Asia Pacific Journal of Mathematics, Vol. 4, No. 1 (2017), 32-37 ISSN 2357-2205

COEFFICIENT ESTIMATES FOR BI-UNIVALENT SAKAGUCHI TYPE FUNCTIONS

S.P. VIJAYALAKSHMI[1,*], T.V. SUDHARSAN[2]

[1]Department of Mathematics, Ethiraj College, Chennai - 600 008, India

[2]Department of Mathematics, SIVET College, Chennai - 600 073, India

[*]Corresponding author: vijishreekanth@gmail.com

Received Feb 27, 2017

Abstract. The object of the present paper is to obtain Coefficient estimates for bi-univalent Sakaguchi type functions defined by subordination.

2010 Mathematics Subject Classification. 30C45, 30C50.

Key words and phrases. vertex degrees; univalent functions; bi-univalent functions, subordination; sakaguchi type functions; starlike functions.

1. Introduction

Let A be the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Also let S denote the subclass of all functions in A which are univalent in Δ (see [6] for details). It is well known that every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \quad (z \in \Delta) \text{ and}$$
$$f(f^{-1}(w)) = w, \quad (|w| < r_0(f), r_0(f) \geq \frac{1}{4}).$$

A function $f \in A$ is said to bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let Σ denote the class of bi-univalent functions defined in the unit disk Δ.

An analytic function f is subordinate to an analytic function g, written $f(z), \prec g(z)$ [6] provided there is an analytic function w defined on Δ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and

Minda [9] unified various well known subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1+\frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function ϕ with positive real part in the unit disk Δ, $\phi(0)=1$, $\phi'(0)>0$, ϕ maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minds starlike functions consists of functions $f \in A$ satisfying the subordination $\frac{zf'(z)}{f(z)} \prec \phi(z)$. Similarly, the class of Ma-Minda convex functions consists of functions $f \in A$ Satisfying the subordination $1+\frac{zf''(z)}{f'(z)} \prec \phi(z)$.

In the sequel, it is assumed that ϕ is an analytic function with positive real part in the unit disk Δ, with $\phi(0)=1$, $\phi'(0)>0$ and $\phi(\Delta)$ is symmetric with respect to the real axis. Such function has a series expansion of the form

$$\phi(z)=1+B_1z+B_2z^2+\cdots, \quad (B_1>0). \tag{1.2}$$

Lewin [8] investigated the class of bi-univalent functions and obtained the bound for the second coefficient. Several authors have investigatd similar problems in this direction (see [3, 10]). Brannan and Taha [4] considered certain sublasses of bi-univalent functions. They introduced bi-starlike functions and bi-convex functions and obtained estimates on the initial coefficients. Srivastava et al. [15] introduced and studied subclasses of bi-univalent functions and obtained bounds for the initial coefficients. Bounds for the initial coefficients of several classes of functions were also investigated in [1, 2, 14].

Denote by S_S^* the subclass of S consisting of functions given by (1.1) satisfying $Re\left[\frac{zf'(z)}{f(z)-f(-z)}\right] > 0$ for $z \in \Delta$. These functions introduced by Sakaguchi [13] are called functions starlike with respect to symmetric points.

Recently Frasin [7] introduced and studied a generalized Sakaguchi type class $S(\alpha, s, t)$ if it satisfies

$$Re\left\{\frac{(s-t)zf'(z)}{f(sz)-f(tz)}\right\} > \alpha \tag{1.3}$$

For some $0 \le \alpha < 1$, $s, t \in C$ with $s \ne t$ and for all $z \in \Delta$. Also denote by $T(\alpha, s, t)$ the subclass of A consisting of all functions $f(z)$ such that $zf'(z) \in S(\alpha, s, t)$. The class $S(\alpha, s, t)$ was introduced and studied by Owa et al. [11, 12]. If $t=-1$, the class $S(\alpha, 1, -1) \equiv S_S(\alpha)$ [13] is called Sakaguchi function of order α (see [5, 11]) where as $S_S(0)=S_S^*$ [13].

Note that $S(\alpha, 1, 0) \equiv S^*(\alpha)$ and $T(\alpha, 1, 0) \equiv C(\alpha)$ which are,respectively, the familiar classes of starlike functions of order α $(0 \le \alpha < 1)$ and convex functions of order α $(0 \le \alpha < 1)$.

In [16] investigated the classes $S_S^*(\phi, s, t)$ and $T(\phi, s, t)$ defined as follows.

Definition 1.1. *Let $\phi(z)=1+B_1z+B_2z^2+\cdots$ be univalent starlike function with respect to 1 which maps the unit disk Δ onto a region in the right half plane which is symmetric* *with respect to the real axis, and let $B_1 > 0$. The function $f \in A$ is in the class $S_s^*(\phi, s, t)$ if*

$$\left\{ \frac{(s-t)zf'(z)}{f(sz)-f(tz)} \right\} \prec \phi(z), \quad s \neq t.$$

Remark 1.1. *$T(\phi, s, t)$ denotes the subclass of A consisting functions $f(z)$ such that $zf'(z) \in S_S^*(\phi, s, t)$. Observe that $S_S^*(\phi, 1, 0) \equiv S_S^*(\phi)$ and $T(\phi(1, 0) \equiv C(\phi)$, which are the classes introduced and studied by Ma and Minda* [9]. *Also note that $S_S^*(\phi, 1, -1) \equiv S_S^*(\phi)$, Shanmugam et al.* [14].

In the present investigation we are going to obtain the estimates on the initial coefficients for bi-univalent sakaguchi type functions.

2. Main Results

Theorem 2.1. *If $f \in S_S^*(\phi, s, t)$ is given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{[3-s^2-st-t^2]B_1^2 + \{(B_1-B_2)(2-s-t)-(s+t)B_1^2\}(2-s-t)}}$$

$$|a_3| \leq \frac{B_1}{(3-s^2-st-t^2)} + \frac{B_1^2}{(2-s-t)^2}$$

Proof. Let $f \in S_S^*(\phi, s, t)$ and $g = f^{-1}$. Then there are analytic functions $u, v : \Delta \to: \Delta$, with $u(0) = v(0) = 0$, satisfying and

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} = \phi(u(z) \quad \text{and}$$

$$\frac{(s-t)wg'(w)}{g(sw)-g(tw)} = \phi(v(w)). \tag{2.1}$$

Define the functions p_1 and p_2 by

$$p_{1(z)} = \frac{1+u(z)}{1-u(z)} = 1 + c_1(z) + c_2z^2 + \cdots$$

and

$$p_{2(z)} = \frac{1+v(z)}{1-v(z)} = 1 + b_1(z) + b_2z^2 + \cdots$$

Or equivalently,

$$u(z) = \frac{p_1(z)-1)}{p_1(z)+1} = \frac{1}{2}\left[c_1z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots\right] \tag{2.2}$$

$$v(z) = \frac{p_2(z)-1)}{p_2(z)+1} = \frac{1}{2}\left[b_1z + \left(b_2 - \frac{b_1^2}{2}\right)z^2 + \cdots\right] \tag{2.3}$$

Then p_1 and p_2 are analytic in Δ with $p_1(0) = 1 = p_2(0)$. Since $u, v : \Delta \to \Delta$,the functions p_1 and p_2 have positive real part in Δ, and $|b_i| \leq 2$ and $|c_i| \leq 2$.

In view of (2.1) , (2.2) and (2.3), clearly

$$
\begin{aligned}
&\frac{(s-t)zf'(z)}{f(sz)-f(tz)} = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) \quad \text{and} \\
&\frac{(s-t)wg'(w)}{g(sw)-g(tw)} = \phi\left(\frac{p_2(w)-1}{p_2(w)+1}\right)
\end{aligned} \tag{2.4}
$$

using (2.2) and (2.3) together with (1.2), it is evident that

$$
\phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^2 + \cdots \tag{2.5}
$$

and

$$
\phi\left(\frac{p_2(w)-1}{p_2(w)+1}\right) = 1 + \frac{1}{2}B_1b_1w + \left(\frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2b_1^2\right)w^2 + \cdots \tag{2.6}
$$

Since $f \in \Sigma$ has the Maclaurin series given by (1.1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$
\begin{aligned}
g(w) &= f^{-1}(w) \\
&= w - a_2w^2 + (2a_2^2 - a_3)w^3 + \cdots
\end{aligned}
$$

Since

$$
\begin{aligned}
\frac{(s-t)zf'(z)}{f(sz)-f(tz)} = &(1 + 2a_2z + 3a_3z^2 + \cdots)\{1 - (s+t)a_2z - (s^2+st+t^2)a_3z^2 \\
&+(s+t)^2a_2^2z^2 + (s^2+st+t^2)^2a_3^2z^4 + \cdots)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{(s-t)wg'(w)}{g(sw)-g(tw)} = &(1 + 2a_2w + 3a_3w^2 + \cdots)\{1 - (s+t)a_2w - (s^2+st+t^2)a_3w^2 \\
&+(s+t)^2a_2^2w^2 + (s^2+st+t^2)^2a_3^2w^4 + \cdots)
\end{aligned}
$$

It follows from (2.4), (2.5) and (2.6) that

$$
a_2(2-s-t) = \frac{1}{2}B_1c_1 \tag{2.7}
$$

$$
a_3[3-(s^2+st+t^2)] = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2 + \frac{(s+t)B_1^2c_1^2}{4(2-s-t)} \tag{2.8}
$$

$$
-a_2(2-s-t) = \frac{1}{2}B_1b_1 \tag{2.9}
$$

$$
(2a_2^2 - a_3)[3-(s^2+st+t^2)] = \frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2b_1^2 + \frac{(s+t)B_1^2b_1^2}{4(2-s-t)} \tag{2.10}
$$

From (2.7) and (2.9),
it follows

$$c_1 = -b_1 \tag{2.11}$$

Now (2.8), (2.9), (2.10) and (2.11) yield

$$a_2^2 = \frac{B_1^{(}b_2 + c_2)}{[4\{[3 - s^2 - st - t^2]B_1^2 + \{(B_1 - B_2)(2 - s - t) - (s + t)B_1^2\}(2 - s - t)\}]}$$

Which, in view of the well-known inequalities that $|b_2| \leq 2$ and $|c_2| \leq 2$ for functions with positive real part, gives us the desired estimate on $|a_2|$.

By subtracting (2.10) from (2.8), further computations using (2.7) and (2.11) lead to

$$|a_3| \leq \frac{B_1}{(3 - s^2 - st - t^2)} + \frac{B_1^2}{(2 - s - t)^2}$$

□

Remark 2.1. *If* $f \in S_S^*(\phi, 1, -1)$ *then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{2B_1^2 + 4(B_1 - B_2)}}$$

and

$$|a_3| \leq \frac{B_1}{2} + \frac{B_1^2}{4}$$

Since $f(z) \in T(\phi, s, t)$ *if and only if* $zf'(z) \in S_S^*(\phi, s, t)$. *Proceeding on similar lines as in theorem 2.1 we obtain* $|a_2|$ *and* $|a_3|$ *for the function belonging to the class* $T(\phi, s, t)$ *which is stated below without proof.*

Theorem 2.2. *If* $f \in T(\phi, s, t)$ *is given by* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{3[3 - s^2 - st - t^2]B_1^2 + 4(2 - s - t)\{(B_1 - B_2)(2 - s - t) - B_1^2(s + t)\}}}$$

and

$$|a_3| \leq \frac{B_1}{3(3 - s^2 - t^2)} + \frac{B_1^2}{4(2 - s - t)^2}$$

Remark 2.2. *If* $f \in T(\phi, 1, -1)$ *then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{2(3B_1^2 - 8(B_2 - B_1))}}$$

and

$$|a_3| \leq \frac{B_1}{3} + \frac{B_1^2}{16}$$

References

[1] R.M. Ali, S.K. Lee, V. Ravichandran, and S. Supramaniam, The Fekete-Szego coefficient functional for transforms of analytic functions, Bull. Iranian Math. Soc., 35 (2009), no. 2, 119-142, 276.

[2] R.M. Ali, V. Ravichandran and N. Seenivasagan, Coefficient bounds for p-valent functions, Appl. Math. Comput., 187 (2007), no. 1, 35-46.

[3] D.A. Brannan, J. Clunie and W.E. Kirwan, Coefficient estimates for a class of star-like functions, Canad. J. Math., 22 (1970), 476-485.

[4] D.A. Brannan and T.S. Taha, On some classes of bi-univalent functions, Studia Univ. Babes-Bolyai Math., 31 (1986), no. 2, 70-77.

[5] N.E. Cho, O.S. Kwon and S. Owa, Certain subclasses of Sakaguchi functions, SEA Bull. Math., 17 (1993), 121-126.

[6] P.L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, 259, Springer, New York, 1983.

[7] B.A. Frasin, Coefficient inequalities for certain classes of Sakaguchi type functions, Int. J. Nonlinear Sci., 10(2) (2010), 206-211.

[8] M. Lewin, On a coefficient problem for bi- univalent functions, Proc. Amer. Math. Soc. 18 (1967), 63-68.

[9] W.C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in Proceedings of the Conference on Complex Analysis (Tianjin, 1992), 157-169, Conf. Proc. Lecture Notes Anal. I Int. Press, Cambridge, MA.

[10] E. Netanyahu, The minimal distance of the image boundary from theorigin and the second coefficient of a univalent function in jzj $<$ 1, Arch. Rational Mech. Anal., 32 (1969), 100-112.

[11] S. Owa, T. Sekine, and R. Yamakawa, Notes on Sakaguchi type functions, RIMS Kokyuroku, 1414 (2005), 76-82.

[12] S. Owa, T. Sekine, and R. Yamakawa, On Sakaguchi type functions, Appl Math. Comput., 187 (2007), 356-361.

[13] K. Sakaguchi, On a certain univalent mapping, J. Math. Soc. Japan, 11 (1959), 72-75.

[14] T.N. Shanmugham, C. Ramachandran, and V. Ravichandran, Fekete- Szego problem for a subclasses of starlike functions with respect to symmetric points, Bull. Korean Math. Soc., 43(3) (2006), 589-598.

[15] H.M. Srivastava, A.K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett., 23 (2010), no. 10, 1188-1192.

[16] Trilok Mathur and Ruchi Mathur, Fekete-Szego inequalities for generalized sakaguchi type functions, Proceedings of the World Congress on Engineering, WCE 2012, London, U.K., 2012