Asia Pacific Journal of Mathematics, Vol. 5, No. 1 (2018), 132-138 ISSN 2357-2205

MAJORIZATION PROBLEM FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS DEFINED BY GENERALIZED OPERATOR

H. E. DARWISH[1], A.Y. LASHIN[1], R.M. EL-ASHWAH[2] AND E. M. MADAR[2,*]

[1]Department of Mathematics Faculty of Science, Mansoura University Mansoura, 35516, EGYPT
[2]Department of Mathematics Faculty of Science, Damietta University, New Damietta, 34517, EGYPT
[*]Corresponding author: entesarmadar@gmail.com

Received Jan 2, 2018

Abstract. In this paper, we investigate majorization properties for certain classes of multivalent analytic functions defined by a generalized operator. Also, we point out some new and known consequences of our main result.
2010 Mathematics Subject Classification. 30C45.
Key words and phrases. rational functions; starlike functions; convex functions; subordination; Majorization.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_k z^{p+k}, \qquad (p \in \mathbb{N} := \{1, 2, 3, ...\}, \tag{1.1}$$

which are analytic and p-valent in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also, let $\mathcal{A}_1 := \mathcal{A}$. For functions $f_j \in \mathcal{A}_p$ given by

$$f_j(z) = z^p + \sum_{k=1}^{\infty} a_{k,j} z^{p+k}, \qquad (j = 1, 2; p \in \mathbb{N}), \tag{1.2}$$

we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = z^p + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^{p+k} = (f_2 * f_1)(z).$$

Let $f(z)$ and $g(z)$ be analytic in Δ. Then we say that the function $f(z)$ is subordinate to $g(z)$ in Δ, if there exists an analytic function $w(z)$ in Δ with

$$w(0) = 0, \quad |w(z)| < 1 \ \ (z \in \Delta),$$

such that

$$f(z) = g(w(z)) \quad (z \in \Delta).$$

We denote this subordination by $f(z) \prec g(z)$. Furthermore, if the function $g(z)$ is univalent in Δ, then $f(z) \prec g(z)$ $(z \in \Delta) \Leftrightarrow f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

Suppose that the functions $f(z)$ and $g(z)$ are analytic in open unit disk Δ. Then we say that the function $f(z)$ is majorized by $g(z)$ in Δ (see [5]) and written as follows

$$f(z) \ll g(z) \quad (z \in \Delta), \tag{1.3}$$

if there exists a function $\phi(z)$, analytic in Δ, such that

$$|\phi(z)| \leq 1 \text{ and } f(z) = \phi(z)g(z) \quad (z \in \Delta).$$

The majorization (1.3) is closely related to the concept of quasi-subordination between analytic functions in Δ.

Let $\alpha_1, \alpha_2, ..., \alpha_q$ and $\beta_1, \beta_2, ..., \beta_s$ $(q, s \in \mathbb{N} \cup \{0\}, q \leq s+1)$ be complex numbers such that $\beta_\ell \neq 0, -1, -2, ...$ for $\ell \in \{1, 2, ..., s\}$. The generalized hypergeometric function ${}_qF_s$ is given by

$${}_qF_s(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k, (\alpha_2)_k, ..., (\alpha_q)_k}{(\beta_1)_k, (\beta_2)_k, ..., (\beta_s)_k} \frac{z^n}{n!}, \quad (z \in \Delta),$$

where $(x)_k$ denotes the pochhammer symbol defined by

$$(x)_k = x(x+1)(x+2)...(x+n-1) \text{ for } n \in \mathbb{N} \text{ and } (x)_0 = 1.$$

Corresponding to a function $\Re_{q,s}(\alpha_1, \beta_1; z)$ defined by

$$\Re_{q,s}(\alpha_1, \beta_1; z) := z^p \; {}_qF_s(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z), \tag{1.4}$$

Selvaraj and Selvakumarn [10] recently defined the following generalized differential operator $G_\lambda^{p,m}(\alpha_1, \beta_1)f : \mathcal{A}_p \to \mathcal{A}_p$ by

$$G_\lambda^{p,0}(\alpha_1, \beta_1)f(z) = f(z) * G_{q,s}^p(\alpha_1, \beta_1; z),$$

$$G_\lambda^{p,1}(\alpha_1, \beta_1)f(z) = (1-\lambda)(f(z) * G_{q,s}^p(\alpha_1, \beta_1; z)) + \frac{\lambda}{p} z(f(z) * G_{q,s}^p(\alpha_1, \beta_1; z))', \tag{1.5}$$

$$G_\lambda^{p,m}(\alpha_1, \beta_1)f(z) = G_\lambda^{p,1}(G_\lambda^{p,m-1}(\alpha_1, \beta_1)f(z)),$$

where $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\lambda \geq 0$.

If $f(z) \in \mathcal{A}_p$, then we have

$$G_\lambda^{p,m}(\alpha_1, \beta_1)f(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{p + \lambda k}{p}\right)^m \frac{(\alpha_1)_k, (\alpha_2)_k, ..., (\alpha_q)_k}{(\beta_1)_k, (\beta_2)_k, ..., (\beta_s)_k} a_n \frac{z^{p+k}}{n!}. \tag{1.6}$$

It can be seen that, by specializing the parameters the operator $G_\lambda^{p,m}(\alpha_1, \beta_1)f(z)$ reduces to many known and new integral and differential operators. For example, when $m = 0$ the operator $G_\lambda^{p,m}(\alpha_1, \beta_1)f(z)$ reduces to the well known Dziok-Srivastava operator [3] and for $p = 1$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, it reduces to the operator introduced by AL-Oboudi

[1]. Further we remark that, when $p = 1$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, $\alpha_2 = 1$ and $\lambda = 1$ the operator $G_\lambda^{p,m}(\alpha_1, \beta_1)f(z)$ reduces to the operator introduced by G. S. Salagean [8].

It can be easily verified from [6] that

$$\lambda z(G_\lambda^{p,m}(\alpha_1, \beta_1)f(z))' = p\, G_\lambda^{p,m+1}(\alpha_1, \beta_1)f(z) - p\,(1-\lambda)G_\lambda^{p,m}(\alpha_1, \beta_1)f(z). \tag{1.7}$$

Using the operator $G_\lambda^{p,m}(\alpha_1, \beta_1)f(z)$ we now define the following class of p-valent analytic functions.

Definition 1. *A function $f(z) \in \mathcal{A}_p$ is said to be in the class $M_{\lambda,m}^{p,j}(A, B; b, \gamma)$ of p-valent functions of complex order $b \neq 0$ in Δ if and only if*

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left(\frac{z(G_\lambda^{p,m}(\alpha_1, \beta_1)f(z))^{j+1}}{(G_\lambda^{p,m}(\alpha_1, \beta_1)f(z))^{j}} - p + j\right)\right\} \prec \frac{1 + [(B-A)\gamma + A]z}{1 + Bz} \tag{1.8}$$

$(z \in \Delta; -1 \leq B < A \leq 1, 0 \leq \gamma < 1; p \in \mathbb{N}, m, j \in \mathbb{N}_0; b \in \mathbb{C} - \{0\}\,; p > |b\lambda(A-B)(1-\gamma) + pB|)$.

It can be seen that, by specializing the parameters the class $M_{\lambda,m}^{p,j}(A, B; b, \gamma)$ reduces to many known subclasses of analytic functions. In particular, when $\gamma = 0$ the class reduces to the class $M_{\lambda,m}^{p,j}(A, B; b)$ which has recently been introduced by Selvaraj and Selvakumaran [10] and when $A = 1$, $B = -1$ and $\gamma = 0$ the class reduces to the class $M_{\lambda,m}^{p,j}(b)$ which has recently been introduced by Selvaraj and Selvakumaran [9]. Further, when $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, we have the following relationships:

(1)$M_{\lambda,0}^{1,0}(1, -1; b, 0) = S(b)$ $(b \in \mathbb{C} - \{0\})$[see Aouf [6]].

(2)$M_{\lambda,0}^{1,1}(1, -1; b, 0) = K(b)$ $(b \in \mathbb{C} - \{0\})$[see Aouf [6]].

(3)$M_{\lambda,0}^{1,0}(1, -1; 1-\alpha, 0) = S^*(\alpha)$ for $0 \leq \alpha < 1$[see Wiatrowski [11]].

The classes $S(b)$ and $K(b)$ are said to be the classes of starlike and convex functions of complex order $b \neq 0$ in Δ which were studied by Nasr and Aouf [6] and Wiatrowski [11] and $S^*(\alpha)$ is the class of starlike functions of order α in Δ.

2. Majorization problem for the class $M_{\lambda,m}^{p,j}(A, B; b, \gamma)$

Theorem 1. *Let the function $f(z)$ be in the class $\mathcal{A}_p$ and suppose that $g(z) \in M_{\lambda,m}^{p,j}(A, B; b, \gamma)$. If $(G_\lambda^{p,m}(\alpha_1, \beta_1)f(z))^{(j)}$ is majorized by $(G_\lambda^{p,m}(\alpha_1, \beta_1)g(z))^{(j)}$ in Δ for $j \in \mathbb{N}_0$, then*

$$\left|(G_\lambda^{p,m+1}(\alpha_1, \beta_1)f(z))^{(j)}\right| \leq \left|(G_\lambda^{p,m+1}(\alpha_1, \beta_1)g(z))^{(j)}\right| \text{ for } |z| \leq r_1, \tag{2.1}$$

where $r_0 = r_0(p, b, \lambda, \gamma, A, B)$ is smallest positive root of the equation

$$|b\lambda(A-B)(1-\gamma) + pB|\, r^3 - (p + 2\lambda\,|B|)r^2 - (|b\lambda(A-B)(1-\gamma) + pB| + 2\lambda)r + p = 0 \tag{2.2}$$

$$(-1 \leq B < A \leq 1; p \in \mathbb{N}, b \in \mathbb{C} - \{0\}).$$

Proof. Let

$$h(z) = 1 + \frac{1}{b}\left(\frac{z(G_\lambda^{p,m}(\alpha_1,\beta_1)f(z))^{(j+1)}}{(G_\lambda^{p,m}(\alpha_1,\beta_1)f(z))^{(j)}} - p + j\right)$$

$$(p \in \mathbb{N}; m, j \in \mathbb{N}_0; b \in \mathbb{C} - \{0\}; p > j). \tag{2.3}$$

Since $g(z) \in M_{\lambda,m}^{p,j}(b,\gamma,A,B)$, we find from (1.8) that

$$h(z) = \frac{1 + [(B-A)\gamma + A]w(z)}{1 + Bw(z)}, \tag{2.4}$$

where $w(z)$ is analytic in Δ, which satisfies the conditions

$$w(0) = 0 \text{ and } |w(z)| < 1 \quad (z \in \Delta).$$

It follows from (2.3) and (2.4) that

$$\frac{z(G_\lambda^{p,m}(\alpha_1,\beta_1)f(z))^{(j+1)}}{(G_\lambda^{p,m}(\alpha_1,\beta_1)f(z))^{(j)}} = \frac{(p-j) + [b(A-B)(1-\gamma) + (p-j)B]w(z)}{1 + Bw(z)}. \tag{2.5}$$

In view of

$$\lambda z(G_\lambda^{p,m}(\alpha_1,\beta_1)f(z))^{(j+1)} = p(G_\lambda^{p,m+1}(\alpha_1,\beta_1)f(z))^{(j)} - (p - p\lambda + \lambda j)(G_\lambda^{p,m}(\alpha_1,\beta_1)f(z))^{(j)}, \tag{2.6}$$

(2.5) immediately yields the following inequality:

$$\left|(G_\lambda^{p,m}(\alpha_1,\beta_1)g(z))^{(j)}\right| \leq \frac{p(1 + |B|\,|z|)}{p - |b\lambda(A-B)(1-\gamma) + pB|\,|z|}\left|(G_\lambda^{p,m+1}(\alpha_1,\beta_1)g(z))^{(j)}\right|. \tag{2.7}$$

Since $(G_\lambda^{p,m}(\alpha_1,\beta_1)f(z))^{(j)}$ is majorized by $(G_\lambda^{p,m}(\alpha_1,\beta_1)g(z))^{(j)}$ in Δ, there exist an analytic function $\phi(z)$
such that

$$(G_\lambda^{p,m}(\alpha_1,\beta_1)f(z))^{(j)} = \phi(z)(G_\lambda^{p,m}(\alpha_1,\beta_1)g(z))^{(j)} \tag{2.8}$$

and $|\phi(z)| \leq 1$ $(z \in \Delta)$. Thuse we have

$$z(G_\lambda^{p,m}(\alpha_1,\beta_1)f(z))^{(j+1)} = z\phi'(z)(G_\lambda^{p,m}(\alpha_1,\beta_1)g(z))^{(j)} + z\phi(z)(G_\lambda^{p,m}(\alpha_1,\beta_1)g(z))^{(j+1)}. \tag{2.9}$$

Using (2.6), in the above equation, we get

$$(G_\lambda^{p,m+1}(\alpha_1,\beta_1)f(z))^{(j)} = \frac{\lambda z}{p}\phi'(z)(G_\lambda^{p,m}(\alpha_1,\beta_1)g(z))^{(j)} + \phi(z)(G_\lambda^{p,m+1}(\alpha_1,\beta_1)g(z))^{(j)}. \tag{2.10}$$

Noting that $\phi(z)$ satisfies (cf. [4], [7])

$$|\phi'(z)| \leq \frac{1-|\phi(z)|^2}{1-|z|^2} \quad (z \in \Delta), \tag{2.11}$$

we see that

$$\begin{aligned} &\left|(G_\lambda^{p,m+1}(\alpha_1,\beta_1)f(z))^{(j)}\right| \leq \\ &\left\{|\phi(z)| + \frac{1-|\phi(z)|^2}{1-|z|^2}\,\frac{\lambda\,|z|\,(1+|B|\,|z|)}{p-|b\lambda(A-B)(1-\gamma)+pB|\,|z|}\right\} \\ &\left|(G_\lambda^{p,m+1}(\alpha_1,\beta_1)g(z))^{(j)}\right|, \end{aligned} \tag{2.12}$$

which, upon setting

$$|z| = r \quad \text{and} \quad |\phi(z)| = \rho \quad (0 \leq \rho \leq 1)$$

leads us to the following inequality:

$$\left|(G_\lambda^{p,m+1}(\alpha_1,\beta_1)f(z))^{(j)}\right| \leq \Psi(r,\rho)\left|(G_\lambda^{p,m+1}(\alpha_1,\beta_1)g(z))^{(j)}\right|, \tag{2.13}$$

where

$$\Psi(r,\rho) = \frac{-\lambda r(1+|B|\,r)\rho^2 + (1-r^2)\rho(p-|b\lambda(A-B)(1-\gamma)+pB|\,r) + \lambda r(1+|B|\,r)}{(1-r^2)(p-|b\lambda(A-B)(1-\gamma)+pB|\,|z|)}. \tag{2.14}$$

In order to determine r_1, we note that

$$\begin{aligned} r_0 &= \max\{r \in [0,1] : \Psi(r,\rho) \leq 1, \forall \rho \in [0,1]\} \\ &= \max\{r \in [0,1] : \chi(r,\rho) \leq 1, \forall \rho \in [0,1]\}, \end{aligned}$$

where

$$\begin{aligned} \chi(r,\rho) &= (1-r^2)(p-|b\lambda(A-B)(1-\gamma)+pB|\,|z|) - \\ &\quad \rho(1-r^2)(p-|b\lambda(A-B)(1-\gamma)+pB|\,r) - \\ &\quad (1-\rho^2)\lambda(1+|B|\,r)r. \end{aligned} \tag{2.15}$$

A simple calculation shows that the inequality $\chi(r,\rho) \geq 0$ is equivalent to

$$u(r,\rho) = (1-r^2)(p-|b\lambda(A-B)(1-\gamma)+pB|\,|z|) - \lambda(1+|B|\,r)r(1+\rho) \geq 0.$$

Obviously the function $u(r,\rho)$ takes its minimum value at $\rho = 1$, i.e.

$$\min\{u(r,\rho) : \rho \in [0,1]\} = u(r,1) = \upsilon(r),$$

where

$$\upsilon(r) = |b\lambda(A-B)(1-\gamma)+pB|\,r^3 - (p+2\lambda\,|B|)r^2 - (|b\lambda(A-B)(1-\gamma)+pB| + 2\lambda)r + p.$$

It follows that $\upsilon(r) \geq 0$ for all $r \in [0, r_0]$, where $r_0(p, b, \lambda, \gamma, A, B)$ is the smallest positive real root of the equation (2.2). In fact, as one can see easily, in case, i.e either $|b\lambda(A-B)(1-\gamma) + pB| \neq 0$, or if it is equal to zero, (2.2) has a unique root in the interval $(0, 1)$ and this is smallest positive root of equation (2.2). This completes the Theorem 1.

As a special case of Theorem 1, when $A = 1$ and $B = -1$, we have □

Corollary 1. [9] *Let the function $f(z)$ be in the class $\mathcal{A}_p$ and suppose that $g(z) \in M^{p,j}_{\lambda,m}(b, \gamma)$. If $(G^{p,m}_{\lambda}(\alpha_1, \beta_1)f(z))^j$ is majorized by $(G^{p,m}_{\lambda}(\alpha_1, \beta_1)g(z))'$ in Δ for $j \in \mathbb{N}_0$, then*

$$\left|(G^{p,m+1}_{\lambda}(\alpha_1, \beta_1)f(z))^j\right| \leq \left|(G^{p,m+1}_{\lambda}(\alpha_1, \beta_1)g(z))^j\right| \text{ for } |z| \leq r_1, \tag{2.16}$$

where

$$r_1 = r_1(p, b, \lambda, \gamma, A, B) := \frac{k - \sqrt{k^2 - 4p\,|p + 2b\lambda(\gamma - 1)|}}{2\,|p + 2b\lambda(\gamma - 1)|} \tag{2.17}$$

$(k := 2\lambda + p + |p + 2b\lambda(\gamma - 1)|\,;\ p \in \mathbb{N};\ b \in \mathbb{C} - \{0\};\ \lambda \geq 0)$.

Setting $A = 1$, $B = -1$, $p = 1$ and $j = 0$ in Theorem 1, we have

Corollary 2. *Let the function $f(z)$ be in the class $\mathcal{A}$ be analytic and univalent in the open disk Δ and suppose that $g(z) \in M^{1,0}_{\lambda,m}(b, \gamma)$. If $(G^{1,m}_{\lambda}(\alpha_1, \beta_1)f(z))$ is majorized by $(G^{1,m}_{\lambda}(\alpha_1, \beta_1)g(z))$ in Δ, then*

$$\left|(G^{1,m+1}_{\lambda}(\alpha_1, \beta_1)f(z))'\right| \leq \left|(G^{1,m+1}_{\lambda}(\alpha_1, \beta_1)g(z))'\right| \text{ for } |z| \leq r_2, \tag{2.18}$$

where

$$r_2 := \frac{k - \sqrt{k^2 - 4\,|1 + 2b\lambda(\gamma - 1)|}}{2\,|1 + 2b\lambda(\gamma - 1)|} \tag{2.19}$$

$$(k := 2\lambda + 1 + |1 + 2b\lambda(\gamma - 1)|\,;\ b \in \mathbb{C} - \{0\};\ \lambda \geq 0).$$

Further putting $\lambda = 1$, $m = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$ in Corollary 2, we get

Corollary 3. [2] *Let the function $f(z)$ be in the class $\mathcal{A}$ be analytic and univalent in the open disk Δ and suppose that $g(z) \in \varphi(b, \gamma)$. If $f(z)$ is majorized by $g(z)$ in Δ, then*

$$|f'(z)| \leq |g'(z)| \text{ for } |z| \leq r_3, \tag{2.20}$$

where

$$r_3 = \frac{3 + |1 + 2b(\gamma - 1)| - \sqrt{9 + 2\,|1 + 2b(\gamma - 1)| + |1 + 2b(\gamma - 1)|^2}}{2\,|1 + 2b(\gamma - 1)|}. \tag{2.21}$$

For $b = 1$, Corollary 3 reduces to the following result:

Corollary 4. [5] *Let the function $f(z) \in \mathcal{A}$ be analytic and univalent in the open disk Δ and suppose that $g(z) \in \varphi^* = \varphi^*(0)$. If $f(z)$ is majorized by $g(z)$ in Δ, then*

$$|f'(z)| \leq |g'(z)| \text{ for } |z| \leq r_4 \tag{2.22}$$

where

$$r_4 = \frac{3 + |(2\gamma - 1)| - \sqrt{9 + 2\,|(2\gamma - 1)| + |(2\gamma - 1)|^2}}{2\,|(2\gamma - 1)|}.$$

References

[1] F. M. Al-Oboudi, On univalent functions defined by a generalized Salagean operator, Int. J. Math. Math. Sci., no. 25-28, (2004), 1429-1436.

[2] O. Altintas, Ö. Özkan and H. M. Srivastava, Majorization by starlike functions of complex order, Complex Variables Theory Appl. 46, no. 3, (2001), 207-218.

[3] J. Dziok and and H. M. Srivastava, Classes of analytic function associated with the generalized hypergeometric function, Appl. Math. Comput. 103, no. 1, (1999), 1-13.

[4] A. W. Goodman, Univalent functions, Vol. I, Mariner, Tampa, FL, 1983.

[5] T. H. MacGregor, Majorization by univalent functions, Duke Math. J. 34, (1967), 95-102.

[6] M. A. Nasr and M. K. Aouf, Starlike function of complex order, J. Natur. Sci. Math. 25, no. 1, (1985), 1-12.

[7] Z. Nehari, Conformal mapping, McGraw-Hill, Inc., New york, Toronto, London, 1952.

[8] G. S. Salagean, Subclasses of univalent functions, in Complex analysis-fifth Romanian Finnish seminar, Port 1 (Bucharest), Lecture Notes in Math., 1013, Springer, Berlin, (1981), 362-372.

[9] C. Selvaraj and K. A. Selvakumarn, Majorization problem for certain classes of analytic functions, International Mathematical Forum, Vol. 6, (2011), no. 6, 289 - 294.

[10] C. Selvaraj and K. A. Selvakumarn, Majorization for certain classes of analytic functions defined by a generalized operator, European Journal of Pure and Applied Mathematics, Vol. 3, No. 6, (2010), 1048-1054.

[11] P. Wiatrowski, The coefficients of certain family of holomorphic functions, Zeszyty Nauk. Uniw. Lodz. Nauki Mat. przyrod. Ser. II No. 39 Mat., (1971), 75-85.