Asia Pacific Journal of Mathematics, Vol. 3, No. 2 (2016), 99-113 ISSN 2357-2205

LAPLACIAN MINIMUM BOUNDARY DOMINATING ENERGY OF GRAPHS

MOHAMMED ALATIF* AND PUTTASWAMY

Department of Mathematics, P.E.S. College of Engineering, Mandya-571 401.Karnataka State, India

*Corresponding author

Abstract. For a graph G, a subset B of $V(G)$ is called a boundary dominating set if every vertex of $V(G)-B$ is vertex boundary dominated by some vertex of B. The boundary domination number $\gamma_b(G)$ of G is the minimum cardinality of minimum boundary dominating set in G. In this paper we computed Laplacian minimum boundary dominating energies of some standard graphs. Upper and lower bounds for $LE_B(G)$ are established.

2010 Mathematics Subject Classification. 05C69, 05C38.

Key words and phrases. boundary dominating set; Laplacian minimum boundary dominating eigenvalues; Laplacian minimum boundary dominating energy of a graph.

1. Introduction

The concept of Boundary domination in graphs was introduced by KM. Kathiresan, G. Marimuthu and M. Sivanandha Saraswathy [11] in the year 2010. Let G be a graph with n vertices and m edges, we mean a simple graph, that is nonempty, finite, having no loops, no multiple and directed edges.

The distance between two vertices u and v is the length of a shortest path joining them. A vertex v is called a boundary neighbor of u if v is a nearest boundary of u. If $u \in V$, then the boundary neighbourhood of u denoted by $N_b(u)$ is defined as $N_b(u) = \{v \in V : d(u,w) \leq d(u,v) \text{ for all } w \in N(u)\}$. The cardinality of $N_b(u)$ is denoted by $deg_b(u)$ in G. The maximum and minimum boundary degree of a vertex in G are denoted respectively by $\Delta_b(G)$ and $\delta_b(G)$. That is $\Delta_b(G) = \max_{u\in V} |N_b(u)|$, $\delta_b(G) = \min_{u\in V} |N_b(u)|$.

A vertex u boundary dominate a vertex v if v is a boundary neighbor of u. A subset B of $V(G)$ is called a boundary dominating set if every vertex of $V - B$ is boundary dominated by some vertex of B. The minimum taken over all boundary dominating sets of a graph G is called the boundary domination number of G and is denoted by $\gamma_b(G)$.

The concept energy of a graph was introduced by I. Gutman [6] in the year 1978. Let

$A(G) = (a_{ij})$ be the adjacency matrix of G. The eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of a matrix $A(G)$, assumed in nonincreasing order, are the eigenvalues of the graph G.
As A is real symmetric, the eigenvalues of G are real with sum equal to zero. The energy $E(G)$ of G is defined to be the sum of the absolute values of the eigenvalues of G, i.e. $E(G) = \sum_{i=1}^{n} |\lambda_i|$. For more details on the mathematical aspects of the theory of graph energy we refer to[7] .
I. Gutman and B. Zhou [8] was defined the Laplacian energy of a graph G in the year 2006 . The Laplacian matrix of the graph G, denoted by $L = (L_{ij})$, is a square matrix of order n whose elements are defined as

$$L_{ij} = \begin{cases} -1 & \text{if } v_i \text{ and } v_j \text{, are adjacent,} \\ 0 & \text{if } v_i \text{ and } v_j \text{, are not adjacent,} \\ d_i & \text{if } i = j. \end{cases}$$

Where d_i is the degree of vertex v_i.
M.R. Rajesh Kanna and G. Sridhara [12] was computed the Laplacian minimum dominating energy of some standard graphs. Let $D(G)$ be the diagonal matrix of vertex degrees of the graph G and let $\omega_1, \omega_2, ..., \omega_n$ be the Laplacian eigenvalues of $LE(G)$. Laplacian energy $LE(G)$ of G is defined as

$$LE(G) = \sum_{i=1}^{n} |\omega_i - \frac{2m}{n}|.$$

Motivated by this paper, we introduce Laplacian minimum boundary dominating energy, denoted by $LE_B(G)$, of a graph G, and computed the Laplacian minimum boundary dominating energies of some standard graphs. Upper and lower bounds for $LE_B(G)$ are established. It is possible that the Laplacian minimum boundary dominating energy that we are considering in this paper may be have some applications in chemistry as well as in other areas. We need the following theorems.

Theorem 1.1. [12]

(a) For $n \geq 2$, the Laplacian minimum dominating energy of Complete graph K_n is $(n-2) + \sqrt{n^2 - 2n + 5}$.
(b) For $n \geq 2$, the Laplacian minimum dominating energy of Star graph $K_{1,n-1}$ is equal to $\frac{(n-2)^2}{n} + \sqrt{n^2 - 2n + 5}$.

2. Main Result

2.1. **The MBD Energy of Graphs.** Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set E. A subset B of $V(G)$ is called a boundary dominating set if every vertex of $V - B$ is boundary dominated by some vertex of B. The boundary domination number $\gamma_b(G)$ of G is the minimum cardinality of a boundary dominating set. Any boundary dominating set with minimum cardinality is called a MBD set. Let B be a MBD set of a graph G. The MBD matrix of G is the $n \times n$ matrix defined by $A_B(G) = a_{ij}$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_j \in N_b(v_i), \\ 1 & \text{if } i = j \text{ and } v_i \in B, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $A_B(G)$ is denoted by

$$f_n(G, \lambda) = \det(\lambda I - A_B(G))$$

The MBD eigenvalues of the graph G are the eigenvalues of $A_B(G)$. Since $A_B(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. The MBD energy of G is defined as $E_B(G) = \sum_{i=1}^{n} |\lambda_i|$.

2.2. **The LMBD Energy of Graphs.** Let $J_B(G)$ be the diagonal matrix of boundary vertex degrees of the graph G. Then $L_B(G) = J_B(G) - A_B(G)$ is called the Laplacian minimum boundary dominating (LMBD) matrix of G . Let $\omega_1, \omega_2, ..., \omega_n$ be the eigenvalues of $L_B(G)$, arranged in non-increasing order. These eigenvalues are called LMBD eigenvalues of G. The LMBD energy of the graph G is defined as

$$LE_B(G) = \sum_{i=1}^{n} |\omega_i - \frac{\xi}{n}|,$$

where $\xi = \sum_{i=1}^{n} deg_b(v_i)$, $deg_b(v)$ is the boundary degree of v and $\frac{\xi}{n}$ is the average boundary degree of G.

In this paper, we interested in studying mathematical aspects of the LMBD energy of a graph. It is possible that the LMBD energy that we are considering in this paper may have some applications in chemistry as well as in other areas. We first compute the LMBD energy of a graph in Figure 1.

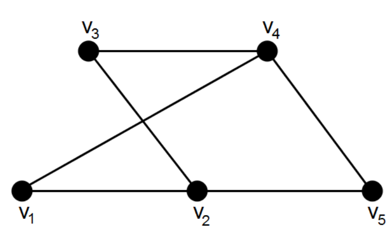

FIGURE 1. G

Example 2.1. *Let G be a graph in fig 1 with vertices set $\{v_1, v_2, v_3, v_4, v_5\}$ and let its Boundary Dominating set be $B_1 = \{v_1, v_4\}$. Then*

$$A_{B_1}(G) = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } J_B(G) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$L_{B_1}(G) = J_B(G) - A_{B_1}(G) = \begin{pmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 2 \end{pmatrix}.$$

Characteristic equation

$$f_n(G, \omega) = \omega^5 - 6\omega^4 + 9\omega^3 + 3\omega^2 - 8\omega - 3 = 0.$$

Hence, the LMBD eigenvalues are
$\omega_1 \approx 3.0000, \omega_2 \approx 2.4142, \omega_3 \approx 1.6180, \omega_4 \approx -0.6180, \omega_5 \approx -0.4142.$
Average boundary degree $= \frac{\xi}{n} = \frac{8}{5}$.
Therefore the LMBD energy of G is

$$LE_{B_1}(G) \approx 6.4644.$$

If we take another Boundary Dominating set in G, namely $B_2 = \{v_2, v_5\}$, then

$$A_{B_2}(G) = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

,

$$L_{B_2}(G) = J_B(G) - A_{B_2}(G) = \begin{pmatrix} 2 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

$$f_n(G, \omega) = \omega^5 - 6\omega^4 + 10\omega^3 - \omega^2 - 6\omega = 0.$$

Hence,the LMBD eigenvalues are
$\omega_1 \approx 3.0000, \omega_2 \approx 2.0000, \omega_3 \approx 1.6180, \omega_1 \approx -0.6180, \omega_5 \approx 0.0000$.
Average boundary degree $= \frac{\xi}{n} = \frac{8}{5}$.
Therefore the LMBD energy of G is

$$LE_{B_2}(G) \approx 4.036.$$

This example illustrates the fact that the LMBD energy of a graph G depends on the choice of the boundary dominating set. i.e. the LMBD energy is not a graph invariant.

3. Properties of LMBD Eigen Values of a Graph

Theorem 3.1. *Let B be a MBD set of a graph G, $\xi = \sum_{i=1}^{n} deg_b(v_i)$ and $\omega_1, \omega_2, ..., \omega_n$ are the eigenvalues of $L_B(G)$, then*
(i) $\sum_{i=1}^{n} \omega_i = \xi - |B|$.
(ii) $\sum_{i=1}^{n} \omega_i^2 = \xi + \sum_{i=1}^{n} (deg_b(v_i) - c_i)^2$

Proof.
(i) By definition, the sum of the principal diagonal elements of $L_B(G)$ is equal to $\sum_{i=1}^{n} deg_b(v_i) - |B| = \xi - |B|$. Also the sum of the eigenvalues of $L_B(G)$ is the trace of $L_B(G)$, it follows that

$$\sum_{i=1}^{n} \omega_i = \xi - |B|.$$

(ii) Similarly the sum of squares of the eigenvalues of $L_B(G)$ is the trace of $(L_B(G))^2$. Then

$$\begin{aligned}
\sum_{i=1}^{n} \omega_i^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} l_{ij} l_{ji} \\
&= \sum_{i=1}^{n} (l_{ii})^2 + \sum_{i \neq j}^{n} l_{ij} l_{ji} \\
&= 2 \sum_{i<j}^{n} (l_{ij})^2 + \sum_{i=1}^{n} (l_{ii})^2 \\
&= \xi + \sum_{i=1}^{n} (deg_b(v_i) - c_i)^2. \qquad where \qquad c_i = \begin{cases} 1 & \text{if } v_i \in B \\ 0 & \text{if } v_i \notin B \end{cases} \\
&= M
\end{aligned}$$

□

Theorem 3.2. *Let G be a graph of order n and size m and let $\omega_1(G)$ be the largest eigenvalue of $A_B(G)$. Then*

$$\omega_1(G) \geq \frac{\xi + \gamma_b}{n}$$

.

Proof. Let G be a graph of order n and let ω_1 be the largest minimum boundary eigenvalue of $A_B(G)$. Then from [3] we have $\omega_1 = \max_{X \neq 0}\{\frac{X^t L_B X}{X^t X}\}$, where X is any nonzero vector and X^t is its transpose and L_B is a matrix . If we tack $X = I = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$. Then we have

$$\omega_1 \geq \frac{I^t L_B I}{I^t I} = \frac{\xi + \gamma_b}{n}.$$

□

Theorem 3.3. *Let G be a graph with a MBD set B. If the the absolute values of LMBD eigenvalues is a rational number, then $\sum_{i=1}^{n} |\omega_i| \equiv \gamma_b(G)\ (mod2)$*

Proof. Let $\omega_1, \omega_2, ..., \omega_n$ are the eigenvalues of $L_B(G)$ of a graph G of which $\omega_1, \omega_2, ..., \omega_r$ are positive and the rest are non-positive, then

$$\begin{aligned}
\sum_{i=1}^{n} |\omega_i| &= (\omega_1 + \omega_2 + ... + \omega_r) - (\omega_{r+1} + \omega_{r+2} + ... + \omega_n) \\
&= 2(\omega_1 + \omega_2 + ... + \omega_r) - (\omega_1 + \omega_2 + ... + \omega_n) \\
&= 2(\omega_1 + \omega_2 + ... + \omega_r) - \sum_{i=1}^{n} \omega_i \\
&= 2(\omega_1 + \omega_2 + ... + \omega_r) - (\xi - |B|) \\
&= 2(\omega_1 + \omega_2 + ... + \omega_r - \frac{1}{2}\xi) + |B|
\end{aligned}$$

Therefore

$$\sum_{i=1}^{n} |\omega_i| \equiv \gamma_b(G)(mod2)$$

Hence the theorem holds true.

□

4. LMBD Energy of Some Standard Graphs

In this section, we investigate the exact values of the LMBD energy of some standard graphs.

Theorem 4.1. *For* $n \geq 2$, $LE_B(K_n) = (n-2) + \sqrt{n^2 - 2n + 5}$.

Proof. Since $\gamma_b(K_n) = \gamma(K_n) = 1$ and $\frac{\xi}{n} = \frac{2m}{n} = n - 1$, then from Theorem 1.1.(a) we get $LE_B(K_n) = LE(K_n) = (n-2) + \sqrt{n^2 - 2n + 5}$. □

Theorem 4.2. *For* $n \geq 2$, $LE_B(K_{1,n}) = \frac{(n-2)^2}{n} + \sqrt{n^2 - 2n + 5}$.

Proof. Since $\gamma_b(K_{1,n}) = \gamma(K_{1,n}) = 1$ and $\frac{\xi}{n} = \frac{2m}{n} = \frac{2(n-1)}{n}$, then from Theorem 1.1.(b) we get $LE_B(K_{1,n}) = LE(K_{1,n}) = \frac{(n-2)^2}{n} + \sqrt{n^2 - 2n + 5}$. □

Theorem 4.3. *For* $r \geq 2$, $LE_B(K_{r,r}) = (r-1) + 2\sqrt{r^2 - 2r + 5}$.

Proof. For the complete bipartite graph $K_{r,r}$, $(r \geq 2)$ with vertex set $V = \{v_1, v_2, ..., v_r, u_1, u_2, ..., u_r\}$, $\gamma_b = 2$, hence the MBD set is $B = \{v_1, u_1\}$. Then

$$A_B(K_{r,r}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 0 \end{pmatrix}_{(2r)\times(2r)}$$

,

$$J(K_{r,r}) = \begin{pmatrix} r-1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & r-1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & r-1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r-1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & r-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & r-1 \end{pmatrix}_{(2r)\times(2r)}$$

and $L_B(K_{r,r}) = J(K_{r,r}) - A_B(K_{r,r})$

$$= \begin{pmatrix} r-2 & -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ -1 & r-1 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & r-1 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & r-1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & r-1 & -1 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & r-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 & -1 & \cdots & r-1 \end{pmatrix}_{(2r)\times(2r)}$$

$$f_n(K_{r,r}, \omega) = (\omega - r)^{2r-4}(\omega^2 - (r-1)\omega - 1)^2 = 0.$$

LMBD eigenvalues are $\omega = r$ and $\omega = \frac{(r-1)\pm\sqrt{r^2-2r+5}}{2}$.
Since $\gamma_b(K_{r,r}) = 2$, then $\frac{\xi}{n} = \frac{n(r-1)}{n} = r - 1$. Where $n = 2r$
Hence, the LMBD energy, $LE_B(K_{r,r})$

$$\begin{aligned} &= |r-(r-1)|(2r-4) + 2\left|\frac{(r-1)+\sqrt{r^2-2r+5}}{2} - (r-1)\right| \\ &+ 2\left|\frac{(r-1)-\sqrt{r^2-2r+5}}{2} - (r-1)\right| \\ &= (2r-4) + 2\left|\frac{-r+1+\sqrt{n^2-2n+5}}{2}\right| + 2\left|\frac{-r+1-\sqrt{n^2-2n+5}}{2}\right| \\ &= (r-1) + 2\sqrt{r^2-2r+5}. \end{aligned}$$

□

Definition 4.4. *The double star graph $S_{n,m}$ is the graph constructed from union $K_{1,n-1}$ and $K_{1,m-1}$ by join whose centers v_0 with u_0. Then $V(S_{n,m}) = V(K_{1,n-1}) \cup V(K_{1,m-1}) = \{v_0, v_1, ..., v_{n-1}, u_0, u_1, ..., u_{m-1}\}$ and $E(S_{n,m}) = \{v_0u_0, v_0v_i, u_0u_j; 1 \leq i \leq n-1, 1 \leq j \leq m-1\}$.*

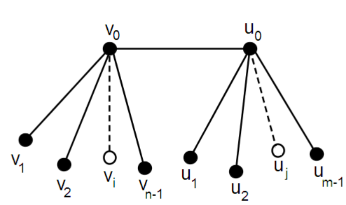

FIGURE 2. Double Star Graph $S_{n,m}$

Theorem 4.5. *For any double star graph $S_{r,r} (r \geq 3)$,*

$$LE_B(S_{r,r}) = (r-1) + 2\sqrt{r^2 - 2r + 5}.$$

Proof. The proof is similar to the proof of Theorem 4.3. □

Definition 4.6. *The gear graph is a wheel graph with vertices added between pair of vertices of the outer cycle. The gear graph G_n has $2n+1$ vertices and $3n$ edges. Let $V(G_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v\}$ and $E(G_n) = \{e_i = v_i u_i, 1 \leq i \leq n\} \cup \{e_i^{'} = v_i v, 1 \leq i \leq n\} \cup \{e_i^{''} = u_i v_{i+1}, 1 \leq i \leq n$, subscripts modulo $n\}$, where v is an external vertex adjacent to every other vertex v_i for $1 \leq i \leq n$.*

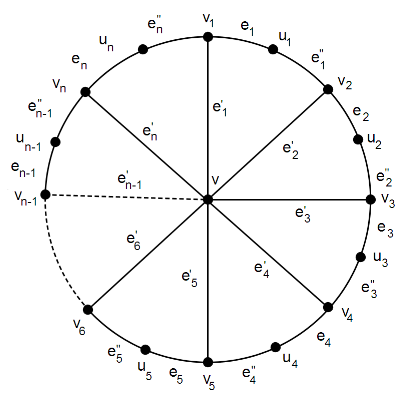

FIGURE 3. G_n

Theorem 4.7. *For any gear graph G_n, $(n \geq 2)$,*

$$LE_B(G_n) = \frac{4n^2 - 4n - 1}{2n+1} + \sqrt{n^2+4} + \sqrt{n^2 - 2n + 5}.$$

Proof. For the gear graph G_n, $(n \geq 2)$ with vertex set $V = \{v, v_1, v_2, ..., v_n, u_1, u_2, ..., u_n\}$, $\gamma_b = 2$, hence the MBD set is $B = \{v, v_1\}$. Then

$$A_B(G_n) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 \end{pmatrix}_{(n)\times(n)}$$

,

$$J(G_n) = \begin{pmatrix} n-1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & n-2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n-2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & n-1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & n-1 \end{pmatrix}_{(n)\times(n)}$$

and $L_B(G_n) = J(G_n) - A_B(G_n)$

$$= \begin{pmatrix} n-2 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & n-3 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & n-2 & 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & n-1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & n-1 \end{pmatrix}_{(n)\times(n)}$$

$$f_n(G_n, \omega) = (\omega - n)^{n-2}(\omega - (n+1))^{n-1}(\omega^2 - (n-1)\omega - 1)(\omega^2 - n\omega - 1) = 0.$$

LMBD eigenvalues are
$\omega = n$, $\omega = n+1$, $\omega = \frac{(n-1)\pm\sqrt{n^2-2n+5}}{2}$ and $\omega = \frac{n\pm\sqrt{n^2+4}}{2}$
Since $\gamma_b(G_n) = 2$, then $\frac{\xi}{2n+1} = \frac{2n^2}{2n+1}$
Hence, the LMBD energy, $LE_B(G_n)$

$$= |n - \frac{2n^2}{2n+1}|(n-2) + |(n+1) - \frac{2n^2}{2n+1}|(n-1) + |\frac{n+\sqrt{n^2+4}}{2} - \frac{2n^2}{2n+1}|$$
$$+ |\frac{n-\sqrt{n^2+4}}{2} - \frac{2n^2}{2n+1}| + |\frac{(n-1)+\sqrt{n^2-2n+5}}{2} - \frac{2n^2}{2n+1}|$$
$$+ |\frac{(n-1)-\sqrt{n^2-2n+5}}{2} - \frac{2n^2}{2n+1}|$$
$$= \frac{n(n-2)}{2n+1} + \frac{(n-1)(3n+1)}{2n+1} + |\frac{-2n^2+n+(2n+1)\sqrt{n^2+4}}{2(2n+1)}|$$
$$+ |\frac{-2n^2+n-(2n+1)\sqrt{n^2+4}}{2(2n+1)}| + |\frac{-2n^2-n-1+(2n+1)\sqrt{n^2-2n+5}}{2(2n+1)}|$$
$$+ |\frac{-2n^2-n-1-(2n+1)\sqrt{n^2-2n+5}}{2(2n+1)}|$$
$$= \frac{4n^2-4n-1}{2n+1} + \sqrt{n^2+4} + \sqrt{n^2-2n+5}.$$

□

5. Bounds on LMBD Energy of Graphs

Theorem 5.1. *Let G be a connected graph of order n and $\xi = \sum_{i=1}^{n} deg_b(v_i)$. Then*

$$LE_B(G) \leq \sqrt{nM} + \xi.$$

Proof. From the Cauchy-Schwartiz inequality

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \left(\sum_{i=1}^{n} (a_i)^2 \right) \left(\sum_{i=1}^{n} (b_i)^2 \right)$$

Now, by setting $a_i = 1$ and $b_i = |\omega_i|$, then

$$\left(\sum_{i=1}^{n} |\omega_i| \right)^2 \leq \left(\sum_{i=1}^{n} 1 \right) \left(\sum_{i=1}^{n} (|\omega_i|)^2 \right) = nM.$$

and

$$\sum_{i=1}^{n} |\omega_i| \leq \sqrt{nM}$$

By Triangle inequality, $|\omega_i - \frac{\xi}{n}| \leq |\omega_i| + |\frac{\xi}{n}| \;\; \forall i = 1, 2, ..., n$

$$i.e, |\omega_i - \tfrac{\xi}{n}| \leq |\omega_i| + \tfrac{\xi}{n} \; \forall i$$

$$\Rightarrow \sum_{i=1}^{n} |\omega_i - \frac{\xi}{n}| \leq \sum_{i=1}^{n} |\omega_i| + \sum_{i=1}^{n} |\frac{\xi}{n}| \leq \sqrt{nM} + \xi$$

therefore

$$LE_B(G) \leq \sqrt{nM} + \xi.$$

□

Theorem 5.2. *Let G be a connected graph of order* n*,* $\xi = \sum_{i=1}^{n} deg_b(v_i)$ *and* B *is a minimum boundary dominating set of G. Then*

$$LE_B(G) \leq \sqrt{nM - \xi(\xi - \gamma_b)}.$$

Proof. Cauchy-Schwartiz inequality is

$$\Big(\sum_{i=1}^{n} a_i b_i \Big)^2 \leq \Big(\sum_{i=1}^{n} (a_i)^2 \Big) \Big(\sum_{i=1}^{n} (b_i)^2 \Big)$$

Put $a_i = 1$ and $b_i = |\omega_i - \frac{\xi}{n}|$ then

$$\Big(\sum_{i=1}^{n} |\omega_i - \tfrac{\xi}{n}| \Big)^2 \leq \Big(\sum_{i=1}^{n} 1 \Big) \Big(\sum_{i=1}^{n} (|\omega_i - \tfrac{\xi}{n}|)^2 \Big)$$

$$\begin{aligned} i.e., (LE_B(G))^2 &\leq n[\sum_{i=1}^{n} \omega_i^2 + \sum_{i=1}^{n} \frac{\xi^2}{n^2} - 2\sum_{i=1}^{n} \frac{\omega_i \xi}{n}] \\ &= n[M + \frac{\xi^2}{n} - 2\frac{\xi}{n}(\xi - |B|)] \\ &= n[M - \frac{\xi^2}{n} + \frac{\xi\gamma_b}{n}] \\ &= nM - \xi^2 + \xi\gamma_b \\ &= nM - \xi(\xi - \gamma_b). \end{aligned}$$

Therefore

$$LE_B(G) \leq \sqrt{nM - \xi(\xi - \gamma_b)}.$$

□

Theorem 5.3. *Let G be a connected graph of order* n *and* $\xi = \sum_{i=1}^{n} deg_b(v_i)$*.If* B *is the minimum boundary dominating set and* $P = |detL_B(G)|$*. Then*

$$LE_B(G) \geq \sqrt{M + n(n-1)P^{\frac{2}{n}} - \xi}$$

.

Proof. Consider

$$\begin{aligned} \Big(\sum_{i=1}^{n} |\omega_i| \Big)^2 &= \Big(\sum_{i=1}^{n} |\omega_i| \Big) . \Big(\sum_{j=1}^{n} |\omega_j| \Big) \\ &= \sum_{i=1}^{n} |\omega_i|^2 \sum_{i \neq j} |\omega_i||\omega_j| \end{aligned}$$

therefore

$$\sum_{i\neq j}|\omega_i||\omega_j| = \left(\sum_{i=1}^{n}|\omega_i| \right)^2 - \sum_{i=1}^{n}|\omega_i|^2 \tag{5.1}$$

Applying inequality between the arithmetic and geometric means for $n(n-1)$

$$\frac{\sum_{i\neq j}|\omega_i||\omega_j|}{n(n-1)} \geq \left[\prod_{i\neq j}|\omega_i||\omega_j|\right]^{\frac{1}{n(n-1)}}$$

$$i.e., \sum_{i\neq j}|\omega_i||\omega_j| \geq n(n-1)\left[\prod_{i\neq j}|\omega_i||\omega_j|\right]^{\frac{1}{n(n-1)}}$$

Using (4.1) we get,

$$\left(\sum_{i=1}^{n}|\omega_i| \right)^2 - \sum_{i=1}^{n}|\omega_i|^2 \geq n(n-1)\left[\prod_{i=1}^{n}|\omega_i|^{2(n-1)}\right]^{\frac{1}{n(n-1)}}$$

$$\left(\sum_{i=1}^{n}|\omega_i| \right)^2 \geq \sum_{i=1}^{n}|\omega_i|^2 + n(n-1)\left[\prod_{i=j}^{n}|\omega_i|^{2(n-1)}\right]^{\frac{1}{n(n-1)}}$$

$$= M + n(n-1)\left[\prod_{i=1}^{n}|\omega_i|\right]^{\frac{2}{n}}$$

therefore

$$\sum_{i=1}^{n}|\omega_i| \geq \sqrt{M + n(n-1)P^{\frac{2}{n}}} \tag{5.2}$$

We know that $|\omega_i| - |\frac{\xi}{n}| \leq |\omega_i - \frac{\xi}{n}| \;\; \forall i$

$$i.e, |\omega_i| - \frac{\xi}{n} \leq |\omega_i - \frac{\xi}{n}| \; \forall i$$

$$\sum_{i=1}^{n}|\omega_i| - \sum_{i=1}^{n}\frac{\xi}{n} \leq \sum_{i=1}^{n}|\omega_i - \frac{\xi}{n}|$$

$$i.e, \sum_{i=1}^{n}|\omega_i| - \xi \leq LE_B(G)$$

$$i.e, LE_B(G) \geq \sum_{i=1}^{n}|\omega_i| - \xi = \sqrt{M + n(n-1)P^{\frac{2}{n}}} - \xi \;\; \text{From (4.2)}$$

Hence

$$LE_B(G) \geq \sqrt{M + n(n-1)P^{\frac{2}{n}}} - \xi$$

. □

Theorem 5.4. *Let G be a connected graph of order n, $\xi = \sum_{i=1}^{n} deg_b(v_i)$ and B is MBD , If $\sum_{i=1}^{n} |\omega_i|$ is a rational number, then*

$$(\gamma_b + 2h - \xi) \leq LE_B(G) \leq (\gamma_b + 2h + \xi).$$

where h is any integer such that $\sum_{i=1}^{n} |\omega_i| \equiv \gamma_b(G)\ (mod 2)$.

Proof. Since

$$\sum_{i=1}^{n} |\omega_i - \frac{\xi}{n}| \leq \sum_{i=1}^{n} |\omega_i| + \xi$$

then

$$\begin{aligned} LE_B(G) &\leq \sum_{i=1}^{n} |\omega_i| + \xi \\ &= |B| + 2h + \xi \\ &= \gamma_b + 2h + \xi \end{aligned}$$

Also,

$$\begin{aligned} LE_B(G) &\geq \sum_{i=1}^{n} |\omega_i| - \xi \\ &= |B| + 2h - \xi \\ &= \gamma_b + 2h - \xi \end{aligned}$$

Hence

$$LE_B(G) \in (\gamma_b + 2h - \xi, \gamma_b + 2h + \xi).$$

□

References

[1] C.Adiga, A.Bayad, I.Gutman and S.A.Srinivas, The minimum covering energy of a graph, Kragujevac Journal of Science, 34(2012), 39-56.

[2] T. AleksiÂťc, Upper bounds for Laplacian energy of graphs. MATCH Commun. Math. Comput. Chem. 60 (2008) 435-439.

[3] R.B.Bapat and S.Pati, Energy of a graph is never an odd integer, Bulletin of Kerala Mathematics Association, 1(2011),129-132.

[4] N. N. M. de Abreu, C. T. M. Vinagre, A. S. BonifÂťacio, I.Gutman, The Laplacian energy of some Laplacian integral graphs. MATCH Commun. Math. Comput. Chem. 60 (2008), 447 - 460.

[5] G. H. Fath-Tabar, A. R. Ashrafi, I. Gutman, Note on Laplacian energy of graphs. Bull. Acad. Serbe Sci. Arts (Cl. Math. Natur.) 137 (2008), 1-10.

[6] I.Gutman, The energy of a graph, Ber. Math-Statist. Sekt. Forschungsz.Graz, 103 (1978), 1-22.

[7] I. Gutman, The Energy of a Graph: Old and New Results, ed. by A. Betten, A. Kohnert, R. Laue, A. Wassermann. Algebraic Combinatorics and Applications. (Springer, Berlin, 2001), pp. 196 - 211

[8] I. Gutman, B. Zhou, Laplacian energy of a graph. Lin. Algebra Appl. 414 (2006), 29 - 37.

[9] I. Gutman, N. M. M. de Abreu, C. T. M. Vinagre, A. S. BonifÂťacio, S.RadenkoviÂťc, Relation between energy and Laplacian energy. MATCH Commun. Math. Comput. Chem. 59 (2008), 343-354.

[10] J. Liu, B. Liu, On relation between energy and Laplacian energy. MATCH Commun. Math. Comput. Chem. 61 (2009), 403-406.

[11] KM. Kathiresan, G. Marimuthu and M. Sivanandha Saraswathy, Boundary domination in graphs, Kragujevac J. Math. 33 (2010), 63-70.

[12] M. R. Rajesh Kanna, B. N. Dharmendra, G. Sridhara, Laplacian Minimum Dominating Energy Of A Graph. International Journal of Pure and Applied Mathematics 89 (2013), No. 4, 565-581.

[13] M. Robbiano, R. JimÂťenez, Applications of a theorem by Ky Fan in the theory of Laplacian energy of graphs. MATCH Commun. Math. Comput. Chem. 62 (2009), 537-552.

[14] D. StevanoviÂťc, I. StankoviÂťc, M. MiloËĞseviÂťc, More on the relation between energy and Laplacian energy of graphs. MATCH Commun. Math. Comput. Chem. 61 (2009), 395-401.

[15] H. Wang, H. Hua, Note on Laplacian energy of graphs. MATCH Commun. Math. Comput. Chem. 59 (2008), 373-380.

[16] B. Zhou, New upper bounds for Laplacian energy. MATCH Commun. Math. Comput. Chem. 62 (2009), 553-560.

[17] B. Zhou, On the sum of powers of the Laplacian eigenvalues of graphs. Lin. Algebra Appl. 429 (2008), 2239-2246.

[18] B. Zhou, I. Gutman, On Laplacian energy of graphs. MATCH Commun. Math. Comput. Chem. 57 (2007), 211-220.