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# SOME SUMMATION THEOREMS FOR CLAUSEN'S TERMINATING HYPERGEOMETRIC SERIES

M. I. QURESHI[1], SAIMA JABEE[1,*] AND SULAKSHANA BAJAJ[2]

[1]Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi-110025, India

[2]Department of Mathematics, Govt. College Kota, Kota, Rajasthan-324001, India

*Corresponding author: saimajabee007@gmail.com

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Abstract. In the present paper, we have obtained two summation theorems for Clausen's terminating hypergeometric series, using series rearrangement technique.

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## 1. Introduction

In the present paper, we shall use the following standard notations:
$\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}$, $\mathbb{Z}_0^- := \{0, -1, -2, -3, \ldots\}$,
$\mathbb{Z}^- := \{-1, -2, -3, \ldots\} = \mathbb{Z}_0^- \backslash \{0\}$ and $\mathbb{Z} = (\mathbb{Z}_0^- \cup \mathbb{N})$.
Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of positive real numbers and $\mathbb{C}$ denotes the set of complex numbers.
The Pochhammer symbol (or the shifted factorial) $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined, in terms of the familiar Gamma function, by
(1.1)

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & ; (\nu = 0; \lambda \in \mathbb{C}\backslash\{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & ; (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}\backslash\mathbb{Z}_0^-) \\ \frac{(-1)^n k!}{(k-n)!} & ; (\lambda = -k; \nu = n; n, k \in \mathbb{N}_0; 0 \le n \le k) \\ 0 & ; (\lambda = -k; \nu = n; n, k \in \mathbb{N}_0; n > k) \\ \frac{(-1)^n}{(1-\lambda)_n} & ; (\nu = -n; n \in \mathbb{N}; \lambda \neq 0, \pm 1, \pm 2, \ldots), \end{cases}$$

it is being understood *conventionally* that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

In the Gaussian hypergeometric series ${}_2F_1(a,b;c;z)$ ,there are two numerator parameters $a$, $b$ and one denominator parameter $c$. A natural generalization of this series is accomplished by introducing any arbitrary number of numerator and denominator parameters. The **non-terminating hypergeometric series** [3, p.42-43]

$$(1.2)\qquad {}_pF_q\left[\begin{array}{c}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{array} z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_p)_n}{(\beta_1)_n\ldots(\beta_q)_n}\frac{z^n}{n!},$$

where $(\lambda)_n$ is the pochhammer symbol defined by equation (1.1), and (1.2) is known as the *generalized Gauss and Kummer series*, or simply, the *generalized hypergeometric series.* Here $p$ and $q$ are positive integers or zero (interpreting an empty product as 1), and we assume that the variable $z$, the numerator parameters $\alpha_1,\ldots,\alpha_p$ and the denominator parameters $\beta_1,\ldots,\beta_q$ take on complex values, provided that

$$(1.3)\qquad \beta_j\neq 0,-1,-2,\ldots,\quad j=1,\ldots,q.$$

Supposing that none of the numerator parameters is zero or a negative integer (otherwise the question of convergence will not arise), and with the usual restriction (1.3), the ${}_pF_q$ series in the definition (1.2)
(i) converges for $|z|<\infty$, if $p\leqq q$
(ii) converges for $|z|<1$, if $p\ =\ q+1$ and
(iii) diverges for all z, $z\neq 0$, if $p>q+1$.
Furthermore, if we set

$$\omega=\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_j.$$

It is known that the ${}_pF_q$ series, with $p=q+1$, is
I. absolutely convergent for $|z|=1$ if $\Re(\omega)>0$,
II. conditionally convergent for $|z|=1$, $z\neq 1$, if $-1<\Re(\omega)\leqq 0$, and
III. divergent for $|z|=1$ if $\Re(\omega)\leqq -1$.

**Whipple's theorem for non-terminating Clausen's hypergeometric series** [1, p.16]

$$(1.4)\qquad {}_3F_2\left[\begin{array}{c}a,\ b,\ c;\\ g\ ,\ f\ ;\end{array} 1\right]=\frac{\pi\ \Gamma(g)\Gamma(f)}{2^{2c-1}\Gamma(\frac{a+g}{2})\Gamma(\frac{a+f}{2})\Gamma(\frac{b+g}{2})\Gamma(\frac{b+f}{2})},$$

$\left(\Re(c)>0;\ a+b=1,\ g+f=2c+1;\ a,b\in\mathbb{C};\ c,g,f\in\mathbb{C}\backslash\mathbb{Z}_0^-\right).$

**Dixon's theorem for non-terminating and terminating series** [2, p.92(Th.33), see also, p.105(Q.N.3)]

(1.5)

$${}_3F_2\left[\begin{array}{c} \alpha, \quad \beta, \quad \gamma\ ; \\ 1+\alpha-\beta, 1+\alpha-\gamma; \end{array} 1\right] = \frac{\Gamma(1+\alpha-\beta)\Gamma(1+\alpha-\gamma)\Gamma(1+\frac{\alpha}{2})\Gamma(1+\frac{\alpha}{2}-\beta-\gamma)}{\Gamma(1+\frac{\alpha}{2}-\beta)\Gamma(1+\frac{\alpha}{2}-\gamma)\Gamma(1+\alpha)\Gamma(1+\alpha-\beta-\gamma)},$$

$\left(\Re(\alpha-2\beta-2\gamma) > -2;\ \beta,\gamma \in \mathbb{C};\ 1+\alpha-\beta,\ 1+\alpha-\gamma,\ 1+\frac{\alpha}{2},\ 1+\frac{\alpha}{2}-\beta-\gamma \in \mathbb{C}\backslash\mathbb{Z}_0^-\right),$

$$(1.6) \qquad = \quad \frac{cos\left(\frac{\pi\alpha}{2}\right)sin\{\pi\left(\beta-\frac{\alpha}{2}\right)\}\Gamma(1-\alpha)\Gamma(1+\alpha-\gamma)\Gamma(\beta-\frac{\alpha}{2})\Gamma(1+\frac{\alpha}{2}-\beta-\gamma)}{sin\{\pi(\beta-\alpha)\}\Gamma(1-\frac{\alpha}{2})\Gamma(1+\frac{\alpha}{2}-\gamma)\Gamma(\beta-\alpha)\Gamma(1+\alpha-\beta-\gamma)}.$$

In equation (1.6), the numerator parameter '$\alpha$' is a non-positive integer.

**Summation property** [3, p.100(eq.1)]

$$(1.7) \qquad \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\Phi(r,s) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\Phi(r-s,s).$$

The series involved are absolutely convergent.

**Pfaff-Kummer's linear transformation** [3, p.33(eq.19)]

$$(1.8) \qquad {}_2F_1\left[\begin{array}{c} a,b; \\ c\ ; \end{array} z\right] = (1-z)^{-a}{}_2F_1\left[\begin{array}{c} a,c-b; \\ c\ ; \end{array} \frac{-z}{1-z}\right],$$

$(a,b \in \mathbb{C};\ c \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $|arg(1-z)| < \pi)$.

**Binomial theorem**

$$(1.9) \qquad {}_1F_0\left[\begin{array}{c} a\ ; \\ -; \end{array} z\right] = (1-z)^{-a}, \quad (|z|<1).$$

**Legendre's duplication formula**

$$(1.10) \qquad \sqrt{(\pi)}\ \Gamma(2z) = 2^{(2z-1)}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \quad (2z \in \mathbb{C}\backslash\mathbb{Z}_0^-).$$

**Particular case of Gauss's multiplication theorem**

$$(1.11) \qquad (\lambda)_{2n} = 2^{2n}\left(\frac{\lambda}{2}\right)_n\left(\frac{\lambda+1}{2}\right)_n, \quad n = 0,1,2,3,\ldots.$$

## 2. Whipple's theorem for terminating series

$$(2.1)\qquad {}_3F_2\left[\begin{array}{c} -m,\ 1+m,\ c; \\ g\ ,\ 2c+1-g\ ; \end{array} 1\right] = \frac{\pi\ \Gamma(g)\Gamma(2c+1-g)}{2^{2c-1}\Gamma(\frac{-m+g}{2})\Gamma(\frac{2c+1-m-g}{2})\Gamma(\frac{1+m+g}{2})\Gamma(\frac{2+2c+m-g}{2})},$$

$\big(m \in \mathbb{N}_0;\ c,\ g,\ 2c+1-g \in \mathbb{C}\backslash\mathbb{Z}_0^-\big)$.

**Proof:-** To obtain terminating hypergeometric series, any one of the numerator parameters should be a negative integer. Since $\Re(c) > 0$, therefore we cannot take $c = -m$ in (1.4), to derive a summation theorem for terminating series. Here $m$ is a non negative integer. By putting $a = -m$ in (1.4), we get the desired result (2.1).

## 3. Another summation theorem for terminating series

$$(3.1)\qquad {}_3F_2\left[\begin{array}{c} -m,\ a,\ 1-a\ ; \\ c,\ 1-c-2m\ ; \end{array} 1\right] = \frac{\left(\frac{c+a}{2}\right)_m \left(\frac{c+1-a}{2}\right)_m}{\left(\frac{c}{2}\right)_m \left(\frac{c+1}{2}\right)_m},$$

$\big(m \in \mathbb{N}_0;\ a, 1-a, c,\ 1-c-2m,\ \frac{a+c+2m}{2}, \frac{1-a+c+2m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\big)$.

**Proof:-** Consider the expression

$$(3.2)\qquad \Psi = (1-z)^{-m-c}{}_2F_1\left[\begin{array}{c} a, 1-a; \\ c\ ; \end{array} z\right].$$

On using Pfaff-Kummer's linear transformation (1.8) in equation (3.2), we get

$$(3.3)\qquad \Psi = (1-z)^{-m-c}{}_2F_1\left[\begin{array}{c} a, 1-a; \\ c\ ; \end{array} z\right] = (1-z)^{-a-m-c}{}_2F_1\left[\begin{array}{c} a, c+a-1; \\ c\ ; \end{array} \frac{z}{z-1}\right].$$

Now expressing left hand side of (3.3) into power series form, we obtain

$$(3.4)\qquad \Psi = (1-z)^{-m-c}{}_2F_1\left[\begin{array}{c} a, 1-a; \\ c\ ; \end{array} z\right] = \sum_{r=0}^{\infty} \frac{(c+m)_r\, z^r}{r!} \sum_{s=0}^{\infty} \frac{(a)_s (1-a)_s\, z^s}{(c)_s\, s!}.$$

Further using summation property (1.7) in equation (3.4) and simplifying further,we get

$$(3.5)\qquad \Psi = \sum_{r=0}^{\infty} \frac{(c+m)_r\, z^r}{r!}{}_3F_2\left[\begin{array}{c} -r,\ a,\ 1-a\ ; \\ c, 1-c-m-r\ ; \end{array} 1\right].$$

Now expressing right hand side of (3.3) into power series form, we obtain

(3.6)

$$\Psi = (1-z)^{-a-m-c}{}_2F_1\left[\begin{array}{c} a, c+a-1; \\ c\ ; \end{array} \frac{z}{z-1}\right] = \sum_{s=0}^{\infty} \frac{(a)_s (c+a-1)_s\, (-z)^s}{(c)_s\, s!} \sum_{r=0}^{\infty} \frac{(a+c+m+s)_r\, z^r}{r!}.$$

On using summation property (1.7) in equation (3.6) and simplifying further, we get

$$\Psi = \sum_{r=0}^{\infty} \frac{(c+m+a)_r\, z^r}{r!} {}_3F_2 \left[ \begin{matrix} -r, \ a, \ c+a-1; \\ c\ , \ a+m+c\ ; \end{matrix} 1 \right]. \tag{3.7}$$

Now equating the coefficients of $z^m$ in equations (3.5) and (3.7), we obtain

$${}_3F_2 \left[ \begin{matrix} -m, \ a, \ 1-a\ ; \\ c, 1-c-2m\ ; \end{matrix} 1 \right] = \frac{(a+c+m)_m}{(c+m)_m} {}_3F_2 \left[ \begin{matrix} -m, \ a, \ c+a-1; \\ c\ , \ a+m+c\ ; \end{matrix} 1 \right]. \tag{3.8}$$

Now applying Dixon's theorem (1.5) and using the algebraic properties of Pochhammer's symbol, we get the desired result (3.1).

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