right). \tag{1.7}$$

Reduction formula between ${}_pF_q$ and G-functions [3, p.215, Entry(5.6.1); 15, p.47, Entry(9)], is given by when $p \leq q+1$, then

(1.8)

$$\frac{\Gamma(a_1)\Gamma(a_2)\dots\Gamma(a_p)}{\Gamma(b_1)\Gamma(b_2)\dots\Gamma(b_q)}\,{}_pF_q\left[\begin{matrix} a_1, a_2, \dots, a_p; \\ b_1, b_2, \dots, b_q; \end{matrix}\; z\right] = G_{p,q+1}^{1,p}\left(-z \,\middle|\, \begin{matrix} 1-a_1, 1-a_2, \dots, 1-a_p \\ 0, 1-b_1, 1-b_2, \dots, 1-b_q \end{matrix}\right)$$

$$= G_{q+1,p}^{p,1}\left(-\frac{1}{z} \,\middle|\, \begin{matrix} 1, b_1, b_2, \dots, b_q \\ a_1, a_2, \dots, a_p \end{matrix}\right),$$

$$\left(p \leq q \text{ and } |z| < \infty, \text{ or } p = q+1 \text{ and } |z| < 1\right).$$

2. Main Results and Their Demonstrations

Our main results (presumably new) is stated here as the following theorem. The results derived in this section is interesting and potentially useful.

The following hypergeometric summation theorem holds true, when any values of parameters and variables leading to the result which do not make sense, are tacitly excluded and each of the series involved in the following result is absolutely convergent. Then

Theorem 2.1.

$${}_pF_{q+1}\left[\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_p; \\ g-m, \beta_1, \beta_2, \dots, \beta_q; \end{matrix}\; z\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(\alpha_1)_r(\alpha_2)_r\dots(\alpha_p)_r z^r}{(g-m)_r(g)_r(\beta_1)_r(\beta_2)_r\dots(\beta_q)_r}\times\right.$$

$$\text{(2.1)} \qquad \times {}_pF_{q+1}\left[\begin{matrix} \alpha_1+r, \alpha_2+r, \ldots, \alpha_p+r; \\ g+r, \beta_1+r, \beta_2+r, \ldots, \beta_q+r; \end{matrix} \; z\right]\Bigg\} ,$$

$$\Bigg(g-m, \alpha_i, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^- \; ; \{\textbf{i=1,2,3,...,p} \; ; \textbf{j=1,2,3,...,q}\} \; ; \; p=q+2, \; |z|<1,$$
$$\textbf{or } p \le (q+1) \; , |z| < \infty, \; \textbf{or } p=q+2, \; z=1, \; \Re(g-m+\textstyle\sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i) > 0,$$
$$\textbf{or } p=q+2, \; z=-1, \; \Re(g-m+\textstyle\sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i) > -1 \; ; \; m \in \mathbb{N}_0\Bigg).$$

Demonstration: In order to derive the theorem (2.1) we consider the following Mellin-Barnes type contour integral

$${}_pF_{q+1}\left[\begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_p; \\ g+m, \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} \; z\right] = \frac{\Gamma(g+m)\Gamma(\beta_1)\Gamma(\beta_2)\ldots\Gamma(\beta_q)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)} \times$$

$$\times \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Gamma(-s)\Gamma(\alpha_1+s)\Gamma(\alpha_2+s)\ldots\Gamma(\alpha_p+s)(-z)^s \mathbf{d}s}{\Gamma(g+m+s)\Gamma(\beta_1+s)\Gamma(\beta_2+s)\ldots\Gamma(\beta_q+s)}$$

$$= \frac{\Gamma(g+2m)\Gamma(\beta_1)\Gamma(\beta_2)\ldots\Gamma(\beta_q)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)} \times$$

$$\times \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Gamma(-s)\Gamma(\alpha_1+s)\Gamma(\alpha_2+s)\ldots\Gamma(\alpha_p+s)}{\Gamma(g+2m+s)\Gamma(\beta_1+s)\Gamma(\beta_2+s)\ldots\Gamma(\beta_q+s)} {}_2F_1\left[\begin{matrix} -m, -s; \\ g+m; \end{matrix} \; 1\right](-z)^s \mathbf{d}s$$

$$= \frac{\Gamma(g+2m)\Gamma(\beta_1)\Gamma(\beta_2)\ldots\Gamma(\beta_q)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)} \times$$

$$\times \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Gamma(-s)\Gamma(\alpha_1+s)\Gamma(\alpha_2+s)\ldots\Gamma(\alpha_p+s)}{\Gamma(g+2m+s)\Gamma(\beta_1+s)\Gamma(\beta_2+s)\ldots\Gamma(\beta_q+s)} \sum_{r=0}^{m}\left\{\frac{(-m)_r(-s)_r}{(g+m)_r \; r!}\right\}(-z)^s \mathbf{d}s$$

$$= \frac{\Gamma(g+2m)\Gamma(\beta_1)\Gamma(\beta_2)\ldots\Gamma(\beta_q)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)} \sum_{r=0}^{m}\Bigg\{\frac{(-m)_r}{(g+m)_r \; r!} \times$$

$$\times \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Gamma(r-s)\Gamma(1-(1-\alpha_1)+s)\ldots\Gamma(1-(1-\alpha_p)+s)(-z)^s \mathbf{d}s}{\Gamma(1-(1-g-2m)+s)\Gamma(1-(1-\beta_1)+s)\ldots\Gamma(1-(1-\beta_q)+s)}\Bigg\}$$

$$= \frac{\Gamma(g+2m)\Gamma(\beta_1)\Gamma(\beta_2)\ldots\Gamma(\beta_q)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)} \times$$

$$\times \sum_{r=0}^{m}\left\{\frac{(-m)_r}{(g+m)_r \; r!} G^{1,p}_{p,q+2}\left(-z \,\middle|\, \begin{matrix} 1-\alpha_1, 1-\alpha_2, \ldots, 1-\alpha_p \\ r; 1-g-2m, 1-\beta_1, 1-\beta_2, \ldots, 1-\beta_q \end{matrix}\right)\right\}$$

Now using the symmetric property (1.7), we get

$$
{}_pF_{q+1}\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ g+m,\beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\; z\right]=\frac{\Gamma(g+2m)\Gamma(\beta_1)\Gamma(\beta_2)\ldots\Gamma(\beta_q)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)}\times
$$

$$
\times\sum_{r=0}^{m}\left\{\frac{(-m)_r(-z)^r}{(g+m)_r\; r!}\left(\frac{-1}{z}\right)^r G^{p,1}_{q+2,p}\left(-\frac{1}{z}\,\middle|\,\begin{matrix}1-r,g+2m,\beta_1,\beta_2,\ldots,\beta_q\\ \alpha_1,\alpha_2,\ldots,\alpha_p\end{matrix}\right)\right\}
$$

Now applying the translation property (1.6), we get

$$
{}_pF_{q+1}\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ g+m,\beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\; z\right]=\frac{\Gamma(g+2m)\Gamma(\beta_1)\Gamma(\beta_2)\ldots\Gamma(\beta_q)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)}\times
$$

$$
\times\sum_{r=0}^{m}\left\{\frac{(-m)_r(-z)^r}{(g+m)_r\; r!}G^{p,1}_{q+2,p}\left(-\frac{1}{z}\,\middle|\,\begin{matrix}1,g+2m+r,\beta_1+r,\beta_2+r,\ldots,\beta_q+r\\ \alpha_1+r,\alpha_2+r,\ldots,\alpha_p+r\end{matrix}\right)\right\}
$$

Now using the reduction formula (1.8), we get

$$
{}_pF_{q+1}\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ g+m,\beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\; z\right]=\frac{\Gamma(g+2m)\Gamma(\beta_1)\Gamma(\beta_2)\ldots\Gamma(\beta_q)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)}\times
$$

$$
\times\sum_{r=0}^{m}\left\{\frac{(-m)_r\Gamma(\alpha_1+r)\Gamma(\alpha_2+r)\Gamma(\alpha_p+r)(-z)^r}{(g+m)_r\;\Gamma(g+2m+r)\Gamma(\beta_1+r)\ldots\Gamma(\beta_q+r)\; r!}\times\right.
$$

$$
\left.\times{}_pF_{q+1}\left[\begin{matrix}\alpha_1+r,\alpha_2+r,\ldots,\alpha_p+r;\\ g+2m+r,\beta_1+r,\beta_2+r,\ldots,\beta_q+r;\end{matrix}\; z\right]\right\}
$$

Therefore

$$
{}_pF_{q+1}\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ g+m,\beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\; z\right]=\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(\alpha_1)_r(\alpha_2)_r\ldots(\alpha_p)_r z^r}{(g+m)_r(g+2m)_r(\beta_1)_r(\beta_2)_r\ldots(\beta_q)_r}\times\right.
$$

$$
\left.\times\;{}_pF_{q+1}\left[\begin{matrix}\alpha_1+r,\alpha_2+r,\ldots,\alpha_p+r;\\ g+2m+r,\beta_1+r,\beta_2+r,\ldots,\beta_q+r;\end{matrix}\; z\right]\right\} \tag{2.2}
$$

$\Big(g,\alpha_i,\beta_j\in\mathbb{C}\backslash\mathbb{Z}_0^-$; {**i=1,2,3,...,p** ; **j=1,2,3,...,q**} ; $p=q+2,\ |z|<1,$

or $p\le(q+1)$, $|z|<\infty,$ **or** $p=q+2,\ z=1,\ \Re(g+m+\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i)>0,$

or $p=q+2,\ z=-1,\ \Re(g+m+\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i)>-1$; $m\in\mathbb{N}_0\Big)$.

Replace g by $g-2m$ in equation (2.2), we get the theorem (2.1)

3. Applications in Clausen Series

In this section, we obtain some special cases of our main summation theorem (2.1). In theorem (2.1) put $p = 3$ and $q = 1$, we get

$$(3.1)\quad {}_3F_2\left[\begin{array}{r} a, b, c; \\ g-m, h; \end{array} z\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(a)_r(b)_r(c)_r(z)^r}{(g-m)_r(g)_r(h)_r}\,{}_3F_2\left[\begin{array}{r} a+r, b+r, c+r; \\ g+r, h+r; \end{array} z\right]\right\},$$

$$\left(a,\ b,\ c,\ g-m,\ h \in \mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ |z|<1;\ m\in\mathbb{N}_0\right).$$

In equation (3.1) put $g = a$ and $z = -1$, we get

$$(3.2)\quad {}_3F_2\left[\begin{array}{r} a, b, c; \\ a-m, h; \end{array} -1\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(b)_r(c)_r(-1)^r}{(a-m)_r(h)_r}\,{}_2F_1\left[\begin{array}{r} b+r, c+r; \\ h+r; \end{array} -1\right]\right\},$$

$$\left(\Re(h-b-c) > m-1\ ;\ a-m,\ b,\ c,\ h\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m\in\mathbb{N}_0\right).$$

In equation (3.2) put $h = 1 + b - c - p$ and use the summation theorem [2, p.1523, Equation (2.2)] of Choi-Rathie and Malani, we get

$$ {}_3F_2\left[\begin{array}{r} a, b, c; \\ a-m, 1+b-c-p; \end{array} -1\right] =$$

$$(3.3)\quad = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(c)_r\Gamma(1+b-c-p)}{2(-1)^r\Gamma(b)(a-m)_r}\sum_{k=0}^{p-r}\left[\binom{p-r}{k}\frac{\Gamma(\frac{b+r+k}{2})}{\Gamma(\frac{b+r+k+2-2c-2p}{2})}\right]\right\},$$

$$\left(\Re(c) < \tfrac{2-p-m}{2}\ ;\ a-m,\ b,\ c,\ 1+b-c-p\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m,\ p\in\mathbb{N}_0\ ;\ p\geq m\right).$$

In equation (3.2) put $h = 1 + b - c + p$ and use the summation theorem [2, p.1524, Equation (2.3)] of Choi-Rathie and Malani, we get

$$ {}_3F_2\left[\begin{array}{r} a, b, c; \\ a-m, 1+b-c+p; \end{array} -1\right] =$$

$$(3.4)\quad = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{\Gamma(1+b-c+p)}{2\Gamma(b)(a-m)_r(1-c)_p}\sum_{k=0}^{p+r}\left[\binom{p+r}{k}\frac{(-1)^k\Gamma(\frac{b+r+k}{2})}{\Gamma(\frac{b-r+k+2-2c}{2})}\right]\right\},$$

$$\left(\Re(c) < \tfrac{p-m+2}{2}\ ;\ a-m,\ b,\ c,\ 1-c-m,\ 1+b-c\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m,\ p\in\mathbb{N}_0\right).$$

In equation (3.2) put $h=b-c-p$ and use the summation theorem [10, p.14, Equation (3.1)] derived by the authors, we get

$$ {}_3F_2\left[\begin{matrix} a,b,c; \\ a-m,b-c-p; \end{matrix}\; -1\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{\Gamma(b-c-p)(c)_r(-1)^r}{2(a-m)_r\Gamma(b)}\times\right. $$

$$ \left.\times\sum_{k=0}^{p-r}\binom{p-r}{k}\left[\frac{\Gamma(\frac{b+r+k}{2})}{\Gamma(\frac{b+r+k-2c-2p}{2})}+\frac{\Gamma(\frac{b+r+k+1}{2})}{\Gamma(\frac{b+r+k+1-2c-2p}{2})}\right]\right\}, \tag{3.5} $$

$$ \left(\Re(c)<\tfrac{1-m-p}{2};\ a-m,\ b,\ c,\ b-c-p\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m,\ p\in\mathbb{N}_0;\ p\geq m\right). $$

In equation (3.2) put $h=b-c+p$ and use the summation theorem [10, p.14, Equation (3.2)] derived by the authors, we get

$$ {}_3F_2\left[\begin{matrix} a,b,c; \\ a-m,b-c+p; \end{matrix}\; -1\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{\Gamma(b-c+p)(c)_r}{2(a-m)_r(1+c)_r(-c)_p\Gamma(b)}\times\right. $$

$$ \left.\times\sum_{k=0}^{p+r}\binom{p+r}{k}\left[\frac{(-1)^k\Gamma(\frac{b+r+k}{2})}{\Gamma(\frac{b-r+k-2c}{2})}+\frac{(-1)^k\Gamma(\frac{b+r+k+1}{2})}{\Gamma(\frac{b-r+k+1-2c}{2})}\right]\right\}, \tag{3.6} $$

$$ \left(\Re(c)<\tfrac{p+1-m}{2};\ a-m,\ b,\ c,\ -c-m,\ b-c\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m,\ p\in\mathbb{N}_0\right). $$

When $g=a$ and $z=\frac{1}{2}$ in equation (3.1), we get

$$ {}_3F_2\left[\begin{matrix} a,b,c; \\ a-m,h; \end{matrix}\; \frac{1}{2}\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(b)_r(c)_r}{2^r\,(a-m)_r(h)_r}\,{}_2F_1\left[\begin{matrix} b+r,c+r; \\ h+r; \end{matrix}\; \frac{1}{2}\right]\right\}, \tag{3.7} $$

$$ \left(a-m,\ b,\ c,\ h\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m\in\mathbb{N}_0\right). $$

In equation (3.7) put $h=\frac{b+c+1-p}{2}$ and use the summation theorem [9, p.491, Entry (7.3.7.2)] of Prudnikov et al., we get

$$ {}_3F_2\left[\begin{matrix} a,b,c; \\ a-m,\frac{b+c+1-p}{2}; \end{matrix}\; \frac{1}{2}\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{2^{c-1}\,(b)_r\,\Gamma(\frac{b+c+1-p}{2})}{(a-m)_r\,\Gamma(c)}\sum_{k=0}^{p}\left[\binom{p}{k}\frac{\Gamma(\frac{c+r+k}{2})}{\Gamma(\frac{b+r+k-p+1}{2})}\right]\right\}, \tag{3.8} $$

$$ \left(a-m,\ b,\ c,\ \tfrac{b+c+1-p}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m,\ p\in\mathbb{N}_0\right). $$

In equation (3.7) put $h=\frac{b+c+1+p}{2}$ and use the summation theorem [14, p.827, Theorem (1)] of Rakha-Rathie, we get

(3.9)

$$ {}_3F_2\left[\begin{matrix} a,b,c; & \frac{1}{2} \\ a-m, \frac{b+c+1+p}{2}; & 2 \end{matrix}\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{2^{c-1}\,(b)_r\,\Gamma(\frac{b+c+1+p}{2})}{(a-m)_r\,\Gamma(c)\,(\frac{b-c+1-p}{2})_p}\sum_{k=0}^{p}\left[\binom{p}{k}\frac{(-1)^k\Gamma(\frac{c+r+k}{2})}{\Gamma(\frac{b+r+k-p+1}{2})}\right]\right\}, $$

$$ \left(a-m,\ b,\ c,\ \tfrac{b+c+1+p}{2},\ \tfrac{b-c+1-p}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ p,\ m\in\mathbb{N}_0\right). $$

In equation (3.7) put $h=\frac{b+c-p}{2}$ and use the summation theorem [11, p.48, Equation (16)] derived by the authors, we get

(3.10)

$$ {}_3F_2\left[\begin{matrix} a,b,c; & \frac{1}{2} \\ a-m, \frac{b+c-p}{2}; & 2 \end{matrix}\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{2^{c-1}(b)_r\,\Gamma(\frac{b+c-p}{2})}{(a-m)_r\,\Gamma(c)}\sum_{k=0}^{p}\binom{p}{k}\left[\frac{\Gamma(\frac{c+r+k}{2})}{\Gamma(\frac{b+r+k-p}{2})}+\frac{\Gamma(\frac{c+r+k+1}{2})}{\Gamma(\frac{b+r+k-p+1}{2})}\right]\right\}, $$

$$ \left(a-m,\ b,\ c,\ \tfrac{b+c-p}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ p,\ m\in\mathbb{N}_0\right). $$

In equation (3.7) put $h=\frac{b+c+p}{2}$ and use the summation theorem [1, p.582, Entry (8.1.1.130)] of Brychkov, we get

$$ {}_3F_2\left[\begin{matrix} a,b,c; & \frac{1}{2} \\ a-m, \frac{b+c+p}{2}; & 2 \end{matrix}\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(b)_r(c)_r\,\Gamma(\frac{1}{2})\Gamma(\frac{b+c+p}{2})}{2^r(a-m)_r\,(\frac{-b+c-p}{2})_p}\times\right. $$

(3.11)
$$ \left.\times\sum_{k=0}^{p}\left(\binom{p}{k}(-2)^{-k}(b+r)_k\left[\frac{1}{\Gamma(\frac{b+r+k+1}{2})\Gamma(\frac{c+r+k-p}{2})}+\frac{1}{\Gamma(\frac{b+r+k}{2})\Gamma(\frac{c+r+k-p+1}{2})}\right]\right)\right\}, $$

$$ \left(a-m,\ b,\ c,\ \tfrac{b+c+p}{2},\ \tfrac{-b+c-p}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ p,m\ \in\mathbb{N}_0\right). $$

In equation (3.7) put $c=1-b-p$ and use the summation theorem [14, p.828, Theorem (6)] of Rakha-Rathie, we get

$$ {}_3F_2\left[\begin{matrix} a,b,1-b-p; & \frac{1}{2} \\ a-m,h; & 2 \end{matrix}\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(b)_r(1-b-p)_r\Gamma(h)}{(a-m)_r\Gamma(h-b)2^{b+p}}\times\right. $$

(3.12)
$$ \left.\times\sum_{k=0}^{p-2r}\left[\binom{p-2r}{k}\frac{\Gamma(\frac{h-b+k}{2})}{\Gamma(\frac{h+b+k+2r}{2})}\right]\right\}, $$

$$ \left(a-m,\ b,\ 1-b-p,\ h,\ h-b\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m,\ p\in\mathbb{N}_0\ ;\ p\geq 2m\right). $$

In equation (3.7) put $c = 1-b+p$ and use the summation theorem [14, p.827, Theorem (5)] of Rakha-Rathie, we get

$$
{}_3F_2\left[\begin{array}{r} a, b, 1-b+p; \\ a-m, h; \end{array} \frac{1}{2}\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(1-b+p)_r\Gamma(h)\Gamma(b-r-p)}{(a-m)_r\Gamma(b)\Gamma(h-b)2^{b-p}}\times\right.
$$

(3.13)
$$
\left.\times\sum_{k=0}^{p+2r}\left[\binom{p+2r}{k}\frac{(-1)^k\Gamma(\frac{h-b+k}{2})}{\Gamma(\frac{h+b+k-2r-2p}{2})}\right]\right\},
$$

$$
\Big(a-m,\ b,\ 1-b,\ h,\ h-b,\ b-m-p \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m,\ p \in \mathbb{N}_0\Big).
$$

In equation (3.7) put $c = -b-p$ and use the summation theorem [12, Equation (3.3)] derived by the authors, we get

$$
{}_3F_2\left[\begin{array}{r} a, b, -b-p; \\ a-m, h; \end{array} \frac{1}{2}\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(b)_r(-b-p)_r\Gamma(h)2^{-b-p-1}}{(a-m)_r\Gamma(h-b)}\times\right.
$$

(3.14)
$$
\left.\times\sum_{k=0}^{p-2r}\binom{p-2r}{k}\left[\frac{\Gamma(\frac{h-b+k}{2})}{\Gamma(\frac{h+b+k+2r}{2})}+\frac{\Gamma(\frac{h-b+k+1}{2})}{\Gamma(\frac{h+b+k+2r+1}{2})}\right]\right\},
$$

$$
\Big(a-m,\ b,\ -b-p,\ h,\ h-b \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m,\ p \in \mathbb{N}_0\ ;\ p \geq 2m\Big).
$$

In equation (3.7) put $c = -b+p$ and use the summation theorem [12, Equation (3.5)] derived by the authors, we get

$$
{}_3F_2\left[\begin{array}{r} a, b, -b+p; \\ a-m, h; \end{array} \frac{1}{2}\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(-b+p)_r\Gamma(h)\Gamma(b-p-r)2^{p-b-1}}{(a-m)_r\Gamma(b)\Gamma(h-b)}\times\right.
$$

(3.15)
$$
\left.\times\sum_{k=0}^{p+2r}\binom{p+2r}{k}\left[\frac{(-1)^k\Gamma(\frac{h-b+k}{2})}{\Gamma(\frac{h+b+k-2p-2r}{2})}+\frac{(-1)^k\Gamma(\frac{h-b+k+1}{2})}{\Gamma(\frac{h+b+k-2r-2p+1}{2})}\right]\right\},
$$

$$
\Big(a-m,\ b,\ -b,\ b-p-m,\ h,\ h-b \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m,\ p \in \mathbb{N}_0\Big).
$$

In equation (3.1) put $g = a$ and $z = 1$ and use the Classical Gauss summation theorem [15, p. 30, Equation 1.2(7)], we get

(3.16)
$$
{}_3F_2\left[\begin{array}{r} a, b, c; \\ a-m, h; \end{array} 1\right] = \frac{\Gamma(h)\Gamma(h-b-c)}{\Gamma(h-b)\Gamma(h-c)}\,{}_3F_2\left[\begin{array}{r} -m, b, c; \\ a-m, 1+b+c-h; \end{array} 1\right],
$$

$$
\Big((h-b-c) \neq 0, \pm 1, \pm 2, \dots\ ;\ \Re(h-b-c) > m\ ;
$$
$$
a-m,\ b,\ c,\ h,\ 1+b+c-h \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\Big).
$$

Now put $h = c + 1$ in the equation (3.16), we get

$$
{}_3F_2\left[\begin{array}{r} a, b, c; \\ a-m, c+1; \end{array} 1\right] = \frac{\Gamma(c+1)\Gamma(1-b)(1+c-a)_m}{\Gamma(c+1-b)(1-a)_m}\ , \tag{3.17}
$$

$$
\big((1-b) \neq 0, \pm 1, \pm 2 \ldots\ ;\ \Re(1-b) > m\ ;\ a-m,\ b,\ c \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\big).
$$

which is a known summation theorem of Mitra [8, p.85, Last Equation].

4. Applications in Gaussian Series

In equation (3.1) put $h = c$ and $z = -1$, we get

$$
{}_2F_1\left[\begin{array}{r} a, b; \\ g-m; \end{array} -1\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(a)_r(b)_r(-1)^r}{(g)_r(g-m)_r}\,{}_2F_1\left[\begin{array}{r} a+r, b+r; \\ g+r; \end{array} -1\right]\right\}\ , \tag{4.1}
$$

$$
\Big(\Re(g-a-b) > m-1\ ;\ a,\ b,\ g-m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\Big).
$$

In equation (4.1) put $g = 1 + a - b - p$ and use the summation theorem [2, p.1523, Equation (2.2)] of Choi-Rathie and Malani, we get

$$
{}_2F_1\left[\begin{array}{r} a, b; \\ 1+a-b-p-m; \end{array} -1\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(-1)^r(b)_r\Gamma(1+a-b-p)}{2\Gamma(a)(1+a-b-p-m)_r}\times\right.
$$

$$
\left.\times\sum_{k=0}^{p-r}\left[\binom{p-r}{k}\frac{\Gamma(\frac{a+r+k}{2})}{\Gamma(\frac{a+r+k+2-2b-2p}{2})}\right]\right\}\ , \tag{4.2}
$$

$$
\Big(\Re(b) < \tfrac{2-p-m}{2}\ ;\ a,\ b,\ 1+a-b-p-m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m,\ p \in \mathbb{N}_0\ ;\ p \geq m\Big).
$$

In equation (4.1) put $g = 1 + a - b + p$ and use the summation theorem [2, p.1524, Equation (2.3)] of Choi-Rathie and Malani, we get

$$
{}_2F_1\left[\begin{array}{r} a, b; \\ 1+a-b+p-m; \end{array} -1\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{\Gamma(1+a-b+p)}{2\Gamma(a)(1-b)_p(1+a-b+p-m)_r}\times\right.
$$

$$
\left.\times\sum_{k=0}^{p+r}\left[\binom{p+r}{k}\frac{(-1)^k\Gamma(\frac{a+r+k}{2})}{\Gamma(\frac{a-r+k+2-2b}{2})}\right]\right\}\ , \tag{4.3}
$$

$$
\Big(\Re(b) < \tfrac{p+2-m}{2}\ ;\ a,\ b,\ 1-b-m,\ 1+a-b+p-m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m,\ p \in \mathbb{N}_0\Big).
$$

In equation (4.1) put $g = a-b-p$ and use the summation theorem [10, p.14, Equation (3.1)] derived by the authors, we get

$$
{}_2F_1\left[\begin{matrix} a,b; \\ a-b-p-m; \end{matrix} \; -1\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(b)_r\Gamma(a-b-p)(-1)^r}{2\Gamma(a)(a-b-p-m)_r}\times\right.
$$

$$
\left.\times\sum_{k=0}^{p-r}\binom{p-r}{k}\left[\frac{\Gamma(\frac{a+r+k}{2})}{\Gamma(\frac{a+r+k-2b-2p}{2})}+\frac{\Gamma(\frac{a+r+k+1}{2})}{\Gamma(\frac{a+r+k+1-2b-2p}{2})}\right]\right\}, \tag{4.4}
$$

$$
\left(\Re(b) < \tfrac{1-p-m}{2}\; ;\; a,\; b,\; a-b-p-m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\; m,\; p \in \mathbb{N}_0\; ;\; p \geq m\right).
$$

In equation (4.1) put $g = a-b+p$ and use the summation theorem [10, p.14, Equation (3.2)] derived by the authors, we get

$$
{}_2F_1\left[\begin{matrix} a,b; \\ a-b+p-m; \end{matrix} \; -1\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(b)_r\Gamma(a-b+p)}{2\Gamma(a)(-b)_p(1+b)_r(a-b+p-m)_r}\times\right.
$$

$$
\left.\times\sum_{k=0}^{p+r}\binom{p+r}{k}\left[\frac{(-1)^k\Gamma(\frac{a+r+k}{2})}{\Gamma(\frac{a-r+k-2b}{2})}+\frac{(-1)^k\Gamma(\frac{a+r+k+1}{2})}{\Gamma(\frac{a-r+k+1-2b}{2})}\right]\right\}, \tag{4.5}
$$

$$
\left(\Re(b) < \tfrac{p+1-m}{2}\; ;\; a,\; b,\; -b-m,\; a-b+p-m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\; m,\; p \in \mathbb{N}_0\right).
$$

In equation (3.1) put $h = c$ and $z = \frac{1}{2}$, we get

$$
{}_2F_1\left[\begin{matrix} a,b; \\ g-m; \end{matrix} \; \frac{1}{2}\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(a)_r(b)_r}{2^r(g)_r(g-m)_r}\,{}_2F_1\left[\begin{matrix} a+r,b+r; \\ g+r; \end{matrix} \; \frac{1}{2}\right]\right\}, \tag{4.6}
$$

$$
\left(a,\; b,\; g-m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\; m \in \mathbb{N}_0\right).
$$

In equation (4.6) put $g = \frac{a+b+1-p}{2}$ and use the summation theorem [9, p.491, Entry (7.3.7.2)] of Prudnikov et al., we get

(4.7)

$$
{}_2F_1\left[\begin{matrix} a,b; \\ \frac{a+b+1-p-2m}{2}; \end{matrix} \; \frac{1}{2}\right] = \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{2^{b-1}\,(a)_r\,\Gamma(\frac{a+b+1-p}{2})}{(\frac{a+b+1-p-2m}{2})_r\,\Gamma(b)}\sum_{k=0}^{p}\left[\binom{p}{k}\frac{\Gamma(\frac{b+r+k}{2})}{\Gamma(\frac{a+r+k+1-p}{2})}\right]\right\},
$$

$$
\left(a, b, \tfrac{1+a+b-p-2m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-;\; p,\; m \in \mathbb{N}_0\right).
$$

In equation (4.6) put $g = \frac{a+b+1+p}{2}$ and use the summation theorem [14, p.827, Theorem (1)] of Rakha-Rathie, we get

(4.8)
$$
{}_2F_1\left[\begin{array}{r}a,b;\\ \frac{a+b+1+p-2m}{2};\end{array}\ \frac{1}{2}\right]=\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{2^{b-1}\,(a)_r\,\Gamma(\frac{a+b+1+p}{2})}{(\frac{a+b+1+p-2m}{2})_r\,\Gamma(a)(\frac{a-b+1-p}{2})_p}\sum_{k=0}^{p}\left[\binom{p}{k}\frac{(-1)^k\Gamma(\frac{b+r+k}{2})}{\Gamma(\frac{a+r+k+1-p}{2})}\right]\right\},
$$
$$
\left(a,b,\tfrac{1+a+b+p-2m}{2},\tfrac{a-b+1-p}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ p,\ m\in\mathbb{N}_0\right).
$$

In equation (4.6) put $g=\frac{a+b-p}{2}$ and use the summation theorem [11, p.48, Equation (16)] derived by the authors, we get

(4.9)
$$
{}_2F_1\left[\begin{array}{r}a,b;\\ \frac{a+b-p-2m}{2};\end{array}\ \frac{1}{2}\right]=\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{2^{b-1}(a)_r\,\Gamma(\frac{a+b-p}{2})}{(\frac{a+b-p-2m}{2})_r\,\Gamma(b)}\sum_{k=0}^{p}\binom{p}{k}\left[\frac{\Gamma(\frac{b+r+k}{2})}{\Gamma(\frac{a+r+k-p}{2})}+\frac{\Gamma(\frac{b+r+k+1}{2})}{\Gamma(\frac{a+r+k-p+1}{2})}\right]\right\},
$$
$$
\left(a,b,\tfrac{a+b-p-2m}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ p,\ m\in\mathbb{N}_0\right).
$$

In equation (4.6) put $g=\frac{a+b+p}{2}$ and use the summation theorem [1, p.582, Entry (8.1.1.130)] of Brychkov, we get

$$
{}_2F_1\left[\begin{array}{r}a,b;\\ \frac{a+b+p-2m}{2};\end{array}\ \frac{1}{2}\right]=\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(a)_r(b)_r\,\Gamma(\frac{1}{2})\Gamma(\frac{a+b+p}{2})}{2^r(\frac{a+b+p-2m}{2})_r\,(\frac{-a+b-p}{2})_p}\right.\times
$$

(4.10)
$$
\left.\times\sum_{k=0}^{p}\left(\binom{p}{k}(-2)^{-k}(a+r)_k\left[\frac{1}{\Gamma(\frac{a+r+k+1}{2})\Gamma(\frac{b+r+k-p}{2})}+\frac{1}{\Gamma(\frac{a+r+k}{2})\Gamma(\frac{b+r+k-p+1}{2})}\right]\right)\right\},
$$
$$
\left(a,b,\tfrac{a+b+p-2m}{2},\tfrac{-a+b-p}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ p,\ m\in\mathbb{N}_0\right).
$$

In equation (4.6) put $b=1-a-p$ and use the summation theorem [14, p.828, Theorem (6)] of Rakha-Rathie, we get

$$
{}_2F_1\left[\begin{array}{r}a,1-a-p;\\ g-m;\end{array}\ \frac{1}{2}\right]=\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(a)_r(1-a-p)_r\Gamma(g)}{(g-m)_r\Gamma(g-a)2^{a+p}}\right.\times
$$

(4.11)
$$
\left.\times\sum_{k=0}^{p-2r}\left[\binom{p-2r}{k}\frac{\Gamma(\frac{g-a+k}{2})}{\Gamma(\frac{g+a+k+2r}{2})}\right]\right\},
$$
$$
\left(a,\ 1-a-p,\ g-a,\ g-m\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m,\ p\in\mathbb{N}_0\ ;\ p\geq 2m\right).
$$

In equation (4.6) put $b=1-a+p$ and use the summation theorem [14, p.827, Theorem (5)] of Rakha-Rathie, we get

$$
{}_2F_1\left[\begin{array}{r}a,1-a+p;\\ g-m;\end{array}\ \frac{1}{2}\right]=\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(1-a+p)_r\Gamma(g)\Gamma(a-r-p)}{(g-m)_r\Gamma(a)\Gamma(g-a)2^{a-p}}\right.\times
$$

$$\times \sum_{k=0}^{p+2r} \left[\binom{p+2r}{k} \frac{(-1)^k \Gamma(\frac{g-a+k}{2})}{\Gamma(\frac{g+a+k-2r-2p}{2})} \right] \Bigg\} , \tag{4.12}$$

$$\left(a,\ 1-a,\ g-a,\ g-m,\ a-m-p \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m,\ p \in \mathbb{N}_0 \right).$$

In equation (4.6) put $b = -a-p$ and use the summation theorem [12, Equation (3.3)] derived by the authors, we get

$${}_2F_1 \left[\begin{matrix} a, -a-p; \\ g-m; \end{matrix} \ \frac{1}{2} \right] = \sum_{r=0}^{m} \Bigg\{ \binom{m}{r} \frac{(a)_r (-a-p)_r \Gamma(g) 2^{-a-p-1}}{(g-m)_r \Gamma(g-a)} \times$$

$$\times \sum_{k=0}^{p-2r} \binom{p-2r}{k} \left[\frac{\Gamma(\frac{g-a+k}{2})}{\Gamma(\frac{g+a+k+2r}{2})} + \frac{\Gamma(\frac{g-a+k+1}{2})}{\Gamma(\frac{g+a+k+2r+1}{2})} \right] \Bigg\} , \tag{4.13}$$

$$\left(a,\ -a-p,\ g-a,\ g-m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m,\ p \in \mathbb{N}_0\ ;\ p \geq 2m \right).$$

In equation (4.6) put $b = -a+p$ and use the summation theorem [12, Equation (3.5)] derived by the authors, we get

$${}_2F_1 \left[\begin{matrix} a, -a+p; \\ g-m; \end{matrix} \ \frac{1}{2} \right] = \sum_{r=0}^{m} \Bigg\{ \binom{m}{r} \frac{(-a+p)_r \Gamma(g) \Gamma(a-p-r) 2^{p-a-1}}{\Gamma(a)\Gamma(g-a)(g-m)_r} \times$$

$$\times \sum_{k=0}^{p+2r} \binom{p+2r}{k} \left[\frac{(-1)^k \Gamma(\frac{g-a+k}{2})}{\Gamma(\frac{g+a+k-2p-2r}{2})} + \frac{(-1)^k \Gamma(\frac{g-a+k+1}{2})}{\Gamma(\frac{g+a+k-2r-2p+1}{2})} \right] \Bigg\} , \tag{4.14}$$

$$\left(a,\ -a,\ a-p-m,\ g-a,\ g-m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m,\ p \in \mathbb{N}_0 \right).$$

We conclude our present investigation by observing that several other corollaries and consequences of hypergeometric summation theorems (2.1), (3.1), (3.2), (3.7), (4.1) and (4.6) can also be deduced in an analogous manner.

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