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TRANSFORMATIONS ASSOCIATED WITH QUADRUPLE HYPERGEOMETRIC FUNCTIONS OF EXTON AND SRIVASTAVA

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Abstract. In this paper, we obtain some new transformations relating quadruple hypergeometric function $F^{(4)}$ of Srivastava and quadruple hypergeometric functions $D_5,\ K_{12},\ K_{13}$ of Exton. Two correct forms of an erroneous transformation of Exton are also given.

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1. INTRODUCTION

During 1970-71, Srivastava gave the following quadruple hypergeometric function $F^{(4)}$ [22, p.70(2.5); 23, p.229(1.1); 24,pp.35-36(1.2), pp.39-40(2.3,2.4); 25,pp.147-148 (36-37), p.232(43-44)]

$$F^{(4)}\left[\begin{array}{lr} a :: b,c\ ;\ d,e\colon f,c;\ g,e; & \\ & x,y,z,t \\ h :: k,m;\ n,p\colon q,m;\ s,p; & \end{array}\right]$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a)_{i+j}(b)_i(d)_j(c)_i(e)_j\ x^iy^j}{(h)_{i+j}(k)_i(n)_j(m)_i(p)_j\ i!\ j!}\ F^{1:2;2}_{1:2;2}\left[\begin{array}{lr} a+i+j\ :\ f,c+i\ ;\ g,e+j; & \\ & z,t \\ h+i+j\ :\ q,m+i;\ s,p+j; & \end{array}\right]$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a)_{i+j}(b)_i(d)_j(c)_i(e)_j\ x^iy^j}{(h)_{i+j}(k)_i(n)_j(m)_i(p)_j\ i!\ j!}\sum_{\ell=0}^{\infty}\sum_{r=0}^{\infty}\frac{(a+i+j)_{\ell+r}(f)_\ell(c+i)_\ell(g)_r(e+j)_r\ z^\ell t^r}{(h+i+j)_{\ell+r}(q)_\ell(m+i)_\ell(s)_r(p+j)_r\ \ell!\ r!}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{r=0}^{\infty} \frac{(a)_{i+j+\ell+r}(c)_{i+\ell}(e)_{j+r}(b)_i(d)_j(f)_\ell(g)_r \; x^i y^j z^\ell t^r}{(h)_{i+j+\ell+r}(m)_{i+\ell}(p)_{j+r}(k)_i(n)_j(q)_\ell(s)_r \; i! \; j! \; \ell! \; r!} \tag{1.1}$$

where $F^{1:2;2}_{1:2;2}$ is Kampé de Fériet's double hypergeometric function in the notation of Srivastava and Panda[26,p.423(26); see also 27,p.23(1.2,1.3)].

Now we generalize (1.1) by increasing the number of numerator and denominator parameters. For the sake of convenience, we write in the following slightly modified notation different from (1.1)

$$F^{(4)}\left[\begin{array}{ll} (a_A)::(b_B);\,(d_D);\,(e_E);\,(g_G):\,(h_H);\,(d_D);\,(m_M);\,(g_G); & \\ & w,x,y,z \\ (n_N)::(p_P);\,(q_Q);\,(r_R)\,;\,(s_S):\,(t_T)\,;\,(q_Q)\,;\,(u_U)\,;\,(s_S)\,; & \end{array}\right]$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{v=0}^{\infty} \frac{[(a_A)]_{i+j+k+v}[(d_D)]_{i+k}[(g_G)]_{j+v}[(b_B)]_i[(e_E)]_j[(h_H)]_k[(m_M)]_v \; w^i x^j y^k z^v}{[(n_N)]_{i+j+k+v}[(q_Q)]_{i+k}[(s_S)]_{j+v}[(p_P)]_i[(r_R)]_j[(t_T)]_k[(u_U)]_v \; i! \; j! \; k! \; v!} \tag{1.2}$$

where the notation (a_A) denotes the array of A parameters given by $a_1, a_2, a_3, \cdots, a_A$ and Pochhammer's symbol $[(a_A)]_m$ is defined by

$$[(a_A)]_m = \prod_{n=1}^{A}\left\{\Big(a_n\Big)_m\right\}$$

where

$$(a_n)_m = \begin{cases} \frac{\Gamma(a_n+m)}{\Gamma(a_n)} \; ; \text{ if } a_n \neq 0,-1,-2,-3,\ldots \\ a_n(a_n+1)(a_n+2)\cdots(a_n+m-1); \text{ if } m = 1,2,3,\ldots \\ 1 \; ; \text{ if } m = 0 \end{cases}$$

with similar interpretation for others.

The quadruple hypergeometric function (1.2) of Srivastava is the unification and generalization of Exton's some triple and quadruple hypergeometric functions[5;11], some triple hypergeometric functions of Jain[13] and Saran[20;21], Lauricella's quadruple hypergeometric functions $F^{(4)}_A$, $F^{(4)}_B$, $F^{(4)}_C$, $F^{(4)}_D$[16,pp.113-114], Chandel's quadruple hypergeometric function ${}^{(2)}_{(1)}E^{(4)}_C$[2,p.120(2.3); see also 3], Carlson's function of four variables R[1,p.453(2.1)], Karlsson's generalized Kampé de Fériet function of four variables $F^{A:B}_{C:D}$[12,p.108(3.7.3); 14,p.265(1)], Exton's quadruple hypergeometric functions ${}^{(2)}_{(1)}E^{(4)}_D$, ${}^{(2)}_{(2)}E^{(4)}_D$[6; see also 12, p.89 (3.4.1,3.4.2)], K_5, K_9, K_{10}, K_{12}, K_{13}, K_{20}, K_{21}[7 and 8; see also 12,pp.78-79], Karlsson's quadruple hypergeometric function ${}^{(k)}F^{(4)}_{CD}$[15,p.212(1.1) with suitable values of k], quadruple hypergeometric functions ${}^{(k)}F^{(4)}_{AC}$, ${}^{(k)}F^{(4)}_{AD}$, ${}^{(k)}F^{(4)}_{BD}$ of Chandel and Gupta[4,equations(1.4, 1.5, 1.6) with suitable values of k].

During 1971-73, Exton[12,p.89(3.3.4.7); see also 9;10] gave the following quadruple hypergeometric function

$$D_5[a,b,c,d,e,f;\ x,y,z,t] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(a)_m(b)_n(c)_p(d)_q(e)_{p+q-m-n}(f)_{m+n-p-q}\ x^m y^n z^p t^q}{m!\ n!\ p!\ q!} \tag{1.3}$$

which is the generalization of Pandey's function G_B[17,pp.115-116].

During 1972-73, Exton[12,p.78(3.3.12,3.3.13)] defined the following two quadruple hypergeometric functions

$$ {}^{(2)}_{(1)}E^{(4)}_{D} : K_{12}[a,a,a,a;\ b,c,d,e;\ f,f,g,g;\ x,y,z,t]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(a)_{m+n+p+q}(b)_m(c)_n(d)_p(e)_q\ x^m y^n z^p t^q}{(f)_{m+n}(g)_{p+q}\ m!\ n!\ p!\ q!} \tag{1.4}$$

which is the generalization of Saran's triple hypergeometric function F_G[20;21; see also 25,p.67(28)]. In another notation of Exton[12,pp.90-91], K_{12} is also denoted by ${}^{(2)}_{(1)}E^{(4)}_{D}$.

$$K_{13}[a,a,a,a;\ b,c,d,e;\ f,f,g,h;\ x,y,z,t]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(a)_{m+n+p+q}(b)_m(c)_n(d)_p(e)_q\ x^m y^n z^p t^q}{(f)_{m+n}(g)_p(h)_q\ m!\ n!\ p!\ q!} \tag{1.5}$$

which is the generalization of Lauricella's triple hypergeometric functions $F_A^{(3)}$, F_8[16,pp.113-114]. In 1954, the notation F_G was used by Saran[20;21] for Lauricella's triple hypergeometric function F_8 in his systematic study of triple hypergeometric functions of Lauricella's set.

Exton[12,p.117(4.1.25)] gave the following transformation

$$D_5\left[a,b,c,d,1-d,1-e;\ \frac{x}{1-x},\frac{y}{1-y},\frac{z}{1-z},\frac{t}{1-t}\right]$$

$$= (1-x)^a(1-y)^b(1-z)^c(1-t)^f\ K_{13}[d+e-1,d+e-1,d+e-1,d+e-1;\ a,b,c,f;\ e,e,d,f;\ x,y,z,t] \tag{1.6}$$

Infact above transformation is incorrect and was obtained from Pochhammer's double loop type contour integral representation for K_{13}.

In our investigations, we shall use the following transformation of Pandey[17,p.116(3.11); see also 18,p.1240(1.9)]

$$G_B\left[1-a,c,d,e;\ b;\ \frac{x}{x-1},\frac{y}{y-1},\frac{z}{z-1}\right]$$

$$= (1-x)^c(1-y)^d(1-z)^e\ F_G[a+b-1,a+b-1,a+b-1,c,d,e;\ a,b,b;\ x,y,z] \tag{1.7}$$

where Pandey's triple hypergeometric function G_B[17,pp.115-116] is defined by

$$G_B[a,b,c,d;\ e;\ x,y,z] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a)_{n+p-m}(b)_m(c)_n(d)_p\ x^m y^n z^p}{(e)_{n+p-m}\ m!\ n!\ p!} \tag{1.8}$$

and Saran's triple hypergeometric function F_G[12; see also 25,p.67(28)] is defined by

$$F_8 : F_G[a,a,a,b,c,d;\ e,f,f;\ x,y,z] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a)_{m+n+p}(b)_m(c)_n(d)_p\ x^m y^n z^p}{(e)_m(f)_{n+p}\ m!\ n!\ p!} \tag{1.9}$$

Euler's linear hypergeometric transformations[19, p.60, Theorems 20 (4,5); see also 25, p.33 (19,20,21)] are given by

$$ {}_2F_1\left[\begin{array}{lc} a,\ b\ ; & \\ & z \\ c\ \ ; & \end{array}\right] = (1-z)^{-a}\ {}_2F_1\left[\begin{array}{lc} a,\ c-b\ ; & \\ & \frac{z}{z-1} \\ c\ \ ; & \end{array}\right] \tag{1.10}$$

$$= (1-z)^{-b}\ {}_2F_1\left[\begin{array}{lc} b,\ c-a\ ; & \\ & \frac{z}{z-1} \\ c\ \ ; & \end{array}\right] \tag{1.11}$$

$$= (1-z)^{c-a-b}\ {}_2F_1\left[\begin{array}{lc} c-a,\ c-b\ ; & \\ & z \\ c\ \ ; & \end{array}\right] \tag{1.12}$$

$$\Big(c \notin \{0,-1,-2,\dots\} \text{ and } |\arg(1-z)| < \pi\Big)$$

2. MAIN HYPERGEOMETRIC TRANSFORMATIONS

We obtain the following transformations by series rearrangement techniques

$$D_5\left[a,b,c,d,e,f;\ \frac{x}{1-x},\frac{y}{1-y},\frac{z}{1-z},\frac{t}{1-t}\right]$$

$$= (1-x)^a(1-y)^b(1-z)^c(1-t)^{d+e+f-1}\ F^{(4)}\left[\begin{array}{l} 1-e-f::\ c\ ;1-f-d;\ a\ ;\ \text{-}: \\ \\ \text{-}\ ::1-f-d;\ 1-f\ ;\text{-};1-e: \end{array}\right.$$

$$\left.\begin{array}{lc} \text{-};1-f-d;\ b\ ;\ \text{-}\ ; & \\ & \frac{z}{1-t},\frac{x}{1-t},\frac{t}{t-1},\frac{y}{1-t} \\ \text{-};\ 1-f\ ;\text{-};1-e; & \end{array}\right] \tag{2.1}$$

$$= (1-x)^a(1-y)^b(1-z)^c(1-t)^{d+e+f-1}\ F^{(4)}\left[\begin{array}{l} 1-e-f::\text{-};1-f-d;\ a\ ;\ \text{-}: \\ \\ \text{-}\ ::\text{-};\ 1-f\ ;\text{-};1-e: \end{array}\right.$$

$$\left.\begin{array}{l} c\,;1-f-d;\,b\,;\,-\,; \\ \\ 1-f-d;\,1-f\,;-;1-e; \end{array}\quad \tfrac{t}{t-1},\tfrac{x}{1-t},\tfrac{z}{1-t},\tfrac{y}{1-t}\right] \tag{2.2}$$

$$= (1-x)^a(1-y)^b(1-z)^c(1-t)^{d+e+f-1}\ F^{(4)}\left[\begin{array}{l} 1-e-f::\,c\,;1-f-d;\,b\,;\,-: \\ \\ -\,::1-f-d;\,1-f\,;-;1-e: \end{array}\right.$$

$$\left.\begin{array}{l} -;1-f-d;\,a\,;\,-\,; \\ \\ -;\,1-f\,;-;1-e; \end{array}\quad \tfrac{z}{1-t},\tfrac{y}{1-t},\tfrac{t}{t-1},\tfrac{x}{1-t}\right] \tag{2.3}$$

$$= (1-x)^a(1-y)^b(1-z)^c(1-t)^{d+e+f-1}\ F^{(4)}\left[\begin{array}{l} 1-e-f::-;1-f-d;\,b\,;\,-: \\ \\ -\,::-;\,1-f\,;-;1-e: \end{array}\right.$$

$$\left.\begin{array}{l} c\,;1-f-d;\,a\,;\,-\,; \\ \\ 1-f-d;\,1-f\,;-;1-e; \end{array}\quad \tfrac{t}{t-1},\tfrac{y}{1-t},\tfrac{z}{1-t},\tfrac{x}{1-t}\right] \tag{2.4}$$

$$= (1-x)^a(1-y)^b(1-z)^c(1-t)^{d+e+f-1}\ F^{(4)}\left[\begin{array}{l} 1-e-f::\,a\,;-;\,c\,;1-f-d: \\ \\ -\,::-;1-e;1-f-d;\ 1-f\,: \end{array}\right.$$

$$\left.\begin{array}{l} b\ ;-;-;1-f-d; \\ \\ -;1-e;-;\ 1-f\,; \end{array}\quad \tfrac{x}{1-t},\tfrac{z}{1-t},\tfrac{y}{1-t},\tfrac{t}{t-1}\right] \tag{2.5}$$

$$= (1-x)^a(1-y)^b(1-z)^c(1-t)^{d+e+f-1}\ F^{(4)}\left[\begin{array}{l} 1-e-f::\,a\ ;-;-;1-f-d: \\ \\ -\,::-;1-e;-;\ 1-f\,: \end{array}\right.$$

$$\left.\begin{array}{l} b\ ;-;\,c\,;1-f-d; \\ \\ -;1-e;1-f-d;\ 1-f\,; \end{array}\quad \tfrac{x}{1-t},\tfrac{t}{t-1},\tfrac{y}{1-t},\tfrac{z}{1-t}\right] \tag{2.6}$$

$$= (1-x)^a(1-y)^b(1-z)^c(1-t)^{d+e+f-1}\ F^{(4)}\left[\begin{array}{l} 1-e-f::\,b\,;-;\,c\,;1-f-d: \\ \\ -\,::-;1-e;1-f-d;\ 1-f\,: \end{array}\right.$$

$$\left.\begin{array}{l} a\ ;-;-;1-f-d; \\ \\ -;1-e;-;\ 1-f\,; \end{array}\quad \tfrac{y}{1-t},\tfrac{z}{1-t},\tfrac{x}{1-t},\tfrac{t}{t-1}\right] \tag{2.7}$$

$$= (1-x)^a(1-y)^b(1-z)^c(1-t)^{d+e+f-1}\ F^{(4)}\left[\begin{array}{l} 1-e-f\text{::}\ b\ \text{;-;-;}\ 1-f-d\text{:} \\ \\ \text{- ::-;}1-e\text{;-;}\ 1-f\ \text{:} \end{array}\right.$$

$$\left.\begin{array}{l} a\ \text{;-;}\ c\ \text{;}1-f-d\text{;} \\ \\ \text{-;}1-e\text{;}1-f-d\text{;}\ \ 1-f\ \text{;} \end{array} \frac{y}{1-t}, \frac{t}{t-1}, \frac{x}{1-t}, \frac{z}{1-t}\right] \tag{2.8}$$

$$= (1-x)^a(1-y)^b(1-z)^c(1-t)^d\times$$

$$\times K_{12}[1-e-f, 1-e-f, 1-e-f, 1-e-f;\ a, b, c, d;\ 1-e, 1-e, 1-f, 1-f;\ x, y, z, t] \tag{2.9}$$

Infact the transformation (2.9) relating D_5 and K_{12} is a known transformation of Exton[12,p.117(4.1.24)] and was obtained from Pochhammer's double loop type contour integral representation for K_{12}.

3. DERIVATIONS OF HYPERGEOMETRIC TRANSFORMATIONS (2.1), (2.2), $\cdots$, (2.9)

Suppose left hand side of Exton's quadruple hypergeometric function D_5 of transformation (2.1) is denoted by S, then its power series form is

$$S = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(a)_m(b)_n(c)_p(d)_q(e)_{p+q-m-n}(f)_{m+n-p-q}\ (\frac{x}{1-x})^m(\frac{y}{1-y})^n(\frac{z}{1-z})^p(\frac{t}{1-t})^q}{m!\ n!\ p!\ q!}$$

$$= \sum_{q=0}^{\infty} \frac{(d)_q(e)_q\ (\frac{t}{t-1})^q}{(1-f)_q\ q!} G_B\left[f-q, c, a, b;\ 1-e-q;\ \frac{z}{z-1}, \frac{x}{x-1}, \frac{y}{y-1}\right] \tag{3.1}$$

Now using the hypergeometric transformation (1.7) in G_B of (3.1), we get

$$S = \sum_{q=0}^{\infty} \frac{(d)_q(e)_q\ (\frac{t}{t-1})^q\ (1-x)^a(1-y)^b(1-z)^c}{(1-f)_q\ q!}\times$$

$$\times F_G[1-e-f, 1-e-f, 1-e-f, c, a, b;\ q-f+1, 1-e-q, 1-e-q;\ z, x, y]$$

$$= (1-x)^a(1-y)^b(1-z)^c \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(1-e-f)_{m+n+p}(d)_q(c)_m(a)_n(b)_p\ z^m x^n y^p (\frac{t}{1-t})^q}{(1-f)_{q+m}(1-e)_{n+p-q}\ m!\ n!\ p!\ q!}$$

$$= (1-x)^a(1-y)^b(1-z)^c \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(1-e-f)_{m+n+p}(c)_m(a)_n(b)_p\ z^m x^n y^p}{(1-f)_m(1-e)_{n+p}\ m!\ n!\ p!}\times$$

$$\times {}_2F_1\left[\begin{array}{l} d,\ e-n-p\ \ ; \\ \\ 1-f+m\ \ \ ; \end{array} \frac{t}{t-1}\right] \tag{3.2}$$

Now using Euler's first linear transformation (1.10) in ${}_2F_1$ of (3.2), we get

$$S = (1-x)^a(1-y)^b(1-z)^c(1-t)^d \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(1-e-f)_{m+n+p}(c)_m(a)_n(b)_p\ z^m x^n y^p}{(1-f)_m(1-e)_{n+p}\ m!\ n!\ p!}\times$$

$$\times {}_2F_1\left[\begin{array}{lr} d,\ 1-e-f+m+n+p & ; \\ & t \\ 1-f+m & ; \end{array}\right]$$

$$= (1-x)^a(1-y)^b(1-z)^c(1-t)^d \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(1-e-f)_{m+n+p+q}(d)_q(c)_m(a)_n(b)_p\ z^m x^n y^p t^q}{(1-f)_{m+q}(1-e)_{n+p}\ m!\ n!\ p!\ q!} \tag{3.3}$$

Now interpreting the definition (1.4) of Exton's function K_{12} in (3.3), we get (2.9).

If we use Euler's second linear transformation (1.12) in ${}_2F_1$ of (3.2), we obtain

$$S = (1-x)^a(1-y)^b(1-z)^c(1-t)^{d+e+f-1}\times$$

$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(1-e-f)_{m+n+p}(c)_m(a)_n(b)_p\ (\frac{z}{1-t})^m(\frac{x}{1-t})^n(\frac{y}{1-t})^p}{(1-f)_m(1-e)_{n+p}\, m!\, n!\, p!}\times$$

$$\times {}_2F_1\left[\begin{array}{lr} 1-d-f+m,\ 1-e-f+m+n+p & ; \\ & \frac{t}{t-1} \\ 1-f+m & ; \end{array}\right]$$

$$= (1-x)^a(1-y)^b(1-z)^c(1-t)^{d+e+f-1}\times$$

$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(1-e-f)_{m+n+p+q}(1-d-f)_{m+q}(c)_m(a)_n(b)_p\ (\frac{z}{1-t})^m(\frac{x}{1-t})^n(\frac{t}{t-1})^q(\frac{y}{1-t})^p}{(1-f)_{m+q}(1-e)_{n+p}(1-d-f)_m\ m!\ n!\ p!\ q!} \tag{3.4}$$

$$= (1-x)^a(1-y)^b(1-z)^c(1-t)^{d+e+f-1}\times$$

$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(1-e-f)_{m+n+p+q}(1-d-f)_{m+q}(a)_n(c)_m(b)_p\ (\frac{t}{t-1})^q(\frac{x}{1-t})^n(\frac{z}{1-t})^m(\frac{y}{1-t})^p}{(1-f)_{m+q}(1-e)_{n+p}(1-d-f)_m\ m!\ n!\ p!\ q!} \tag{3.5}$$

$$= (1-x)^a(1-y)^b(1-z)^c(1-t)^{d+e+f-1}\times$$

$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(1-e-f)_{m+n+p+q}(1-d-f)_{m+q}(a)_n(c)_m(b)_p\ (\frac{z}{1-t})^m(\frac{y}{1-t})^p(\frac{t}{t-1})^q(\frac{x}{1-t})^n}{(1-f)_{m+q}(1-e)_{n+p}(1-d-f)_m\ m!\ n!\ p!\ q!} \tag{3.6}$$

$$= (1-x)^a(1-y)^b(1-z)^c(1-t)^{d+e+f-1}\times$$

$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(1-e-f)_{m+n+p+q}(1-d-f)_{m+q}(b)_p(c)_m(a)_n\ (\frac{t}{t-1})^q(\frac{y}{1-t})^p(\frac{z}{1-t})^m(\frac{x}{1-t})^n}{(1-f)_{m+q}(1-e)_{n+p}(1-d-f)_m\ m!\ n!\ p!\ q!} \tag{3.7}$$

Now interpreting the definition (1.2) of Srivastava's quadruple hypergeometric function $F^{(4)}$ in (3.4), (3.5), (3.6), (3.7), we get (2.1), (2.2), (2.3), (2.4) respectively.

Similarly if we change the order of summation indices in (3.4), (3.5), (3.6) and (3.7), we obtain (2.5), (2.6), (2.7) and (2.8) respectively.

4. CORRECT FORMS OF HYPERGEOMETRIC TRANSFORMATION (1.6)

Suppose K_{13} of (1.6) is denoted by T, then its quadruple power series form is

$$T = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(e+d-1)_{m+n+p+q}(a)_m(b)_n(c)_p(f)_q\ x^m y^n z^p t^q}{(e)_{m+n}(d)_p(f)_q\ m!\ n!\ p!\ q!} \tag{4.1}$$

$$= \sum_{p=0}^{\infty} \frac{(e+d-1)_p(c)_p\ z^p}{(d)_p\ p!} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty} \frac{(e+d-1+p)_{m+n+q}(f)_q(a)_m(b)_n\ t^q\ x^m\ y^n}{(e)_{m+n}(f)_q\ m!\ n!\ q!}$$

$$= \sum_{p=0}^{\infty} \frac{(e+d-1)_p(c)_p\ z^p}{(d)_p\ p!} F_G[e+d+p-1, e+d+p-1, e+d+p-1, f, a, b;\ f, e, e;\ t, x, y] \tag{4.2}$$

$$= \sum_{p=0}^{\infty} \frac{(e+d-1)_p(c)_p\ z^p}{(d)_p\ p!} F_G[e+d+p-1, e+d+p-1, e+d+p-1, d+p, a, b;\ d+p, e, e;\ t, x, y] \tag{4.3}$$

Since we can not apply Pandey's transformation (1.7) in (4.2) because parameters are restricted, therefore we shall apply (1.7) in (4.3) with restricted parameter $d+p$ in place of f

$$T = (1-x)^{-a}(1-y)^{-b}(1-t)^{-d} \sum_{p=0}^{\infty} \frac{(e+d-1)_p(c)_p\ (\frac{z}{1-t})^p}{(d)_p\ p!} \times$$

$$\times G_B\left[1-d-p, d+p, a, b;\ e;\ \frac{t}{t-1}, \frac{x}{x-1}, \frac{y}{y-1}\right]$$

$$= (1-x)^{-a}(1-y)^{-b}(1-t)^{-d} \sum_{p=0}^{\infty} \frac{(e+d-1)_p(c)_p\ (\frac{z}{1-t})^p}{(d)_p\ p!} \times$$

$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty} \frac{(1-d-p)_{n+q-m}(d+p)_m(a)_n(b)_q\ (\frac{t}{t-1})^m(\frac{x}{x-1})^n(\frac{y}{y-1})^q}{(e)_{n+q-m}\ m!\ n!\ q!}$$

$$= (1-x)^{-a}(1-y)^{-b}(1-t)^{-d} \times$$

$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(1-d)_{n+q-m-p}(d)_{m+p}(a)_n(b)_q(e+d-1)_p(c)_p\ (\frac{z}{t-1})^p(\frac{t}{t-1})^m(\frac{x}{x-1})^n(\frac{y}{y-1})^q}{(e)_{n+q-m}(d)_p\ m!\ n!\ p!\ q!} \tag{4.4}$$

which can not be written in terms of Exton's quadruple hypergeometric functions D_1, D_2, D_3, D_4 and D_5[12,pp.88-89; see also 9;10].

Similarly we can express K_{13} from (4.1) into another form

$$T = \sum_{q=0}^{\infty} \frac{(e+d-1)_q\ t^q}{q!} F_G[e+d-1+q, e+d-1+q, e+d-1+q, c, a, b;\ d, e, e;\ z, x, y] \tag{4.5}$$

We can not apply the Pandey's transformation (1.7) in F_G of (4.5) due to parameter $e+d-1+q$.

Now writing again K_{13} in the following form

$$T = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(e+d-1)_{m+n+p+q}(a)_m(b)_n(c)_p\ x^m y^n z^p t^q}{(e)_{m+n}(d)_p\ m!\ n!\ p!\ q!}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(e+d-1)_{m+n+p}(c)_p(a)_m(b)_n\ x^m y^n z^p (1-t)^{(1-d-e-m-n-p)}}{(e)_{m+n}(d)_p\ m!\ n!\ p!}$$

$$= (1-t)^{(1-e-d)} F_G\left[e+d-1, e+d-1, e+d-1, c, a, b;\ d, e, e;\ \frac{z}{1-t}, \frac{x}{1-t}, \frac{y}{1-t}\right] \quad (4.6)$$

Now we can apply the transformation (1.7) in F_G of (4.6), we get

$$K_{13}[e+d-1, e+d-1, e+d-1, e+d-1;\ a, b, c, f;\ e, e, d, f;\ x, y, z, t]$$

$$= (1-x-t)^{-a}(1-y-t)^{-b}(1-z-t)^{-c}(1-t)^{1-e-d+a+b+c}\times$$

$$\times G_B\left[1-d, c, a, b;\ e;\ \frac{z}{z+t-1}, \frac{x}{x+t-1}, \frac{y}{y+t-1}\right] \quad (4.7)$$

which is the correct form of incorrect transformation (1.6) of Exton.

Now making suitable adjustment of parameters in Exton's transformation (2.9), we can write easily

$$D_5\left[a, b, c, d, 1-d, 1-e;\ \frac{x}{1-x}, \frac{y}{1-y}, \frac{z}{1-z}, \frac{t}{1-t}\right] = (1-x)^a(1-y)^b(1-z)^c(1-t)^d\times$$

$$\times K_{12}[e+d-1, e+d-1, e+d-1, e+d-1;\ a, b, c, d;\ d, d, e, e;\ x, y, z, t] \quad (4.8)$$

which is another correct form of incorrect transformation (1.6) of Exton.

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