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REGULAR, INTRA-REGULAR AND DUO Γ-SEMIRINGS

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Abstract. In this paper we give several characterizations of a regular Γ-semiring , an intra-regular Γ-semiring and a duo Γ-semiring by using ideals, interior-ideals, quasi-ideals and bi-ideals of a Γ-semiring.

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§1 Introduction

The notion of a quasi-ideal was firstly introduced for semigroups in [15] and for rings in [16] by Steinfeld. Iseki in [6] discussed some characterizations of quasi-ideals for a semiring without zero. Using quasi-ideals Shabir, Ali, Batool in [14] characterize a class of semirings. Chinram in [2] generalizes the concept of quasi-ideals to a Γ-semigroup and discussed some of its properties. Also in [1] Chinram gave the different characterizations of quasi-ideals in a Γ-semiring while the concept of a Γ-semiring was coined by Rao in [13]. Authors were studied quasi-ideals and minimal quasi-ideals in Γ-semirings in [7] and quasi-ideals in regular Γ-semirings in [8] .

The notion of a bi-ideal was first introduced for semigroups by Good and Hughes in [4]. The concept of a bi-ideal for a ring was given by Lajos [9] . Also in [10,11] Lajos discussed some characterizations of bi-ideals in semigroups. Shabir , Ali , Batool in [14] gave some properties of bi-ideals in a semiring.

The concept of a regular ring was introduced by J. von Neumann in [12] and gave the definition of a regular ring such as a ring R is regular if for any $b \in R$ there exists $x \in R$ such that $b = bxb$. Analogously the concept of a regular semigroup was introduced by Green in [5] and a regular semiring was introduced by Zelznikov[17] respectively. This concept of regularity was extended to Γ-semiring by Rao [13] and then studied by Dutta and Sardar in [3].

In this paper efforts are made to prove various characterizations of a regular Γ-semiring , intra-regular Γ-semiring and a duo Γ-semiring by using ideals, interior-ideals, quasi-ideals and bi-ideals of a Γ-semiring.

§2. Preliminaries

First we recall some definitions of the basic concepts of Γ-semirings that we need in sequel. For this we follow Dutta and Sardar [3].

Definition 2.1: *Let S and Γ be two additive commutative semigroups. S is called a Γ-semiring if there exists a mapping $S\times\Gamma\times S \longrightarrow S$ denoted by $a\alpha b$ for all $a,b\in S$ and $\alpha\in\Gamma$ satisfying the following conditions:*

(i) $a\alpha\,(b+c) = (a\ \alpha b) + (a\ \alpha c)$

(ii) $(b+c)\,\alpha a = (b\ \alpha a) + (c\ \alpha a)$

(iii) $a(\alpha+\beta)c = (a\ \alpha c) + (a\ \beta c)$

(iv) $a\alpha\,(b\beta c) = (a\alpha b)\,\beta c$; for all $a,b,c\in S$ *and for all* $\alpha,\beta\in\ \Gamma$.

Definition 2.2: *An element $0\in S$ is said to be an absorbing zero if*
$0\alpha a = 0 = a\alpha 0, a+0 = 0+a = a$ for all $a\in S$ *and* $\alpha\in\Gamma$.

Now onwards S denotes a Γ-semiring with absorbing zero unless otherwise stated.

Definition 2.3: *A non-empty subset T of S is said to be sub- Γ-semiring of S if (T,+) is a subsemigroup of (S,+) and $a\alpha b\in\ T$* for all $a,b\in\ T$and $\alpha\in\Gamma$.

Definition 2.4: *A non-empty subset T of S is called a left (respectively right) ideal of S if T is a subsemigroup of (S,+) and $x\alpha a\in T$ (respectively $a\alpha x\in T$)* for all $a\in T,\ x\in S$ and $\alpha\in\Gamma$.

Definition 2.5: *If T is both left and right ideal of S, then T is known as an ideal of S.*

A quasi-ideal Q in a Γ-semiring S is defined as follows.

Definition 2.6 [7]**:** *A subsemigroup Q of $(S,+)$ is a quasi-ideal of S if $(S\Gamma Q)\cap(Q\Gamma S)\subseteq Q$.*

Example: Consider a Γ-semiring $S = M_{2\times2}(N_0)$,where N_0 denotes the set of natural numbers with zero and Γ= S. Define $A\alpha B=$ usual matrix product of A,α and B; for all $A,\alpha\ ,B\ \in S$. Then
$Q=\left\{\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}\mid a\in N_0\right\}$ is a quasi-ideal of a Γ-semiring S.

Definition 2.7 [8]**:** *A nonempty subset B of S is a bi-ideal of S if B is a sub Γ-semiring of S and $B\Gamma S\Gamma B\subseteq B$.*

Example : Let N be the set of natural numbers and $\Gamma = 2N$. Then N and Γ both are additive commutative semigroup. An image of a mapping $N\times\Gamma\times N\longrightarrow N$ is denoted by

$a\alpha b$ and defined as $a\alpha b$ = product of a , α , b; for all $a, b \in S$ and $\alpha \in \Gamma$. Then N forms a Γ-semiring. $B = 3N$ is a bi-ideal of N.

Now we define a generalized bi-ideal and an interior-ideal of a Γ- semiring S.

Definition 2.8: *A nonempty subset B of* a Γ- semiring S *is a generalized bi-ideal of* S *if* $B\Gamma S\Gamma B \subseteq B$.

Definition 2.9: *A nonempty subset I of S is an interior-ideal of* a Γ- semiring S *if I is a subsemigroup of S and $S\Gamma I\Gamma S \subseteq I$.*

Result 2.10: *For each nonempty subset X of S following statements hold.*

(i) *$S\Gamma X$ is a left ideal of S.*

(ii) *$X\Gamma S$ is a right ideal of S.*

(iii) *$S\Gamma X\Gamma S$ is an ideal of S.*

Result 2.11: *For $a \in S$ following statements hold.*

(i) *$S\Gamma a$ is a left ideal of S.*

(ii) *$a\Gamma S$ is a right ideal of S.*

(iii) *$S\Gamma a\Gamma S$ is an ideal of S.*

§3. Regular Γ-Semiring

An element a of a Γ-semiring S is said to be regular if $a \in a\Gamma S\Gamma a$.

If all elements of Γ-semiring S are regular, then S is known as a regular Γ-semiring.

Following theorem was proved in [8] by authors.

Theorem 3.1: In S following statements are equivalent.

(1) S is regular

(2) For every left ideal L and a right ideal R of S, $R\Gamma L = R \cap L$.

(3) For every left ideal L and a right ideal R of S,

(i) $R^2 = R\Gamma R = R$ (ii) $L^2 = L\Gamma L = L$

(iii) $R\Gamma L = R \cap L$ is a quasi-ideal of S.

(4) The set of all quasi-ideals of S is a regular Γ-semigroup

(5) Every quasi-ideal of S is of the form $Q\Gamma S\Gamma Q = Q$.

Theorem 3.2: Following statements are equivalent in S.

(1) S is regular.

(2) For any bi-ideal B *of* S, $B\Gamma S\Gamma B = B$.

(3) For any quasi-ideal Q *of* S , $Q\Gamma S\Gamma Q = Q$.

Proof : (1) $\Rightarrow$ (2) Let B be a bi-ideal of S. Let $b \in B$. As S is regular, $b \in b\Gamma S\Gamma b \subseteq B\Gamma S\Gamma B$. Therefore $B \subseteq B\Gamma S\Gamma B$. Hence $B = B\Gamma S\Gamma B$.

(2) $\Rightarrow$ (3) As every quasi-ideal is a bi-ideal, implication (2) $\Rightarrow$ (3) holds.

(3) $\Rightarrow$ (1) Let R be a right ideal and L be a left ideal of S. Then $R \cap L$ is a quasi-ideal of S. Hence by assumption $R \cap L = (R \cap L)\,\Gamma S \Gamma\,(R \cap L) \subseteq (R\Gamma S)\,\Gamma L \subseteq\ R\Gamma L$. Therefore $R \cap L = R\Gamma L$. Thus S is a regular Γ-semiring. ■

Theorem 3.3: If S is a Γ- semiring, then following statements are equivalent

(1) S is regular.

(2) For every bi-ideal B and an ideal I of S, $B \cap I = B\Gamma$IΓB.

(3) For every quasi-ideal Q and an ideal I of S, $Q \cap I = Q\Gamma$IΓQ.

Proof : (1) $\Rightarrow$ (2) Let B be a bi-ideal and I be an ideal of S. Now $B\Gamma I\Gamma B \subseteq B\Gamma S\Gamma B \subseteq B$ and $B\Gamma I\Gamma B \subseteq I$. Therefore $B\Gamma$IΓB $\subseteq B \cap I$. For the reverse inclusion, let $a \in B \cap I$. As S is regular, $a \in a\Gamma S\Gamma a$. Then $a\Gamma S\Gamma a \subseteq (a\Gamma S\Gamma a)\Gamma S\Gamma(a\Gamma S\Gamma a) \subseteq (B\Gamma S\Gamma B)\,\Gamma(S\Gamma I\Gamma S)\Gamma B \subseteq B\Gamma I\Gamma B$. Therefore $a \in B\Gamma I\Gamma B$. Hence we have $B \cap I \subseteq B\Gamma I\Gamma B$. Thus we get $B\Gamma I\Gamma B = B \cap I$.

(2) $\Rightarrow$ (3) Implication follows as every quasi-ideal of S is a bi-ideal.

(3) $\Rightarrow$ (1) Let R be a right ideal and L be a left ideal of S. Then by assumption we have, $R = R\cap$S $= R\Gamma S\Gamma R \subseteq R\Gamma R$ and $L\cap S = L\Gamma S\Gamma L \subseteq L\Gamma L$. Also $R\cap L = R\Gamma L$ is a quasi-ideal of S. Hence by Theorem 3.1, S is a regular Γ-semiring. ■

Proof of following theorem is straightforward.

Theorem 3.4: In S following statements are equivalent.

(1) S is regular.

(2) For every bi-ideal B and left ideal L of S, $B \cap L \subseteq B\Gamma L$.

(3) For every quasi-ideal Q and left ideal L of S, $Q \cap L \subseteq Q\Gamma$L.

(4) For every bi-ideal B and right ideal R *of* S, $B \cap R \subseteq R\Gamma B$.

(5) For every right ideal R and quasi-ideal Q of S, $R \cap Q \subseteq R\Gamma Q$.

(6) For every left ideal L , every right ideal R and every bi-ideal B of S, $L \cap R \cap B \subseteq R\Gamma B\Gamma L$.

(7) For every left ideal ,every right ideal R and every quasi-ideal Q of S, $L\cap R\cap B \subseteq R\Gamma Q\Gamma L$.

Theorem 3.5 : In S following conditions are equivalent.

(1) S is regular.

(2) $I \cap Q = Q\Gamma I\Gamma Q$, for an ideal I and a quasi-ideal Q of S.

(3) $I \cap Q = Q\Gamma I\Gamma Q$, for an interior ideal I and a quasi-ideal Q of S.

Proof : (1) $\Rightarrow$ (2) Let Q be a quasi-ideal and I be an ideal of S. Now $Q\Gamma I\Gamma Q \subseteq Q\Gamma S\Gamma Q \subseteq Q\Gamma S$ by Result 2.6. Similarly we get $Q\Gamma I\Gamma Q \subseteq S\Gamma Q$. Therefore $Q\Gamma I\Gamma Q \subseteq (S\Gamma Q) \cap (Q\Gamma S) \subseteq Q$, since Q is a quasi-ideal. Also $Q\Gamma I\Gamma Q \subseteq I$ as I is an ideal. Therefore $Q\Gamma I\Gamma$Q $\subseteq Q \cap I$. For the reverse inclusion, let $a \in Q \cap I$. As S is regular , $a \in a\Gamma S\Gamma a$. We have $a \in (a\Gamma S\Gamma a)\,\Gamma S\Gamma\,(a\Gamma S\Gamma a) \subseteq (Q\Gamma S\Gamma Q)\,\Gamma(S\Gamma I\Gamma S)\Gamma Q \subseteq Q\Gamma I\Gamma Q$. Hence $Q \cap I \subseteq$

$Q\Gamma I\Gamma Q$.Therefore $Q\Gamma I\Gamma Q = Q \cap I$.
(2) $\Rightarrow$ (1) Let Q be a quasi-ideal of S. By (2) $Q\Gamma S\Gamma Q = Q \cap S$. Hence $Q\Gamma S\Gamma Q = Q$. Therefore S is regular by Theorem 3.2.
(1) $\Rightarrow$ (3) Let Q be a quasi-ideal and I be an interior ideal of S. Now $Q\Gamma I\Gamma Q \subseteq Q\Gamma S\Gamma Q \subseteq Q\Gamma S$ by Result 2.6. Similarly we get $Q\Gamma I\Gamma Q \subseteq S\Gamma Q$. Therefore $Q\Gamma I\Gamma Q \subseteq (S\Gamma Q)\cap(Q\Gamma S) \subseteq Q$. Also $Q\Gamma I\Gamma Q \subseteq I$ as I is an interior ideal. Therefore $Q\Gamma \mathrm{I}\Gamma \mathrm{Q} \subseteq Q \cap I$. For the reverse inclusion, let $a \in Q \cap I$. As S is regular, $a \in a\Gamma S\Gamma a$. Therefore $a \in (a\Gamma S\Gamma a)\,\Gamma S\Gamma\,(a\Gamma S\Gamma a) \subseteq (Q\Gamma S\Gamma Q)\,\Gamma(S\Gamma I\Gamma S)\Gamma Q \subseteq Q\Gamma I\Gamma Q$. Therefore $Q \cap \mathrm{I} \subseteq Q\Gamma I\Gamma Q$. Hence $Q\Gamma I\Gamma Q = Q \cap I$.
(3) $\Rightarrow$ (1) Let Q be a quasi-ideal of S. By (3) $Q\Gamma S\Gamma Q = Q \cap S$. Hence $Q\Gamma S\Gamma Q = Q$. By Theorem 3.2, S is regular. ■

Theorem 3.6 : In S following statements are equivalent.
(1) S is regular.
(2) $Q \cap L \subseteq Q\Gamma L$, for a quasi-ideal Q and a left ideal L of S.
(3) $Q \cap R \subseteq R\Gamma Q$, for a quasi-ideal Q and a right ideal R of S.

Theorem 3.7 :- S is regular if and only if $R \cap Q \cap L \subseteq R\Gamma Q\Gamma L$, for a right ideal R , quasi-ideal Q and a left ideal L of S.

Proof : Suppose that S is a regular Γ-semiring. Let R be a right ideal , Q be a quasi-ideal and L be a left ideal of S. Let $a \in R \cap Q \cap L$. As S is regular , $a \in a\Gamma S\Gamma a$ Therefore $a \in (a\Gamma S\Gamma a)\,\Gamma S\Gamma a \subseteq (R\Gamma S)\,\Gamma Q\Gamma\,(S\Gamma L) \subseteq R\Gamma Q\Gamma L$, shows that $R \cap Q \cap L \subseteq R\Gamma Q\Gamma L$. Conversely, let R be a right ideal and L be a left of S. By assumption $R \cap S \cap L \subseteq R\Gamma S\Gamma L$. Therefore $R \cap L \subseteq R\Gamma L$. Thus we have $R \cap L = R\Gamma L$. Hence S is regular by Theorem 3.1.
■

§4. Intra-regular Γ-semiring

Now we give a definition of an intra-regular Γ-semiring.

Definition 4.1: A Γ-semiring S is said to be an intra-regular Γ-semiring if for any $x \in S$, $x \in S\Gamma x\Gamma x\Gamma S$.

Theorem 4.2: S is intra-regular if and only if each right ideal R and left ideal L of S satisfy $R \cap L \subseteq L\Gamma R$.

Proof : Suppose S is an intra-regular Γ-semiring and R and L be a right ideal and a left ideal of S respectively. Let $a \in R \cap L$. As S is intra-regular , $a \in S\Gamma a\Gamma a\Gamma S$. Now $S\Gamma a\Gamma a\Gamma S = (S\Gamma a)\,\Gamma(a\Gamma \mathrm{S}) \subseteq (S\Gamma L)\,\Gamma(R\Gamma S) \subseteq L\Gamma R$. Therefore $R \cap L \subseteq L\Gamma R$. Conversely, for $a \in S$, $(a)_l = N_0 a + S\Gamma a$, $(a)_r = N_0 a + a\Gamma S$. By assumption $(a)_r \cap (a)_l \subseteq (a)_l\Gamma(a)_r$. Then $(a)_r \cap (a)_l \subseteq (a)_l\Gamma(a)_r = (N_0 a + S\Gamma a)\,\Gamma\,(N_0 a + a\Gamma S)$. Also by assumption we have $(a)_r \subseteq S\Gamma a + S\Gamma a\Gamma S$ and $(a)_l \subseteq a\Gamma S + S\Gamma a\Gamma S$. Hence we have $(a)_r \subseteq S\Gamma a + S\Gamma a\Gamma S \subseteq S\Gamma a\Gamma a\Gamma S$. Therefore we get $a \in S\Gamma a\Gamma a\Gamma S$. Thus any $a \in S$ is an intra-regular element of S. Therefore S is an intra-regular Γ-semiring. ■

Theorem 4.3: In S following statements are equivalent.

(1) S is an intra-regular Γ-semiring.

(2) For bi-ideals B_1 and B_2 of S, $B_1 \cap B_2 \subseteq S\Gamma B_1 \Gamma B_2 \Gamma S$.

(3) For every bi-ideal B and quasi-ideal Q of S, $B \cap Q \subseteq (S\Gamma Q\Gamma B\Gamma S) \cap (S\Gamma B\Gamma Q\Gamma S)$.

(4) For every quasi-ideals Q_1 and Q_2 of S, $Q_1 \cap Q_2 \subseteq S\Gamma Q_1 \Gamma Q_2 \Gamma S$.

Proof : (1) $\Rightarrow$ (2) Suppose S is intra-regular. Let B_1 and B_2 be bi-ideals of S. Let $a \in B_1 \cap B_2$. As S is intra-regular, $a \in S\Gamma a\Gamma a\Gamma S$. $a \in S\Gamma a\Gamma a\Gamma S =\subseteq S\Gamma B_1 \Gamma B_2 \Gamma S$. Therefore $B_1 \cap B_2 \subseteq S\Gamma B_1 \Gamma B_2 \Gamma S$.

(2) $\Rightarrow$ (3) , (3) $\Rightarrow$ (4) Implications follow as every quasi-ideal is a bi-ideal.

(4) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Then R and L both are quasi-ideals of S. By (4) , $R \cap L \subseteq S\Gamma L\Gamma R\Gamma S = (S\Gamma L)\,\Gamma\,(R\Gamma S) \subseteq L\Gamma R$. Therefore we get $R \cap L \subseteq L\Gamma R$. Thus by Theorem 4.2, S is an intra-regular Γ-semiring.

Thus we have proved (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$(4) $\Rightarrow$ (1). ■

Theorem 4.4: In S following statements are equivalent.

(1) S is an intra-regular Γ-semiring.

(2) For a left ideal L and a bi-ideal B of S, $L \cap B \subseteq L\Gamma B\Gamma S$.

(3) For a left ideal L and a quasi-ideal Q of S, $L \cap Q \subseteq L\Gamma Q\Gamma S$.

(4) For a right ideal R and a bi-ideal B of S, $R \cap B \subseteq S\Gamma B\Gamma R$.

(5) For a right ideal R and a quasi-ideal Q of S, $R \cap Q \subseteq S\Gamma Q\Gamma R$.

Proof : (1) $\Rightarrow$ (2) Suppose S is intra-regular. Let L be a left ideal and B be a bi-ideal of S. Let $a \in B \cap L$. As S is intra-regular, $a \in S\Gamma a^2\Gamma S$. $a \in S\Gamma a\Gamma a\Gamma S =\subseteq S\Gamma L\Gamma B\Gamma S \subseteq L\Gamma B\Gamma S$. Hence $B \cap L \subseteq L\Gamma B\Gamma S$.

(2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) As every quasi-ideal is a bi-ideal, implications follow.

(3) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Then R is a quasi-ideal of S. By (3) , $R \cap L \subseteq L\Gamma R\Gamma S \subseteq L\Gamma R$. Therefore we get $R \cap L \subseteq L\Gamma R$. Thus by Theorem 4.2, S is an intra-regular Γ-semiring.

(1) $\Rightarrow$ (4) Suppose S is intra-regular. Let R be a right ideal and B be a bi-ideal of S. Let $a \in B \cap R$. As S is intra-regular, $a \in S\Gamma a\Gamma a\Gamma S$. $a \in S\Gamma a\Gamma a\Gamma S \subseteq S\Gamma B\Gamma R\Gamma S \subseteq S\Gamma B\Gamma R$, which shows that $B \cap R \subseteq S\Gamma B\Gamma R$.

(5) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. By (5) , $R \cap L \subseteq S\Gamma L\Gamma R \subseteq L\Gamma R$ as L is a quasi-ideal of S. Therefore we get $R \cap L \subseteq L\Gamma R$, shows that S is an intra-regular Γ-semiring by Theorem 4.2. ■

Theorem 4.5: In S following statements are equivalent.

(1) S is intra-regular.

(2) $K \cap B \cap R \subseteq K\Gamma B\Gamma$R , for a bi-ideal B, a right ideal R and an interior ideal K of S.

(3) $I \cap B \cap R \subseteq I\Gamma B\Gamma \mathrm{R}$, for a bi-ideal B , a right ideal R and an ideal I of S.

(4) $K \cap Q \cap R \subseteq K\Gamma Q\Gamma \mathrm{R}$, for a quasi-ideal Q , a right ideal R and an interior ideal K of S.

(5) $I \cap Q \cap R \subseteq I\Gamma Q\Gamma \mathrm{R}$, for a quasi-ideal Q , a right ideal R and an ideal I of S.

Proof : (1) $\Rightarrow$ (2) Suppose S is intra-regular. Let R be a right ideal , K be an interior ideal and B be a bi-ideal of S. Let $a \in K \cap B \cap R$. As S is intra-regular, $a \in S\Gamma a\Gamma a\Gamma S$. $a \in S\Gamma a\Gamma a\Gamma S \subseteq (S\Gamma K\Gamma S)\,\Gamma B\Gamma\,(R\Gamma S\Gamma S) \subseteq K\Gamma B\Gamma \mathrm{R}$. Thus we have $K\cap B\cap R \subseteq K\Gamma B\Gamma \mathrm{R}$.

(2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5), As every ideal is an interior ideal, implications follow.

(2) $\Rightarrow$ (4), (3) $\Rightarrow$ (5) Clearly implications follow , since quasi-ideal is a bi-ideal. (5)$\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. As L is a quasi-ideal of S, by (5) we have $S \cap L \cap R \subseteq S\Gamma L\Gamma R \subseteq L\Gamma R$. Therefore we have $R \cap L \subseteq L\Gamma R$. By Theorem 4.2, S is an intra-regular Γ-semiring. ■

Theorem 4.6: In S following statements are equivalent.

(1) S is intra-regular.

(2) $I \cap B \cap L \subseteq L\Gamma B\Gamma I$, for a bi-ideal B , a left ideal L and an interior ideal I of S.

(3) $I \cap B \cap L \subseteq L\Gamma B\Gamma I$, for a bi-ideal B , a left ideal L and an ideal I of S.

(4) $I \cap Q \cap L \subseteq L\Gamma Q\Gamma I$, for a quasi-ideal Q , a left ideal L and an interior ideal I of S.

(5) $I \cap Q \cap L \subseteq L\Gamma Q\Gamma I$, for a quasi-ideal Q , a left ideal L and an ideal I of S.

Proof : (1) $\Rightarrow$ (2) Suppose S is intra-regular. Let L be a left ideal , K be an interior ideal and B be a bi-ideal of S. Let $a \in K \cap B \cap L$. As S is intra-regular, $a \in S\Gamma a\Gamma a\Gamma S$. $a \in S\Gamma a\Gamma a\Gamma S \subseteq (S\Gamma S\Gamma L)\,\Gamma B\Gamma\,(S\Gamma K\Gamma S) \subseteq L\Gamma B\Gamma \mathrm{K}$. Thus we have $K \cap B \cap L \subseteq L\Gamma B\Gamma \mathrm{K}$.

(2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) Clearly implications follow , since an ideal is an interior ideal.

(2) $\Rightarrow$ (4), (4) $\Rightarrow$ (5) As every quasi-ideal is a bi-ideal, implications follow.

(5) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. As right ideal R is a quasi-ideal, and S itself is an ideal of S , $S \cap R \cap L \subseteq L\Gamma R\Gamma S$ by (5). Therefore $L\Gamma R\Gamma S \subseteq L\Gamma R$. Thus we get $R \cap L \subseteq L\Gamma R$. Therefore S is an intra-regular Γ-semiring by Theorem 4.2. ■

§5. Regular and Intra-regular Γ-semiring

Theorem 5.1: For S following statements are equivalent.

(1) S is regular and intra-regular

(2) Each right ideal R and left ideal L of S satisfy $R \cap L = R\Gamma L \subseteq L\Gamma R$.

(3) Each bi-ideal B of S satisfy $B = B^2 = B\Gamma B$.

(4) Each quasi-ideal Q of S satisfy $Q = Q^2 = Q\Gamma Q$.

Proof : (1) $\Leftrightarrow$ (2) Proof follows from the Theorems 3.1 and 4.2.

(1) $\Rightarrow$ (3) Suppose S is regular and intra-regular. Let B be a bi-ideal of S. Then $B^2 = B\Gamma B \subseteq B$. For the reverse inclusion , let $a \in B$. As S is regular, $a \in a\Gamma S\Gamma a$. Hence $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma(a\Gamma S\Gamma a) \subseteq a\Gamma S\Gamma(S\Gamma a\Gamma a\Gamma S)\Gamma S\Gamma a \subseteq.$ $(B\Gamma S\Gamma B)\Gamma(B\Gamma S\Gamma B) \subseteq B\Gamma$B. Therefore $B \subseteq B\Gamma B$. Thus we get $B = B\Gamma B = B^2$.

(3) $\Rightarrow$ (4) As every quasi-ideal is a bi-ideal, implication follows.

(4) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Then $R \cap L$ is a quasi-ideal of S. By (4) , $R \cap L = (R \cap L\ )^2 = (R \cap L)\Gamma(R \cap L) \subseteq L\Gamma R$. This shows S is an intra-regular Γ-semiring by Theorem 4.2. Similarly $R \cap L = (R \cap L\ )^2 = (R \cap L)\Gamma(R \cap L) \subseteq R\Gamma L$. Hence we get $R \cap L = R\Gamma L$. Therefore S is a regular Γ-semiring by Theorem 3.1. ■

Theorem 5.2: In S following statements are equivalent.

(1) S is a regular and an intra-regular Γ-semiring.

(2) For bi-ideals B_1 and B_2 of S, $B_1 \cap B_2 \subseteq (B_1\Gamma B_2) \cap (B_2\Gamma B_1)$.

(3) For every bi-ideal B and quasi-ideal Q of S, $B \cap Q \subseteq (Q\Gamma B) \cap (B\Gamma Q)$.

(4) For quasi-ideals Q_1 and Q_2 of S, $Q_1 \cap Q_2 \subseteq (Q_1\Gamma Q_2) \cap (Q_2\Gamma Q_1)$.

(5) For every quasi-ideal Q and generalized bi-ideal G of S, $G \cap Q \subseteq (G\Gamma Q) \cap (Q\Gamma G)$.

(6) For every left ideal L and bi-ideal B of S, $B \cap L \subseteq (B\Gamma L) \cap (L\Gamma B)$

(7) For every left ideal L and quasi-ideal Q of S, $Q \cap L \subseteq (Q\Gamma L) \cap (L\Gamma Q)$

(8) For every right ideal R and bi-ideal B of S, $B \cap R \subseteq (B\Gamma R) \cap (R\Gamma B)$.

(9) For every quasi-ideal Q and right ideal R of S, $R \cap Q \subseteq (R\Gamma Q) \cap (Q\Gamma R)$. (10) For every left ideal L and right ideal R of S, $R \cap L \subseteq (R\Gamma L) \cap (L\Gamma R)$.

Proof : (1) $\Rightarrow$ (2) Suppose S is regular and intra-regular. Let B_1 and B_2 be bi-ideals of S. Let $a \in B_1 \cap B_2$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Hence $a \in a\Gamma S\Gamma a \subseteq (a\Gamma S\Gamma S\Gamma a)\Gamma(a\Gamma S\Gamma S\Gamma a) \subseteq (B_1\Gamma S\Gamma B_1)\Gamma(B_2\Gamma S\Gamma B_2) \subseteq B_1\Gamma B_2$.

Similarly we can show that $a \in B_2\Gamma B_1$.Therefore $a \in B_1 \cap B_2$ implies $a \in B_1\Gamma B_2$ and $a \in B_2\Gamma B_1$, which shows $B_1 \cap B_2 \subseteq (B_1\Gamma B_2) \cap (B_2\Gamma B_1)$.

(2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) Follows from every quasi-ideal is a bi-ideal.

(4) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Then R and L both are quasi-ideals of S. By (4) , $R \cap L \subseteq (R\Gamma L) \cap (L\Gamma R)$. $R \cap L \subseteq L\Gamma R$ implies S is an intra-regular Γ-semiring by Theorem 4.2. Also $R \cap L \subseteq R\Gamma L$. Therefore we get $R \cap L = R\Gamma L$. Hence by Theorem 3.1, S is a regular Γ-semiring.

(1) $\Rightarrow$ (5) Suppose S is regular and intra-regular. Let G be a generalized bi-ideal and Q be quasi-ideal of S. Let $a \in G \cap Q$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Therefore $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma(a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma S\Gamma a)\Gamma(a\Gamma S\Gamma S\Gamma a)$ $\subseteq (G\Gamma S\Gamma G)\Gamma(Q\Gamma S\Gamma Q) \subseteq G\Gamma Q$. Hence $a \in G\Gamma Q$. Similarly (ii) we can show $a \in Q\Gamma G$.Therefore $a \in G \cap Q$ implies $a \in G\Gamma Q$ and $a \in Q\Gamma G$, which gives $G \cap Q \subseteq$

$(G\Gamma Q)\cap(Q\Gamma G)$.
(5) $\Rightarrow$ (1)Let L be a left ideal and R be a right ideal of S respectively. As R is a generalized bi-ideal and L is a quasi-ideal of S, proof follows from (4) $\Rightarrow$(1). (1) $\Rightarrow$ (6) Suppose S is regular and intra-regular. Let B be a bi-ideal and L be a left ideal of S. Let $a\in B\cap L$. As S is regular and intra-regular, $a\in a\Gamma S\Gamma a$ and $a\in S\Gamma a\Gamma a\Gamma S$. $a\in a\Gamma S\Gamma a\subseteq a\Gamma S\Gamma(a\Gamma S\Gamma a)\subseteq (a\Gamma S\Gamma S\Gamma a)\Gamma(a\Gamma S\Gamma S\Gamma a)\subseteq(B\Gamma S\Gamma B)\Gamma(S\Gamma S\Gamma S\Gamma L)\subseteq B\Gamma L$. Therefore we get $a\in B\Gamma L$. Similarly we can show $a\in L\Gamma B$. Therefore $a\in B\cap L$ implies $a\in B\Gamma L$ and $a\in L\Gamma B$. Hence $B\cap L\subseteq(B\Gamma L)\cap(L\Gamma B)$.
(6) $\Rightarrow$ (7) As every quasi-ideal is a bi-ideal, implication follows.
(7) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S respectively. As R is a quasi-ideal of S, proof follows from (4) $\Rightarrow$ (1).
(1) $\Rightarrow$ (8) Suppose S is regular and intra-regular. Let R be right ideal and B be a bi-ideals of S. Let $a\in B\cap R$. As S is regular and intra-regular, $a\in a\Gamma S\Gamma a$ and $a\in S\Gamma a\Gamma a\Gamma S$. Therefore $a\in a\Gamma S\Gamma a\subseteq a\Gamma S\Gamma(a\Gamma S\Gamma a)\subseteq (a\Gamma S\Gamma S\Gamma a)\Gamma(a\Gamma S\Gamma S\Gamma a)\subseteq(B\Gamma S\Gamma B)\Gamma(R\Gamma S\Gamma S\Gamma S)\subseteq B\Gamma R$. Therefore we get $a\in B\Gamma R$. Similarly we can show $a\in R\Gamma B$. Therefore $a\in B\cap R$ implies $a\in B\Gamma R$ and $a\in R\Gamma B$, which gives $B\cap R\subseteq(B\Gamma R)\cap(R\Gamma B)$.
(8) $\Rightarrow$ (9), (9) $\Rightarrow$ (10) Implications follow as every left ideal is a quasi-ideal.
(10) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Proof follows from (4) $\Rightarrow$ (1).
■

Theorem 5.3: In S following statements are equivalent.
(1) S is a regular and intra-regular Γ-semiring.
(2) For bi-ideals B_1 and B_2 of S, $B_1\cap B_2\subseteq(B_1\Gamma B_2\Gamma B_1)\cap(B_2\Gamma B_1\Gamma B_2)$
(3) For a quasi-ideal Q and a bi-ideal B of S, $Q\cap B\subseteq(B\Gamma Q\Gamma B)\cap(Q\Gamma B\Gamma Q)$.
(4) For quasi-ideals Q_1 and Q_2 of S, $Q_1\cap Q_2\subseteq(Q_1\Gamma Q_2\Gamma Q_1)\cap(Q_2\Gamma Q_1\Gamma Q_2)$.
(5) For a bi-ideal B and a left ideal L of S, $B\cap L\subseteq B\Gamma L\Gamma B$.
(6) For a quasi-ideal Q and a left ideal L of S, $Q\cap L\subseteq Q\Gamma L\Gamma Q$.
(7) For a bi-ideal B and a right ideal R of S, $B\cap R\subseteq B\Gamma R\Gamma B$.
(8) For a quasi-ideal Q and a right ideal R of S, $Q\cap R\subseteq Q\Gamma R\Gamma Q$.
(9) For a quasi-ideal Q and a generalized bi-ideal G of S, $Q\cap G\subseteq(Q\Gamma G\Gamma Q)\cap(G\Gamma Q\Gamma G)$.
Proof : (1)$\Rightarrow$ (2) Suppose S is regular and intra-regular. Let B_1 and B_2 be bi-ideals of S. Let $a\in B_1\cap B_2$. As S is regular and intra-regular, $a\in a\Gamma S\Gamma a$ and $a\in S\Gamma a\Gamma a\Gamma S$. Hence $a\in a\Gamma S\Gamma a\subseteq a\Gamma S\Gamma(a\Gamma S\Gamma a)\subseteq(a\Gamma S\Gamma a)\Gamma(a\Gamma S\Gamma a)\Gamma(a\Gamma S\Gamma a)\subseteq(B_1\Gamma S\Gamma B_1)\Gamma(B_2\Gamma S\Gamma B_2)\Gamma(B_1\Gamma S\Gamma B_1)\subseteq B_1\Gamma B_2\Gamma B_1$. Therefore $B_1\cap B_2\subseteq B_1\Gamma B_2\Gamma B_1$. In the same manner we can show $B_1\cap B_2\subseteq B_2\Gamma B_1\Gamma B_2$. Thus we get $B_1\cap B_2\subseteq(B_1\Gamma B_2\Gamma B_1)\cap(B_2\Gamma B_1\Gamma B_2)$.
(2)$\Rightarrow$ (3), (3)$\Rightarrow$ (4) Implications follow as every quasi-ideal is a bi-ideal.
(4) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Then $R\cap L$ is a quasi-ideal

of S. By (4) , $(R \cap L) \cap (R \cap L) \subseteq ((R \cap L)\,\Gamma\,(R \cap L)\,\Gamma\,(R \cap L)) \subseteq L\Gamma R\Gamma R \subseteq L\Gamma R$. Hence $R \cap L \subseteq L\Gamma R$, which shows that S is an intra-regular Γ-semiring by Theorem 4.2. Also $R \cap L \subseteq ((R \cap L)\,\Gamma\,(R \cap L)\,\Gamma\,(R \cap L))$ implies $R \cap L \subseteq R\Gamma L$. Therefore $R \cap L = R\Gamma L$. Thus S is a regular Γ-semiring by Theorem 3.1.

(1)⇒ (5) Suppose S is regular and intra-regular. Let B be a bi-ideal and L be a left ideal of S. Let $a \in B \cap L$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Therefore $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma(a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma a)\,\Gamma\,(S\Gamma a)\,\Gamma\,(a\Gamma S\Gamma a) \subseteq (B\Gamma S\Gamma B)\,\Gamma\,(S\Gamma L)\,\Gamma\,(B\Gamma S\Gamma B) \subseteq B\Gamma L\Gamma B$. Hence we have $B \cap L \subseteq B\Gamma L\Gamma B$.

(5) ⇒ (6) As every quasi-ideal is a bi-ideal, implication follows.

(6) ⇒ (7) Proof is similar to (4)⇒ (1) (1) ⇒(7) ⇒ (8) ⇒ (1) can be proved similar to (1) ⇒ (5) ⇒ (6) ⇒(1). Proof of (1) ⇒ (9) is similar to (1) ⇒ (2) and proof of (9) ⇒ (1) is parallel to (1) ⇒ (4) ⇒ (1).

Thus we have shown that (1) ⇒ (2) ⇒ (3) ⇒(4) ⇒ (1), (1) ⇒ (5) ⇒ (6) ⇒(1) and (1) ⇒ (7) ⇒ (8) ⇒ (1) and (1) ⇒ (9) ⇒(1). ■

Theorem 5.4: In S following statements are equivalent.

(1) S is regular and intra-regular.

(2) $B \cap R \cap L \subseteq B\Gamma R\Gamma L$, for a bi-ideal B , right ideal R and a left ideal L of S.

(3) $Q \cap R \cap L \subseteq Q\Gamma R\Gamma L$, for a quasi-ideal Q , right ideal R and left ideal R of S.

Proof : (1) ⇒ (2) Suppose S is regular and intra-regular. Let B be a bi-ideal , R be a right ideal and L be a left ideal of S. Let $a \in B \cap R \cap L$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Hence $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma(a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma S\Gamma a)\,\Gamma\,(a\Gamma S)\Gamma(S\Gamma a) \subseteq (B\Gamma S\Gamma B)\,\Gamma\,(R\Gamma S)\,\Gamma(S\Gamma L) \subseteq B\Gamma R\Gamma L$. Therefore $B \cap R \cap L \subseteq B\Gamma R\Gamma L$.

(2) ⇒ (3) As every quasi-ideal is a bi-ideal, implication follows.

(3) ⇒ (1) Let L be a left ideal and R be a right ideal of S respectively. As R is a quasi-ideal and S is a right ideal of , by (3) $R \cap S \cap L \subseteq R\Gamma S\Gamma L \subseteq R\Gamma L$. Hence S is a regular Γ-semiring by Theorem 3.1. Similarly L is a quasi-ideal and S is a left ideal of S gives $L \cap R \cap S \subseteq L\Gamma R\Gamma S \subseteq L\Gamma R$ by (3). Thus $R \cap L \subseteq L\Gamma R$, which shows S is an intra-regular Γ-semiring by Theorem 4.2. ■

§6. Duo Γ-semiring

Now we define a duo Γ-semiring as follows.

Definition 6.1: *A Γ-semiring S is said to be a left (right) duo Γ-semiring if every left (right) ideal of S is a right (left) ideal.*

A Γ-semiring S is said to be a duo Γ-semiring if every one sided ideal of S is a two sided ideal. That is a Γ-semiring S is said to be a duo Γ-semiring if it is both left and right duo.

Theorem 6.2: If S is a regular Γ-semiring , then S is left duo if and only if for any two left ideals A and B of S, $A \cap B = A\Gamma B$.

Proof : Let S be a regular Γ-semiring . Assume S is left duo. Let A and B be any two left ideals of S. As S is left duo, A is a right ideal of S. Then by the Theorem 3.1, $A \cap B = A\Gamma B$. Conversely, suppose the given condition hold. Let L be a left ideal of S. Then by assumption $L\Gamma S = L \cap S \subseteq L$. This shows that L is a right ideal of S. Therefore S is a left duo Γ-semiring. ■

Proof of the following theorem is analogous to proof of theorem 6.2.

Theorem 6.3: If S is a regular Γ-semiring , then S is right duo if and only if for any two right ideals A and B of S, $A \cap B = A\Gamma B$.

Theorem 6.4 : A regular Γ-semiring S is a left duo if and only if every quasi-ideal of S is a right ideal of S.

Proof : Let S be a regular Γ-semiring. Suppose S is left duo. Let Q be any quasi-ideal of S. Then there exists a right ideal R and a left ideal L of S such that $Q = R \cap L$. Therefore $Q = R \cap L$ is a right ideal of S. Conversely, let L be a left ideal of S. Then L is a quasi-ideal of S. Hence by assumption L is a right ideal of S. Therefore S is a left duo Γ-semiring. ■

Proofs of the following theorems are similar to proof of theorem 6.4.

Theorem 6.5: A regular Γ-semiring S is right duo if and only if every quasi-ideal of S is a left ideal of S.

Theorem 6.6: A regular Γ-semiring S is duo if and only if every quasi-ideal of S is an ideal of S.

Theorem 6.7: A regular Γ-semiring S is duo if and only if every bi-ideal of S is a ideal of S.

Theorem 6.8: In S following conditions are equivalent.

(1) S is a regular duo Γ-semiring.

(2) $I \cap B = I\Gamma B\Gamma I$, for every ideal I and bi-ideal B of S.

(3) $I \cap Q = I\Gamma Q\Gamma I$, for every ideal I and quasi-ideal Q of S.

Proof : (1) $\Rightarrow$ (2) Suppose S is a regular duo Γ-semiring. Let I be an ideal and B be a bi-ideal of S. Then by the Theorem 6.7, B is an ideal of S. Therefore $I\Gamma B\Gamma I \subseteq I$ and $I\Gamma B\Gamma I \subseteq B$, since I and B are ideals of S. Hence $I\Gamma B\Gamma I \subseteq I \cap B$. For the reverse inclusion , let $a \in I \cap B$. S is regular implies $a \in a\Gamma S\Gamma a$. $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma(a\Gamma S\Gamma a) \subseteq (I\Gamma S)\Gamma B\Gamma(S\Gamma I) \subseteq I\Gamma B\Gamma I$. Therefore $I \cap B \subseteq I\Gamma B\Gamma I$. Hence $I \cap B = I\Gamma B\Gamma I$.

(2) $\Rightarrow$ (3) As every quasi-ideal of S is a bi-ideal of S, implication follows.

(3) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Hence $S \cap L = S\Gamma L\Gamma S$ and $S \cap R = S\Gamma R\Gamma S$ by (3). Therefore $L = S\Gamma L\Gamma S$ and $R = S\Gamma R\Gamma S$. Now $L\Gamma S = S\Gamma L\Gamma S\Gamma S \subseteq S\Gamma L\Gamma S = L$ and $S\Gamma R = S\Gamma S\Gamma R\Gamma S \subseteq S\Gamma R\Gamma S = R$. Hence $L\Gamma S \subseteq L$ and $\Gamma R \subseteq R$, shows

that L is a right ideal and R is a left ideal of S. Therefore S is a duo Γ-semiring by definition. As S is a duo Γ-semiring , $R \cap L = R\Gamma L\Gamma R$ by (3). $R \cap L = R\Gamma L\Gamma R \subseteq R\Gamma L$. This shows $R \cap L = R\Gamma L$. By Theorem 3.1, S is a regular. ■

Theorem 6.9: If S is a Γ- semiring then following statements are equivalents.

(1) S is a regular duo Γ-semiring.

(2) For every bi-ideals A and B of S, $A \cap B = A\Gamma B$.

(3) For every bi-ideal B and quasi-ideal Q of S, $B \cap Q = B\Gamma Q$.

(4) For every bi-ideal B and right ideal R of S, $B \cap R = B\Gamma R$.

(5) For every quasi-ideal Q and bi-ideal B of S, $Q \cap B = Q\Gamma \mathrm{B}$.

(6) For every quasi-idealsl Q_1 and Q_2 of S, $Q_1 \cap Q_2 = Q_1\Gamma Q_2$.

(7) For every quasi-ideal Q and right ideal R of S, $Q \cap R = Q\Gamma R$.

(8) For every left ideal L and bi-ideal B of S, $L \cap B = L\Gamma B$.

(9) For every left ideal L and right ideal R of S, $L \cap R = L\Gamma R$.

Proof : We can prove the equivalence of statements such as (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1), (1) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (1) and (1) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (1) . Proof of each implication is straightforward so omitted. ■

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