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THE k-DISTANCE NEIGHBORHOOD POLYNOMIAL OF SOME GRAPH OPERATIONS

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Abstract. The k-distance neighborhood polynomial (N_k-polynomial) of a graph G defined as $N_k(G,x) = \sum_{k=0}^{diam(G)} \left(\sum_{i=1}^{n} d_k(v_i) \right) x^k$, where $daim(G)$ is the diameter of G and $d_k(v) = |\{u \in V(G) : d(v,u) = k\}|$. N_k-polynomial of a graph G is the ordinary generating function for the number of k-distance neighborhood of the vertices of G. In this paper, Exact formulas of the N_k-polynomial for corona product and union of two graphs are presented. It is shown that for any graph G of order n size m and $diam(G) = 2$, $N_k(G,x) = n + 2mx + (n^2 - n - 2m)x^2$. Also, the N_k-polynomials of the complement graph $\overline{G}$ of some graph G are obtained.

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1. Introduction

In this paper, we consider finite simple graphs. A graph $G = (V,E)$ is a simple graph, if it having no loops no multiple and directed edges. As usual, we denote by $n = |V|$ and $m = |E|$ to the number of vertices and edges in a graph G, respectively, unless we refer otherwise. The distance $d(u,v)$ between any two vertices u and v of G is the length of a minimum path connecting them. For a vertex $v \in V(G)$ and a positive integer k, the open k-distance neighborhood of v in a graph G, denote $N_k(v/G)$, ($N_k(v)$, if no confuse), is $N_k(v/G) = \{u \in V(G) : d(u,v) = k\}$ and the k-distance degree of a vertex v, denote $d_k(v/G)$, ($d_k(v)$, if no confuse), is $d_k(v) = |N_k(v)|$. It is clearly that $d_1(v) = d(v)$. A graph G is called non-trivial if it has at least one edge. The complement of a graph G, denoted $\overline{G}$, is a graph with vertex set V and edge set $\overline{E}$, such that $e \in \overline{E}$, if and only if $e \notin E$. A graph G is self-complementary if G is isomorphic to its complement. $\overline{K_n}$ is the empty or total disconnected graph with n vertices, i.e., the graph with n vertices no two of which are adjacent.

A graph G is called d-regular graph if the degree $d_1(v)$ of each vertex v in G is equal to d. For a vertex v of G, the eccentricity $e(v) = \max\{d(v,u) : u \in V(G)\}$. The radius of G is $rad(G) = \min\{e(v) : v \in V(G)\}$ and the diameter of G is $diam(G) = \max\{e(v) : v \in V(G)\}$. For any terminology or notation not mention here, we refer the reader to the books [3, 7].

A graph can be characterized by a number (index), a matrix or a polynomial. The characterization of graphs by a single topological index is usually impossible. For example, it is possible to find infinite pairs of graphs with the same Wiener index [14]. On the other hand, it is possible to characterize graphs by matrices. A well-known example of such matrices is an adjacency matrix [7]. But the characterization of graphs by polynomials is a new branch of research in modern graph theory. For more details in the topological indices and polynomials of a graph the reader referred to [1, 2, 8, 9, 10, 11, 14, 13, 15, 16, 17], and the references therein.

Ahmed M. Naji and Soner N. D. [12], in (2016), have been introduced a new type of graph topological polynomial, based on distance and degree, called k-distance neighborhood polynomial of a graph. Which, for simplicity of notion, referred as N_k-polynomial and defined by

$$N_k(G,x) = \sum_{k=0}^{e(v_i)} \left(\sum_{i=1}^{n} |N_k(v_i)| \right) x^k$$

where $N_k(v) = \{u \in V(G) : d(v,u) = k\}$ and $e(v) = \max\{d(v,u) : u \in V(G)\}$. They have been obtained some basic properties of N_k-polynomial of graphs and they presented the exact formulas for the N_k-polynomial of some well-known graphs (namely, a path P_n, a cycle C_n, a complete graph K_n, a star graph K_n, a wheel W_n, a complete bipartite $K_{r,s}$ and a complete multipartite $K_{n_1,\ldots,n_r}$). They also established the N_k-polynomial for some graph operations namely cartesian product and join of graphs.

In this paper, we are extending the results of N_k-polynomial of graphs by present the exact formulas of the N_k-polynomial for corona product and union of graphs. Also, we compute the N_k-polynomial of the complement graph $\overline{G}$ of some graphs G. Since $N_k(v) = 0$, when $k > e(v)$ for every $v \in V(G)$. Then we will rewrite here the N_k-polynomial of a graph as following

$$N_k(G,x) = \sum_{k=0}^{diam(G)} (\sum_{i=0}^{n} d_k(v_i))x^k.$$

2. The N_k-polynomial of corona product of graphs

Definition 2.1. [6] *Let G and H be two graphs on $V(G)$ and $V(H)$ disjoint sets of n_1 and n_2 vertices, respectively. The corona $G \circ H$ of G and H is defined as the graph obtained by taking one copy of G and n_1 copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H.*

The corona of two graphs was first introduced by Frucht and Harary in [6]. It is clear from the definition of $G \circ H$ that $n = |V(G \circ H)| = n_1 + n_1 n_2$, $m = |E(G \circ H)| = m_1 + n_1(n_2 + m_2)$ and $diam(G \circ H) = diam(G) + 2$, where m_1 and m_2 are the sizes of G and H, respectively. The corona product $G \circ H$ of any two graphs G and H is (obviously) a non-commutative operation. In the following results, H^j, for $1 \leq j \leq n_1$, denotes the copy of a graph H which joining to a vertex v_j of a graph G.

Theorem 2.2. *Let G and H be connected graphs of orders n_1 and n_2 and sizes m_1 and m_2, respectively. Then*

$$N_k(G \circ H, x) = \left(1 + 2n_2 x + n_2^2 x^2\right) N_k(G, x) + \left(n_1 n_2 + 2n_1 m_2 x - n_1(n_2 + 2m_2)x^2\right).$$

Proof. Let G and H be connected graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively and let $N_k(G, x) = \sum_{k=0}^{D} b_k x^k$ be the N_k-polynomial of a graph G, where $D = diam(G)$. Then to compute the N_k-polynomial of $G \circ H$, it be must compute the coefficients of $N_k(G \circ H, x)$. Let $N_k(G \circ H, x) = \sum_{k=0}^{diam(G \circ H)} a_k x^k$. Then from the properties of N_k-polynomial of a graph we have, $a_0 = n_1 + n_1 n_2$ and $a_1 = m_1 + n_1(n_2 + m_2)$.
Since for every $v \in V(G \circ H)$ either $v \in V(G)$ or $v \in V(H^j)$, for some $1 \leq j \leq n_1$, it follows that for $0 \leq k \leq diam(G \circ H)$,

$$a_k = \sum_{v \in V(G \circ H)} d_k(v/G \circ H) = \sum_{v \in V(G)} d_k(v/G \circ H) + \sum_{j=1}^{n_1} \sum_{v \in V(H^j)} d_k(v/G \circ H).$$

Thus, $d_2(v/G \circ H) = d_2(v/G) + n_2 d_1(v/G)$, for every $v \in V(G)$.
and $d_2(v/G \circ H^j) = (n_2 - 1) - d_1(v/H^j) + d_1(v_j/G)$, for every $v \in H^j$ and $1 \leq j \leq n_1$.

Hence,

$$\begin{aligned}
a_2 &= \sum_{v\in V(G)} d_2(v/G\circ H) + \sum_{j=1}^{n_1}\sum_{v\in V(H^j)} d_2(v/G\circ H)\\
&= \sum_{v\in V(G)} \Big(d_2(v/G) + n_2 d_1(v/G)\Big) + \sum_{j=1}^{n_1}\sum_{v\in V(H^j)} \Big[(n_2-1) - d_1(v/H_j) + d_1(v_j/G)\Big]\\
&= \Big(b_2 + n_2 b_1\Big) + \Big(n_1 n_2(n_2-1) - 2n_1 m_2 + n_2 b_1\Big)\\
&= b_2 + 2n_2 b_1 + n_1 n_2^2 - n_1 n_2 - 2n_1 m_2.
\end{aligned}$$

Similarly, $d_3(v/G\circ H) = d_3(v/G) + n_2 d_2(v/G)$, for every $v \in V(G)$.
and $d_3(v/G\circ H^j) = d_2(v_j/G) + n_2 d_1(v_j/G)$, for every $v \in H^j$ and $1 \le j \le n_1$. Hence,

$$\begin{aligned}
a_3 &= \sum_{v\in V(G)} d_3(v/G\circ H) + \sum_{j=1}^{n_1}\sum_{v\in V(H^j)} d_3(v/G\circ H)\\
&= \sum_{v\in V(G)} \Big(d_3(v/G) + n_2 d_2(v/G)\Big) + \sum_{j=1}^{n_1}\sum_{v\in V(H^j)} \Big(d_2(v_j/G) + n_2 d_1(v_j/G)\Big)\\
&= (b_3 + n_2 b_2) + (n_2 b_2 + n_2^2 b_1)\\
&= b_3 + 2n_2 b_2 + n_2^2 b_1.
\end{aligned}$$

By continuous in the same process we obtain,
$a_k = b_k + 2n_2 b_{k-1} + n_2^2 b_{k-2}$, for every $3 \le k \le D$.
For $k \ge D+1$, $N_k(v) \cap V(G) = \phi$, for every $v \in V(G)$, thus
$d_{D+1}(v/G\circ H) = n_2 d_D(v/G)$, for every $v \in V(G)$.
and $d_{D+1}(v/G\circ H^j) = d_D(v_j/G) + n_2 d_{D-1}(v_j/G)$, for every $v \in H^j$ and $1 \le j \le n_1$.
Thus, $a_{D+1} = 2n_2 b_D + n_2^2 b_{D-1}$. Similarly, $d_{D+2}(v/G\circ H) = 0$, for every $v \in V(G)$.
and $d_{D+2}(v/G\circ H^j) = n_2 d_D(v_j/G)$, for every $v \in H^j$ and $1 \le j \le n_1$.
Thus, $a_{D+2} = n_2^2 b_D$. Hence,

$$\begin{aligned}
N_k(G\circ H, x) &= a_0 + a_1 x + a_2 x^2 + ... + a_D x^D + a_{D+1}x^{D+1} + a_{D+2}x^{D+2}\\
&= (n_1 + n_1 n_2) + (b_1 + 2n_1 n_2 + 2n_1 m_2)x + (b_2 + 2n_2 b_1 + n_1 n_2^2 - n_1 n_2 - 2n_1 m_2)x^2\\
&\quad + (b_3 + 2n_2 b_2 + n_2^2 b_1)x^3 + ... + (b_k + 2n_2 b_{k-1} + n_2^2 b_{k-2})x^k + ...+\\
&\quad (b_D + 2n_2 b_{D-1} + n_2^2 b_{D-2})x^D + (2n_2 b_D + n_2^2 b_{D-1})x^{D+1} + (n_2^2 b_D)x^{D+2}\\
&= (n_1 + b_1 x + ... + b_D x^D) + 2n_2 x(n_1 + b_1 x + ... + b_D x^D) + n_2^2 x^2(n_1 + b_1 x + ...\\
&\quad + b_D x^D) + n_1 n_2 + 2n_1 m_2 x - (n_1 n_2 + 2n_1 m_2)x^2\\
&= N_k(G,x) + 2n_2 x N_k(G,x) + n_2^2 x^2 N_k(G,x) + (n_1 n_2 + 2n_1 m_2 x - (n_1 n_2 + 2n_1 m_2)x^2)\\
&= (1 + 2n_2 x + n_2^2 x^2)N_k(G,x) + (n_1 n_2 + 2n_1 m_2 x - (n_1 n_2 + 2n_1 m_2)x^2).
\end{aligned}$$

□

Corollary 2.3. *let G be a graph of order n. Then*

$$N_k(G \circ K_1) = (1 + 2x + x^2)N_k(G, x) + (n - nx^2).$$

Corollary 2.4. *let G be a graph of order n. Then*

$$N_k(G \circ \overline{K_p}) = (1 + 2px + p^2x^2)N_k(G, x) + (np - npx^2).$$

3. The N_k-polynomial of some complement graphs

In this section, we investigate the N_k-polynomial of a complement graph $\overline{G}$ of a graph G. The following are some fundamental results which will be required for many of our arguments in this section and which are finding in [4].

Definition 3.1. *We say that the graph G has property X if for every edge xy, there exists a vertex z, such that z not adjacent to either x or y.*

Theorem 3.2. *(1) If G is disconnected graph, then G has property X.*
(2) If G is connected graph and $diam(G) \geq 4$, then G has property X.

Proposition 3.3. *If the graph G has property X, then $\overline{G}$ is connected graph with diameter equals to two.*

Also, we need the following results to prove the main results.

Lemma 3.4. *Let G be a graph of order n, size m and $diam(G) = 2$. Then*

$$N_k(G, x) = n + 2mx + (n^2 - n - 2m)x^2.$$

Proof. Let G be a graph of order n, size m and $diam(G) = 2$. Then by the properties of N_k-polynomial of G $deg(N_k(G, x)) = 2$, i.e., $N_k(G, x) = a_0 + a_1x + a_2x^2$. Again by the properties of N_k-polynomial, $a_0 = n$ and $a_1 = 2m$.
Since $diam(G) = 2$. Then $d_2(v/G) = d_1(v/\overline{G}) = (n - 1) - d(v/G)$, for every $v \in V(G)$, and hence

$$\begin{aligned} a_2 &= \sum_{v \in V(G)} d_2(v/G) \\ &= \sum_{v \in V(G)} \Big((n-1) - d_1(v/G)\Big) \\ &= n(n-1) - 2m = n^2 - n - 2m. \end{aligned}$$

Therefore, $N_k(G, x) = n + 2mx + (n^2 - n - 2m)x^2$. □

Theorem 3.5. *Let G be a graph with property X. Then*

$$N_k(\overline{G}, x) = n + (n^2 - n - 2m)x + 2mx^2.$$

Proof. Let G be a graph of order n size m and having property X. Then by Proposition 3.3, $diam(\overline{G}) = 2$. Let $\overline{m}$ be the size of $\overline{G}$. Then $\overline{m} = \frac{n(n-1)}{2} - m$. Hence, by Lemma 3.4,

$$\begin{aligned} N_k(\overline{G}, x) &= n + 2\overline{m}x + (n^2 - n - 2\overline{m})x^2 \\ &= n + (n^2 - n - 2m)x + 2mx^2. \end{aligned}$$

□

Corollary 3.6. *If G is a connected graph with $diam(G) \geq 4$, then*

$$N_k(\overline{G}, x) = n + (n^2 - n - 2m)x + 2mx^2.$$

Corollary 3.7. *If G is disconnected graph, then*

$$N_k(\overline{G}, x) = n + (n^2 - n - 2m)x + 2mx^2.$$

Corollary 3.8. *If G is a connected self-complement graph with $diam(G) \geq 2$, then*

$$N_k(G, x) = n + \frac{n(n-1)}{2}\left(x + x^2\right).$$

Since $diam(\overline{G}) = 2$, for any d-regular graph G with $diam(G) \geq 3$, see [3], then by Lemma 3.4, the following result is holding.

Proposition 3.9. *Let G be d-regular graph with $diam(G) \geq 3$. Then*

$$N_k(\overline{G}, x) = n + (n^2 - dn - n)x + dnx^2.$$

Lemma 3.10. [5] *A regular self-complementary graph G must have $4d+1$ vertices and degree $2d$ and $diam(G) = 2$.*

Theorem 3.11. *If G is a d-regular self-complementary graph, then*

$$N_k(G, x) = (2d + 1) + (2d^2 + d)(x + x^2).$$

Proof. Let G be d-regular self-complementary graph. Then by Lemma 3.10, $n = 4s + 1$, for some $s \geq 2$, $d = 2s$ and $diam(G) = 2$. Thus, $n = 2d + 1$ and by Corollary 3.8, $N_K(G, x) = n + \frac{n(n-1)}{2}(x + x^2)$. Therefore, $N_k(G, x) = (2d + 1) + (2d^2 + d)(x + x^2)$. □

4. THE N_k-POLYNOMIAL OF THE UNION OF GRAPHS

The distance $d(v,u)$ between any two vertices v and u in a graph G is the length of the shortest (v,u)-path, if v and u are adjacent or infinity otherwise. Since $d_k(v)_G = 0$ for every $k > diam(G)$ or $k > e(v)$, $e(v)$ is the eccentricity of a vertex v in G. Then, in this section, we consider for a graph G having $G_1, G_2, ..., G_r$ components, for $r \geq 2$, that $e(v)_G = e(v)_{G_i}$, such $v \in V(G_i)$ for some $1 \leq i \leq r$. and $diam(G) = \max\{diam(G_i) : 1 \leq i \leq r\}$. Then, under this consideration, we have the following results.

Theorem 4.1. *For any connected graphs G and H,*

$$N_k(G \cup H, x) = N_k(G, x) + N_k(H, x).$$

Proof. Let G and H be connected graphs of orders n_1 and n_2, respectively. Suppose, without loss of generality, that $diam(G) \leq diam(H)$. Then, by the consideration above, $diam(G \cup H) = diam(H)$. Since $n = |V(G \cup H)| = n_1 + n_2$ and

$$d_k(v/G \cup H) = \begin{cases} d_k(v/G), & \text{if } v \in V(G); \\ d_k(v/H), & \text{if } v \in V(H). \end{cases}$$

it follows that

$$\begin{aligned} N_k(G \cup H, x) &= \sum_{k=0}^{diam(G \cup H)} \Big(\sum_{i=0}^{n} d_k(v_i/G \cup H) \Big) x^k \\ &= \sum_{k=0}^{diam(G \cup H)} \Big(\sum_{v \in V(G)} d_k(v/G \cup H) + \sum_{v \in V(H)} d_k(v/G \cup H) \Big) x^k \\ &= \sum_{k=0}^{diam(G \cup H)} \Big(\sum_{v \in V(G)} d_k(v/G) + \sum_{v \in V(H)} d_k(v/H) \Big) x^k \\ &= \sum_{k=0}^{diam(G)} \Big(\sum_{v \in V(G)} d_k(v/G) + \sum_{v \in V(H)} d_k(v/H) \Big) x^k \end{aligned}$$

$$+\sum_{k=diam(G)+1}^{diam(H)}\Big(\sum_{v\in V(G)} d_k(v/G)+\sum_{v\in V(H)} d_k(v/H)\Big)x^k$$

$$=\sum_{k=0}^{diam(G)}\Big(\sum_{v\in V(G)} d_k(v/G)\Big)x^k+\sum_{k=0}^{diam(G)}\Big(\sum_{v\in V(H)} d_k(v/H)\Big)x^k$$

$$+0+\sum_{k=diam(G)+1}^{diam(H)}\Big(\sum_{v\in V(H)} d_k(v/H)\Big)x^k$$

$$=N_k(G,x)+\sum_{k=0}^{diam(G)}\Big(\sum_{v\in V(H)} d_k(v/H)\Big)x^k$$

$$+\sum_{k=diam(G)+1}^{diam(H)}\Big(\sum_{v\in V(H)} d_k(v/H)\Big)x^k$$

$$=N_k(G,x)+\sum_{k=0}^{diam(H)}\Big(\sum_{v\in V(H)} d_k(v/H)\Big)x^k$$

$$=N_k(G,x)+n_k(H,x).$$

□

In the following result, by mathematical induction, we generalize Theorem 4.1.

Theorem 4.2. *For $r\geq 2$, let $G_1, G_2, ..., G_r$ be connected graphs. Then*

$$N_k(\bigcup_{i=1}^{r} G_i, x)=\sum_{i=1}^{r} N_k(G_i,x).$$

Theorem 4.3. *For $n\geq 1$, $N_k(G,x)=n$ if and only if $G=\overline{K_n}$.*

Proof. If $G=\overline{K_n}$, for $n\geq 1$, then we can rewrite G as $G=\bigcup_{i=0}^{n} K_1$. Since $N_k(K_1,x)=1$, it follows that by Theorem 4.2, that $N_k(G,x)=\sum_{i=0}^{n} N_k(K_1,x)=n$.

Conversely, Let $N_k(G,x)=n$ and assume, to the contrary, that $G=\overline{K_n}$, for $n\geq 1$. Then G has at least an edge. Hence by the properties of N_k-polynomial of a graph, the coefficient of x not zero in $N_K(G,x)$, a contradiction. Therefore, $G=\overline{K_n}$. □

Theorem 4.4. *Let G be a graph with r isolated vertices and let H bee the induced subgraph of G that induced by the competent $V-I_0$, where I_0 is the set of isolated vertices. Then*

$$N_k(H,x)=N_k(G,x)-r.$$

Proof. Since we can rewrite G as $G = H \cup \overline{K_r}$, it follows by Theorems 4.2 and 4.3,

$$\begin{aligned} N_k(G,x) &= N_k(H \cup \overline{K_r}, x) \\ &= N_k(H,x) + N_k(\overline{K_r}, x) \\ &= N_k(H,x) + r. \end{aligned}$$

□

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