Asia Pacific Journal of Mathematics, Vol. 4, No. 1 (2017), 1-10 ISSN 2357-2205

SURFACES OF REVOLUTION IN THE THREE DIMENSIONAL SIMPLY ISOTROPIC SPACE $\mathbb{I}_3^1$

MURAT KEMAL KARACAN[1,*], DAE WON YOON[2] AND BAHADDIN BUKCU[3]

[1]Usak University, Faculty of Sciences and Arts, Department of Mathematics, 1 Eylul Campus, 64200, Usak-Turkey

[2]Department of Mathematics Education and RINS, Gyeongsang National University Jinju 660-701, South Korea

[3]Gazi Osman Pasa University, Faculty of Sciences and Arts, Department of Mathematics, 60250, Tokat-Turkey

[*]Corresponding author: murat.karacan@usak.edu.tr

Received Nov 7, 2016

Abstract. In this paper, we classify surfaces of revolution in the three dimensional simply isotropic space $\mathbb{I}_3^1$ satisfying some algebraic equations in terms of the coordinate functions and the Laplacian operators with respect to the first and the second fundamental forms of the surface. We also give explicit forms of these surfaces.

2010 Mathematics Subject Classification. 53A35, 53B30.

Key words and phrases. simply isotropic space; surfaces of revolution; Laplacian operator.

1. Introduction

Let $\mathbf{x} : \mathbf{M} \rightarrow \mathbb{E}^m$ be an isometric immersion of a connected n-dimensional manifold in the m-dimensional Euclidean space $\mathbb{E}^m$. Denote by $\mathbf{H}$ and Δ the mean curvature and the Laplacian of $\mathbf{M}$ with respect to the Riemannian metric on $\mathbf{M}$ induced from that of $\mathbb{E}^m$, respectively. Takahashi ([16]) proved that the submanifolds in $\mathbb{E}^m$ satisfying $\Delta\mathbf{x} = \boldsymbol{\lambda}\mathbf{x}$, that is, all coordinate functions are eigenfunctions of the Laplacian with the same eigenvalue $\boldsymbol{\lambda} \in \mathbb{R}$ are either the minimal submanifolds of $\mathbb{E}^m$ or the minimal submanifolds of hypersphere $\mathbb{S}^{m-1}$ in $\mathbb{E}^m$.

As an extension of Takahashi theorem, Garay studied in [11] hypersurfaces in $\mathbb{E}^m$ whose coordinate functions are eigenfunctions of the Laplacian, but not necessarily associated to the same eigenvalue. He considered hypersurfaces in $\mathbb{E}^m$ satisfying the condition

$$\Delta\mathbf{x} = \mathbf{A}\mathbf{x}, \tag{1.1}$$

where $\mathbf{A} \in Mat\ (m, \mathbb{R})$ is an $m \times m$- diagonal matrix, and proved that such hypersurfaces are minimal ($\mathbf{H} = 0$) in $\mathbb{E}^m$and open pieces of either round hyperspheres or generalized right spherical cylinders.

Related to this, Dillen, Pas and Verstraelen ([9]) investigated surfaces in $\mathbb{E}^3$ whose immersions satisfy the condition

$$\Delta \mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}, \tag{1.2}$$

where $\mathbf{A} \in Mat\ (3, \mathbb{R})$ is a 3×3-real matrix and $\mathbf{B} \in \mathbb{R}^3$. In other words, each coordinate function is of 1-type in the sense of Chen ([8]). For the Lorentzian version of surfaces satisfying (1.2), Alias, Ferrandez and Lucas ([1]) proved that the only such surfaces are minimal surfaces and open pieces of Lorentz circular cylinders, hyperbolic cylinders, Lorentz hyperbolic cylinders, hyperbolic spaces or pseudo-spheres.

The notion of an isometric immersion $\mathbf{x}$ is naturally extended to smooth functions on submanifolds of Euclidean space or pseudo-Euclidean space. The most natural one of them is the Gauss map of the submanifold. In particular, if the submanifold is a hypersurface, the Gauss map can be identified with the unit normal vector field to it. Dillen, Pas and Verstraelen ([10]) studied surfaces of revolution in the three dimensional Euclidean space $\mathbb{E}^3$ such that its Gauss map $\mathbf{G}$ satisfies the condition

$$\Delta \mathbf{G} = \mathbf{A}\mathbf{G}, \tag{1.3}$$

where $\mathbf{A} \in Mat\ (3, \mathbb{R})$. Baikoussis and Verstraelen ([4]) studied the helicoidal surfaces in $\mathbb{E}^3$. Choi ([6]) completely classified the surfaces of revolution satisfying the condition (1.3) in the three dimensional Minkowski space $\mathbb{E}^3_1$. The authors ([7,17]) classified surfaces of revolution satisfying (1.2) and (1.3) in the three dimensional Minkowski space and pseudo-Galilean space.

The main purpose of this paper is to complete classification of surfaces of revolution in the three dimensional simply isotropic space $\mathbb{I}^1_3$ in terms of the position vector field and the Laplacian operator.

2. Preliminaries

A simply isotropic space $\mathbb{I}^1_3$ is a Cayley–Klein space defined from the three dimensional projective space $\mathcal{P}(\mathbb{R}^3)$ with the absolute figure which is an ordered triple (w, f_1, f_2), where w is a plane in $\mathcal{P}(\mathbb{R}^3)$ and f_1, f_2 are two complex-conjugate straight lines in w. The homogeneous coordinates in $\mathcal{P}(\mathbb{R}^3)$ are introduced in such a way that the absolute plane w is given by $x_0 = 0$ and the absolute lines f_1, f_2 by $x_0 = x_1 + ix_2 = 0$, $x_0 = x_1 - ix_2 = 0$. The intersection point $\mathbb{F}(0 : 0 : 0 : 1)$ of these two lines is called the absolute point. The group of motions of the simply isotropic space is a six-parameter group given in the affine coordinates $x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}, z = \frac{x_3}{x_0}$ by

$$\begin{aligned} \overline{x} &= a + x\cos\theta - y\sin\theta \\ \overline{y} &= b + x\sin\theta + y\cos\theta \\ \overline{z} &= c + +c_1 x + c_2 y + z, \end{aligned} \tag{2.1}$$

where $a, b, c, c_1, c_2, \theta \in \mathbb{R}$. Such affine transformations are called isotropic congruence transformations or i-motions [12].

Isotropic geometry has different types of lines and planes with respect to the absolute figure. A line is called non-isotropic (resp. completely isotropic) if its point at infinity does not coincide (coincides) with the point $\mathbb{F}$. A plane is called non-isotropic (resp. isotropic) if its line at infinity does not contain $\mathbb{F}$. Completely isotropic lines and isotropic planes in this affine model appear as vertical, i.e., parallel to the z-axis. Finally, the metric of the simply isotropic space $\mathbb{I}_3^1$ is given by

$$ds^2 = dx^2 + dy^2.$$

A surface $\mathbf{M}$ immersed in $\mathbb{I}_3^1$ is called admissible if it has no isotropic tangent planes. For such a surface, the coefficients E, F, G of its first fundamental form are calculated with respect to the induced metric and the coefficients L, M, N of the second fundamental form, with respect to the normal vector field of a surface which is always completely isotropic. The (isotropic) Gaussian and mean curvature are defined by

$$\mathbf{K} = k_1 k_2 = \frac{LN - M^2}{EG - F^2}, \quad 2\mathbf{H} = k_1 + k_2 = \frac{EN - 2FM + GL}{EG - F^2}, \tag{2.2}$$

where k_1, k_2 are principal curvatures, i.e., extrema of the normal curvature determined by the normal section (in completely isotropic direction) of a surface. Since $EG - F^2 > 0$, for the function in the denominator we often put $W^2 = EG - F^2$.The surface $\mathbf{M}$ is said to be isotropic flat (resp. isotropic minimal) if $\mathbf{K}$ (resp.$\mathbf{H}$) vanishes [14].

It is well known in terms of local coordinates $\{u, v\}$ of $\mathbf{M}$ the Laplacian operators $\Delta^{\mathbf{I}}$, $\Delta^{\mathbf{II}}$ of the first and the second fundamental form on $\mathbf{M}$ are defined by ([5,13])

$$\Delta^{\mathbf{I}}\mathbf{x} = -\frac{1}{\sqrt{|EG - F^2|}} \left[\frac{\partial}{\partial u} \left(\frac{G\mathbf{x}_u - F\mathbf{x}_v}{\sqrt{|EG - F^2|}} \right) - \frac{\partial}{\partial v} \left(\frac{F\mathbf{x}_u - E\mathbf{x}_v}{\sqrt{|EG - F^2|}} \right) \right] \tag{2.3}$$

and

$$\Delta^{\mathbf{II}}\mathbf{x} = -\frac{1}{\sqrt{|LN - M^2|}} \left[\frac{\partial}{\partial u} \left(\frac{N\mathbf{x}_u - M\mathbf{x}_v}{\sqrt{|LN - M^2|}} \right) - \frac{\partial}{\partial v} \left(\frac{M\mathbf{x}_u - L\mathbf{x}_v}{\sqrt{|LN - M^2|}} \right) \right]. \tag{2.4}$$

3. Surfaces of Revolution in $\mathbb{I}_3^1$

Now we adapt the above notion to isotropic spaces. Considering the i-motions given by (2.1), the Euclidean rotations in the simply isotropic space $\mathbb{I}_3^1$ is given by the normal form (in affine coordinates)

$$\begin{aligned} \overline{x} &= x\cos\theta - y\sin\theta \\ \overline{y} &= x\sin\theta + y\cos\theta \\ \overline{z} &= z, \end{aligned} \tag{3.1}$$

where $\theta \in \mathbb{R}$ [2].

First of all, we consider a plane curve α parametrized by $\alpha(u) = (f(u), 0, g(u))$ or isotropic curve $\alpha(u) = (0, f(u), g(u))$ around the z-axis by Euclidean rotation (3.1), where g is a positive function and f is a smooth function on an open interval I. Then by $i-$motion, the surface $\mathbf{M}$ of revolution can be written as

$$\mathbf{x}(u,v) = (f(u)\cos v, f(u)\sin v, g(u)) \tag{3.2}$$

or

$$\mathbf{x}(u,v) = (-f(u)\sin v, f(u)\cos v, g(u)) \tag{3.3}$$

for any $v \in \mathbb{R}$ [2].

4. Surfaces of Revolution Satisfying $\Delta^{\mathbf{I}}\mathbf{x} = \mathbf{A}\mathbf{x}$

In this section, we classify surfaces of revolution given by (3.2) in $\mathbb{I}_3^1$ satisfying the equation

$$\Delta^{\mathbf{I}}\mathbf{x} = \mathbf{A}\mathbf{x}, \tag{4.1}$$

where $\mathbf{A} = (a_{ij}), i, j = 1, 2, 3$ and

$$\Delta^{\mathbf{I}}\mathbf{x} = \left(\Delta^{\mathbf{I}}\mathbf{x}_1, \Delta^{\mathbf{I}}\mathbf{x}_2, \Delta^{\mathbf{I}}\mathbf{x}_3\right),$$

where

$$\mathbf{x}_1 = f(u)\cos v,\ \mathbf{x}_2 = f(u)\sin v,\ \mathbf{x}_3 = g(u).$$

First of all, let $\mathbf{M}$ be a surface of revolution in $\mathbb{I}_3^1$ defined by (3.2). Assume that the rotated curve α is parametrized by arc-length. Then it is rewritten as the form:

$$\alpha(u) = (u, 0, g(u)),$$

where $\alpha'(u) \neq 0$. In this case, the parametrization of $\mathbf{M}$ is given by

$$\mathbf{x}(u,v) = (u\cos v, u\sin v, g(u)), \tag{4.2}$$

where g is a positive function. For this surface of revolution, the coefficients of the first and second fundamental form are

$$E = 1, F = 0, G = u^2, \tag{4.3}$$

$$L = g'', M = 0, N = ug'. \tag{4.4}$$

The Gaussian curvature $\mathbf{K}$ and the mean curvature $\mathbf{H}$ are

$$\mathbf{K} = \frac{g'(u)g''(u)}{u}, \quad \mathbf{H} = \frac{g'(u) + ug''(u)}{2u}, \tag{4.5}$$

where $u \neq 0$, respectively.

Proposition 1. *Surfaces of revolution given by (4.2) in the three dimensional simply isotropic space* $\mathbb{I}_3^1$ *are isotropic flat* ($\mathbf{K} = 0$), *iff* $g(u) = c_1 u + c_2$ *for some constants* c_1 *and* c_2.

Proposition 2. *Surfaces of revolution given by (4.2) in the three dimensional simply isotropic space* $\mathbb{I}_3^1$ *have isotropic constant mean curvature iff* $g(u) = c_1 \ln u + a\frac{u^2}{2} + c_2$, *where* a, c_1, c_2 *are constants. If* $a = 0$, *then surfaces of revolution are isotropic minimal.*

Proposition 3. *Surfaces of revolution given by (4.2) in the three dimensional simply isotropic space* $\mathbb{I}_3^1$ *have isotropic constant Gaussian curvature iff*

$$g(u) = \pm\frac{u\sqrt{a^2 + c_1}}{2} \pm \frac{\ln\left(\sqrt{a}u + \sqrt{au^2 + c_1}\right)}{2\sqrt{a}} + c_2,$$

where a, c_1, c_2 *are constant.*

Suppose that the surface has non zero Gaussian curvature, so $g'(u)g''(u) \neq 0, \forall u \in I$. By a straightforward computation, the Laplacian operator on $\mathbf{M}$ with the help of (4.3) and (2.3) turns out to be

$$\Delta^{\mathbf{I}}\mathbf{x} = \left(0, 0, -\frac{g'(u) + ug''(u)}{u}\right). \tag{4.6}$$

Suppose that $\mathbf{M}$ satisfies (4.1). Then from (4.2) and (4.3), we have

$$\begin{aligned} a_{11}u\cos v + a_{12}u\sin v + a_{13}g(u) &= 0 \\ a_{21}u\cos v + a_{22}u\sin v + a_{23}g(u) &= 0 \\ a_{31}u\cos v + a_{32}u\sin v + a_{33}g(u) &= -\frac{g' + ug''}{u} \end{aligned} \tag{4.7}$$

Since the functions $\cos v, \sin v$ and the constant function are linearly independent, by (4.7) we get $a_{11} = a_{12} = a_{13} = a_{21} = a_{22} = a_{23} = a_{31} = a_{32} = 0$, $a_{33} = \lambda$. Consequently the matrix $\mathbf{A}$ satisfies

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

and (4.7) is rewritten as the following:

$$\lambda g(u) = -\frac{g'(u) + ug''(u)}{u}. \tag{4.8}$$

From (4.8), we obtain

$$\lambda = -\frac{g'(u) + ug''(u)}{ug(u)}. \tag{4.9}$$

From (4.9), we discuss two cases according to λ.

Case 1: If $\lambda = 0$, then we easly get $\mathbf{A} = diag(0,0,0)$ and $g'(u) + ug''(u) = 0$, which implies the mean curvature $\mathbf{H}$ vanishes identically because of (4.5). Therefore, the surfaces of revolution are isotropic minimal.

Proposition 4. *Surface of revolution given by (4.2) in the three dimensional simply isotropic space $\mathbb{I}_3^1$ is harmonic iff the surface $\mathbf{M}$ is isotropic minimal.*

Case 2: If $\lambda \neq 0$, then we have $g(u) = c_1 P + c_2 Q$, where $P = BesselJ\ [0, u\sqrt{\lambda}]$, $Q = BesselY[0, u\sqrt{\lambda}]$. The functions $BesselJ\ [0, u\sqrt{\lambda}]$ and $Q = BesselY[0, u\sqrt{\lambda}]$ have numerical solutions according to constant λ and the parameter u . Thus the matrix $\mathbf{A}$ satisfies

$$\mathbf{A} = \lambda \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.10}$$

and the parametrization of $\mathbf{M}$ is given by

$$\mathbf{x}(u, v) = (u \cos v, u \sin v, c_1 P + c_2 Q) .$$

Suppose that the surfaces of revolution $\mathbf{M}$ given by (3.2) and (3.3) satisfies (4.1). Then, we have the following result:

Theorem 1. *(Classification). Let $\mathbf{M}$ be non harmonic admissible surface of revolution given by (3.2) in the three dimensional simply isotropic space $\mathbb{I}_3^1$. The surface $\mathbf{M}$ satisfies the condition $\Delta^{\mathbf{I}}\mathbf{x} = \mathbf{A}\mathbf{x}$, $\mathbf{A} = (\mathbf{a}_{ij}) \in Mat(3, R)$, then it is an open part of the following surface:*

$$\mathbf{x}(u, v) = (u \cos v, u \sin v, c_1 P + c_2 Q) ,$$

where $c_1, c_2 \in \mathbb{R}$.

5. Surfaces of Revolution Satisfying $\Delta^{\mathbf{II}}\mathbf{x} = \mathbf{A}\mathbf{x}$

In this section, we classify surfaces of revolution with non-degenerate second fundamental form in $\mathbb{I}_3^1$ satisfying the equation

$$\boldsymbol{\Delta}^{\mathbf{II}}\mathbf{x} = \mathbf{A}\mathbf{x}, \tag{5.1}$$

where $\mathbf{A} = (\mathbf{a}_{ij}) \in Mat(3, R)$ and

$$\Delta^{\mathbf{II}}\mathbf{x}_i = \left(\Delta^{\mathbf{II}}\mathbf{x}_1, \Delta^{\mathbf{II}}\mathbf{x}_2, \Delta^{\mathbf{II}}\mathbf{x}_3\right),$$

where

$$\mathbf{x}_1 = u\cos v,\ \mathbf{x}_2 = u\sin v,\ \mathbf{x}_3 = g(u).$$

In this case, the function g is non constant, g' and g'' are nonzero everywhere. By a straightforward computation, the Laplacian operator on $\mathbf{M}$ with the help of (4.2), (4.4) and (2.4) turns out to be

$$\Delta^{\mathbf{II}}\mathbf{x}_i = \begin{pmatrix} \frac{\cos v}{2}\left(\frac{1}{g'(u)} + \frac{-g''(u)+ug'''(u)}{u(g''(u))^2}\right), \\ \frac{\sin v}{2}\left(\frac{1}{g'(u)} + \frac{-g''(u)+ug'''(u)}{u(g''(u))^2}\right), \\ \left(-\frac{3}{2} + \frac{g'(u)(-g''(u)+ug'''(u))}{2u(g''(u))^2}\right) \end{pmatrix}. \tag{5.2}$$

Suppose that $\mathbf{M}$ satisfies (5.1). Then from (4.2) and (4.4), we have

$$\begin{aligned} a_{11}u\cos v + a_{12}u\sin v + a_{13}g(u) &= \frac{\cos v}{2}\left(\frac{1}{g'(u)} + \frac{-g''(u)+ug'''(u)}{u\left(g''(u)\right)^2}\right) \\ a_{21}u\cos v + a_{22}u\sin v + a_{23}g(u) &= \frac{\sin v}{2}\left(\frac{1}{g'(u)} + \frac{-g''(u)+ug'''(u)}{u\left(g''(u)\right)^2}\right) \\ a_{31}u\cos v + a_{32}u\sin v + a_{33}g(u) &= \left(-\frac{3}{2} + \frac{g'(u)\left(-g''(u)+ug'''(u)\right)}{2u\left(g''(u)\right)^2}\right). \end{aligned} \tag{5.3}$$

Since the functions $\cos v, \sin v$ and the constant function are linearly independent, by (5.3) we get $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$, $a_{11} = a_{22} = \lambda$. Consequently the matrix $\mathbf{A}$ satisfies

$$\mathbf{A} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

where $a_{33} = \mu$ and (5.3) is rewritten as the following:

$$\lambda u\cos v = \frac{\cos v}{2}\left(\frac{1}{g'(u)} + \frac{-g''(u)+ug'''(u)}{u\left(g''(u)\right)^2}\right), \tag{5.4}$$

$$\lambda u \sin v = \frac{\sin v}{2}\left(\frac{1}{g'(u)} + \frac{-g''(u) + ug'''(u)}{u\left(g''(u)\right)^2}\right), \tag{5.5}$$

$$\mu g(u) = \left(-\frac{3}{2} + \frac{g'(u)\left(-g''(u) + ug'''(u)\right)}{2u\left(g''(u)\right)^2}\right). \tag{5.6}$$

From (5.4), (5.5) and (5.6), we obtain

$$\begin{aligned} \lambda &= \frac{1}{2u}\left(\frac{1}{g'(u)} + \frac{-g''(u) + ug'''(u)}{u\left(g''(u)\right)^2}\right), \\ \mu &= \frac{1}{2g(u)}\left(-3 + \frac{g'(u)\left(-g''(u) + ug'''(u)\right)}{u\left(g''(u)\right)^2}\right). \end{aligned} \tag{5.7}$$

From (5.7), we have

$$\begin{aligned} \frac{-g''(u) + ug'''(u)}{u\left(g''(u)\right)^2} &= 2u\lambda + \frac{1}{g'(u)}, \\ \frac{-g''(u) + ug'''(u)}{u\left(g''(u)\right)^2} &= \frac{2\mu g(u)}{g'(u)} + \frac{3}{g'(u)}. \end{aligned} \tag{5.8}$$

Combining the first and the second equation of (5.8), we obtain

$$g'(u) - \frac{\mu}{\lambda u} g(u) = \frac{2}{\lambda u}, \tag{5.9}$$

where $\lambda \neq 0$. If we solve the above system of ordinary differential equations, we get

$$g(u) = -\frac{2}{\mu} + cu^{\frac{\mu}{\lambda}}, \tag{5.10}$$

where $\lambda \neq 0$, $\mu \neq 0$, $c \in \mathbb{R}$. Thus the matrix $\mathbf{A}$ satisfies

$$\mathbf{A} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}. \tag{5.11}$$

Then, the parametrization of $\mathbf{M}$ is given by

$$\mathbf{x}(u, v) = \left(u \cos v, u \sin v, -\frac{2}{\mu} + cu^{\frac{\mu}{\lambda}}\right),$$

where $\lambda \neq 0, \mu \neq 0$.

Theorem 2. *(Classification). Let $\mathbf{M}$ be a non $\mathbf{II}$-harmonic surface of revolution with non-degenerate second fundamental form given by (3.2) in the three dimensional simply Isotropic space $\mathbb{I}_3^1$. The surfaces $\mathbf{M}$ satisfies the condition $\Delta^{\mathbf{II}}\mathbf{x} = \mathbf{A}\mathbf{x}$, $\mathbf{A} \in Mat(3, R)$, then it is an open part of the following surface:*

$$\mathbf{x}(u, v) = \left(u \cos v, u \sin v, -\frac{2}{\mu} + cu^{\frac{\mu}{\lambda}}\right),$$

where λ, μ are non-zero constants and $c \in \mathbb{R}$.

We discuss two cases according to constants λ, μ.

Case 1: Let $\lambda \neq 0, \mu = 0$, from (5.9), we obtain

$$g'(u) = \frac{2}{\lambda u}. \tag{5.12}$$

Their general solutions are

$$g(u) = c_1 + \frac{2 \ln u}{\lambda},$$

where $c_1 \in \mathbb{R}$. Thus the matrix $\mathbf{A}$ satisfies

$$\mathbf{A} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.13}$$

Then, the parametrization of $\mathbf{M}$ is given by

$$\mathbf{x}(u, v) = \left(u \cos v, u \sin v, c_1 + \frac{2 \ln u}{\lambda} \right). \tag{5.14}$$

Case 2: Let $\lambda = 0, \mu \neq 0$, from (5.9), we obtain

$$g(u) = -\frac{2}{\mu}. \tag{5.15}$$

Thus the matrix $\mathbf{A}$ satisfies

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu \end{bmatrix}. \tag{5.16}$$

Then, the parametrization of $\mathbf{M}$ is given by

$$\mathbf{x}(u, v) = \left(u \cos v, u \sin v, -\frac{2}{\mu} \right). \tag{5.17}$$

Theorem 3. *(Classification). Let* $\mathbf{M}$ *be a non* $\mathbf{II}$*-harmonic surface of revolution with non-degenerate second fundamental form given by (3.2) in the three dimensional simply Isotropic space* $\mathbb{I}_3^1$*. The surfaces* $\mathbf{M}$ *satisfies the condition* $\Delta^{\mathbf{II}}\mathbf{x} = \mathbf{A}\mathbf{x}$*,* $\mathbf{A} \in Mat(3, R)$*, then it is an open part of the following surface or plane :*

$$\mathbf{x}(u, v) = \left(u \cos v, u \sin v, c_1 + \frac{2 \ln u}{\lambda} \right),$$

$$\mathbf{x}(u, v) = \left(u \cos v, u \sin v, -\frac{2}{\mu} \right).$$

Definition 1. *A surface of in the three dimensional simple isotropic space is said to be* $\mathbf{II}$*-harmonic if it satisfies the condition* $\Delta^{\mathbf{II}}\mathbf{x} = \mathbf{0}$.

Proposition 5. *There is no* **II**$-$*harmonic surface of revolution in the three dimensional simply Isotropic space* $\mathbb{I}_3^1$.

Acknowledgement 1. *The second author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF)funded by the Ministry of Education (2015R1D1A1A01060046).*

References

[1] L. J. Alias, A. Ferrandez and P. Lucas, Surfaces in the 3-dimensional Lorentz-Minkowski space satisfying $\Delta\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}$, Pac. J. Math. 156 (1992), 201–208.

[2] M.E.Aydin, Classification results on surfaces in the isotropic 3-space, arXiv:1601.03190 [math.DG]

[3] M.E.Aydin, A generalization of translation surfaces with constant curvature in the isotropic space, J. Geom, 107(3)(2016), 603-615.

[4] C.Baikoussis and L. Verstraelen, On the Gauss map of helicoidal surfaces, Rend. Sem. Math. Messina Ser. II 2(16) (1993), 31–42.

[5] M.Bekkar,H. Zoubir, Surfaces of Revolution in the 3-Dimensional Lorentz-Minkowski Space Satisfying $\Delta\mathbf{x}^i = \boldsymbol{\lambda}^i\mathbf{x}^i$,Int. J. Contemp. Math. Sciences, 3 (2008) no. 24, 1173 - 1185

[6] S.M.Choi, On the Gauss map of surfaces of revolution in a 3-dimensional Minkowski space, Tsukuba J. Math. 19 (1995), 351–367.

[7] M.Choi, Y. H.Kim and D.W.Yoon, Some classification of surfaces of revolution in Minkowski 3-space, J. Geom. 104 (2013), 85–106

[8] B.Y. Chen, A report on submanifold of finite type, Soochow J. Math. 22 (1996),117–337.

[9] F. Dillen, J. Pas and L. Vertraelen, On surfaces of finite type in Euclidean 3-space, Kodai Math. J. 13 (1990), 10–21.

[10] F. Dillen, J. Pas and L. Vertraelen, On the Gauss map of surfaces of revolution, Bull. Inst. Math. Acad. Sinica 18 (1990), 239–246.

[11] O. J. Garay, An extension of Takahashi's theorem, Geom. Dedicata 34 (1990), 105–112.

[12] H. Sachs, Isotrope Geometrie des Raumes, Vieweg Verlag, Braunschweig, 1990.

[13] B.Senoussi, M. Bekkar, Helicoidal surfaces with $\Delta^J\mathbf{r} = \mathbf{A}\mathbf{r}$ in 3-dimensional Euclidean space,Stud. Univ. Babe s-Bolyai Math. 60(2015), No. 3, 437-448

[14] Z.M. Sipus, Translation Surfaces of onstant curvatures in a simply isotropic space, Period Math. Hung. 68(2014), 160–175.

[15] K. Strubecker, Differentialgeometrie des isotropen Raumes III, Flachentheorie, Math. Zeitsch.48 (1942), 369-427.

[16] T. Takahashi, Minimal immersions of Riemannian manifolds, J. Math. Soc. Japan, 18 (1966), 380–385.

[17] D.W.Yoon, Surfaces of Revolution in the three dimensional pseudo-galilean space, Glasnik Matematicki, 48 (68) (2013), 415 – 428.