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SOME PROPERTIES OF QUASI-ARMENDARIZ RINGS AND THEIR GENERALIZATIONS

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Abstract. Let R be a ring and $(S, \leq)$ a strictly ordered monoid. The generalized power series ring $[[R^{S,\leq}]]$ with coefficients in R and exponents in S is a common generalization of polynomial rings, power series rings, Laurent polynomial rings, group rings, and Malcev-Neumann Laurent series rings.We initiate the study of the S-quasi-Armendariz condition on R, a generalization of the standard quasi-Armendariz condition from polynomials to generalized power series. The class of quasi-Armendariz rings includes semiprime rings, Armendariz rings, right (left) $p.q.$-Baer rings and right (left) PP rings. The S-quasi-Armendariz rings are closed under direct product. Also it is shown that, if R is a left APP-ring, then R is S-quasi-Armendariz. The a necessary and sufficient condition is given for rings under which the ring R is reflexive if and only if $[[R^{S,\leq}]]$ is reflexive ring and $r_{[[R^{S,\leq}]]}(f[[R^{S,\leq}]])$ is pure as a right ideal in $[[R^{S,\leq}]]$ for any element $f \in [[R^{S,\leq}]]$. We conclude some characterizations for generalized power series ring to be semiprime, quasi-Baer ring.

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1. **Preliminaries**

All rings considered here are associative with identity. Any concept and notation not defined here can be found in Ribenboim ([17]−[20]), Elliott and Ribenboim [5]. We will write monoids multiplicatively unless otherwise indicated. If R is a ring and X is a nonempty subset of R, then the left (right) annihilator of X in R is denoted by $\ell_R(X)(r_R(X))$.

Let $(S, \leq)$ be an ordered set. Recall that $(S, \leq)$ is artinian if every strictly decreasing sequence of elements of S is finite, and that $(S, \leq)$ is narrow if every subset of pairwise order-incomparable elements of S is finite. Thus, $(S, \leq)$ is artinian and narrow if and only if every nonempty subset of S has at least one but only a finite number of minimal elements. Let

S be a commutative monoid. Unless stated otherwise, the operation of S will be denoted additively, and the neutral element by 0. The following definition is due to Elliott and Ribenboim [5].

Let $(S, \leq)$ is a strictly ordered monoid (that is, $(S, \leq)$ is an ordered monoid satisfying the condition that, if $s, s', t \in S$ and $s < s'$, then $s + t < s' + t$), and R a ring. Let $[[R^{S,\leq}]]$ be the set of all maps $f : S \to R$ such that $supp(f) = \{s \in S | f(s) \neq 0\}$ is artinian and narrow. With pointwise addition, $[[R^{S,\leq}]]$ is an abelian additive group. For every $s \in S$ and $f, g \in [[R^{S,\leq}]]$, let $X_s(f, g) = \{(u, v) \in S \times S | u + v = s, f(u) \neq 0, g(v) \neq 0\}$. It follows from Ribenboim [20, 4.1] that $X_s(f, g)$ is finite. This fact allows one to define the operation of convolution:

$$(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)g(v).$$

Clearly, $\text{supp}(fg) \subseteq supp(f) + supp(g)$, thus by Ribenboim [18, 3.4] $\text{supp}(fg)$ is artinian and narrow, hence $fg \in [[R^{S,\leq}]]$. With this operation, and pointwise addition, $[[R^{S,\leq}]]$ becomes an associative ring, with identity element e, namely $e(0) = 1, e(s) = 0$ for every $0 \neq s \in S$. Which is called the ring of generalized power series with coefficients in R and exponents in S. Many examples and results of rings of generalized power series are given in Ribenboim ([17]−[20]), Elliott and Ribenboim [5] and Varadarajan ([12], [13]). For example, if $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the usual order, then $[[R^{\mathbb{N}\cup\{0\},\leq}]] \cong R[[x]]$, the usual ring of power series. If S is a commutative monoid and $\leq$ is the trivial order, then $[[R^{S,\leq}]] \cong R[S]$, the monoid ring of S over R. Further examples are given in Ribenboim [18]. To any $r \in R$ and $s \in S$, we associate the maps $c_r, e_s \in [[R^{S,\leq}]]$ defined by

$$c_r(x) = \begin{cases} r, & x = 0, \\ 0, & \text{otherwise}, \end{cases} \qquad e_s(x) = \begin{cases} 1, & x = s, \\ 0, & \text{otherwise}. \end{cases}$$

It is clear that $r \mapsto c_r$ is a ring embedding of R into $[[R^{S,\leq}]], s \mapsto e_s$, is a monoid embedding of S into the multiplicative monoid of the ring $[[R^{S,\leq}]]$, and $c_r e_s = e_s c_r$. Recall that a monoid S is torsion-free if the following property holds: If $s, t \in S$, if k is an integer, $k \geq 1$ and $ks = kt$, then $s = t$.

In this paper we give a new concept of S-quasi-Armendariz ring, which are a common generalization of quasi-Armendariz rings and S-Armendariz rings. We prove that, if R is a left APP-ring, then R is S-quasi-Armendariz. Moreover, a ring R is reflexive ring if and only if $[[R^{S,\leq}]]$ is reflexive ring and (1) $r_R(a)R)$ is pure as a right ideal in R for any element $a \in R$; (2) $r_{[[R^{S,\leq}]]}(f[[R^{S,\leq}]])$ is pure as a right ideal in $[[R^{S,\leq}]]$ for any element $f \in [[R^{S,\leq}]]$ in that case R is S-quasi-Armendariz ring, where $(S, \leq)$ be a strictly ordered monoid. Also as a Corollary, a ring R is a quasi-Baer ring if and only if $[[R^{S,\leq}]]$ is quasi-Baer ring and we give a lattice structure to the right (left) annihilators of a ring and characterize S-quasi-Armendariz rings as those rings R for which an analogue of the Hirano map is a lattice isomorphism from the right (left) annihilators of R to the right (left) annihilators of $[[R^{S,\leq}]]$.

2. Generalization of quasi-Armendariz rings

We start by the following definition:

Definition 2.1. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. We say a ring R, S-quasi-Armendariz, if whenever $f, g \in [[R^{S,\leq}]]$ satisfy $f[[R^{S,\leq}]]g = 0$, then $f(u)Rg(v) = 0$ for each $u, v \in S$.*

The following result appeared in [24, Lemma 2.1].

Lemma 2.2. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. Then $[[R^{S,\leq}]]$ is reduced if and only if R is reduced.*

Reduced rings are semicommutative. From Proposition 2.4 reduced rings are S-quasi-Armendariz for any torsion free and cancellative monoid S. In [23, Corollary 2.3] it was claimed that all semicommutative rings are McCoy. However, Hirano's claim that, if R is semicommutative then $R[x]$ is semicommutative, but this was later shown to be false in [2, Example 2]. Moreover, Nielsen [15] gave an example to show that a semicommutative ring R need not be right McCoy, we also prove that the polynomial ring $R[x]$ over it actually is not semicommutative. By Liu [24], A ring R is called S-Armendariz ring, if for each $f, g \in [[R^{S,\leq}]]$ such that $fg = 0$ implies that $f(u)g(v) = 0$ for each $u, v \in S$ and it was shown that generalized power series rings over semicommutative rings are semicommutative. Here we have the following.

Lemma 2.3. [24, Proposition 2.7] *Let $(S, \leq)$ be a strictly ordered monoid and R be an S-Armendariz ring. Then R is semicommutative if and only if $[[R^{S,\leq}]]$ is semicommutative.*

Proposition 2.4. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S and R a reduced ring. Then R is an S-quasi-Armendariz.*

Proof. Let $0 \neq f, g \in [[R^{S,\leq}]]$ be such that $f[[R^{S,\leq}]]g = 0$. By Ribenboim [18], there exists a compatible strict total order $\leq'$ on S, which is finer than $\leq$. We will use transfinite induction on the strictly totally ordered set $(S, \leq)$ to show that $f(u)Rg(v) = 0$ for any $u \in supp(f)$ and $v \in supp(g)$. Let s and t denote the minimum elements of $\text{supp}(f)$ and $\text{supp}(g)$ in the $\leq'$ order, respectively. If $u \in supp(f)$ and $v \in supp(g)$ are such that $u + v = s + t$, then $s \leq' u$ and $t \leq' v$. If $s <' u$ then $s + t <' u + v = s + t$, a contradiction. Thus $u = s$. Similarly, $v = t$. Hence for any $r \in R, 0 = (fc_r g)(s + t) = \sum_{(u,v)\in X_{s+t}(f,c_r g)} f(u)rg(v) = f(s)rg(t)$.

Now suppose that $w \in S$ is such that for any $u \in supp(f)$ and $v \in supp(g)$ with $u + v <' w$, $f(u)Rg(v) = 0$. We will show that $f(u)Rg(v) = 0$ for any $u \in supp(f)$ and $v \in supp(g)$ with $u + v = w$. We write $X_w(f, g) = \{(u, v) \mid u + v = w, u \in supp(f), v \in supp(g)\}$ as $\{(u_i, v_i) \mid i = 1, 2, \ldots, n\}$ such that

$$u_1 <' u_2 <' \cdots <' u_n.$$

Since S is cancellative, $u_1 = u_2$ and $u_1 + v_1 = u_2 + v_2 = w$ imply $v_1 = v_2$. Since $\leq'$ is a strict order, $u_1 <' u_2$ and $u_1 + v_1 = u_2 + v_2 = w$ imply $v_2 <' v_1$. Thus we have

$$v_n <' \cdots <' v_2 <' v_1.$$

Now, for any $r \in R$,

$$0 = (fc_r g)(w) = \sum_{(u,v) \in X_w(f, c_r g)} f(u)rg(v) = \sum_{i=1}^{n} f(u_i)rg(v_i). \qquad (1)$$

For any $i \geq 2, u_1 + v_i <' u_i + v_i = w$, and thus, by induction hypothesis, we have $f(u_1)Rg(v_i) = 0$. Since R is reduced, by Lemma 2.2 this implies $g(v_i)Rf(u_1) = 0$. Hence, multiplying (1) on the right by $f(u_1)g(v_1)$, we obtain

$$\Big(\sum_{i=1}^{n} f(u_i)rg(v_i)\Big) f(u_1)g(v_1) = f(u_1)g(v_1)rf(u_1)g(v_1) = 0.$$

Then $(f(u_1)rg(v_1))^2 = 0$. Since R is reduced, we have $f(u_1)rg(v_1) = 0$. Now (1) becomes

$$\sum_{i=2}^{n} f(u_i)rg(v_i) = 0. \qquad (2)$$

Multiplying $f(u_2)g(v_2)$ on (2) from the right-hand side, we obtain $f(u_2)rg(v_2) = 0$ by the same way as the above. Continuing this process, we can prove $f(u_i)rg(v_i) = 0$ for any $r \in R$, for $i = 1, 2, \ldots, n$. Thus $f(u)Rg(v) = 0$ for any $u \in supp(f)$ and $v \in supp(g)$ with $u+v = w$. Therefore, by transfinite induction, $f(u)Rg(v) = 0$ for any $u \in supp(f)$ and $v \in supp(g)$. □

Corollary 2.5. [24, Lemma 3.1] *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S, and R a reduced ring. Then R is S-Armendariz.*

Proposition 2.6. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. If R is reduced semicommutative ring, then R is S-Armendariz if and only if R is S-quasi-Armendariz.*

Proof. Apply Lemma 2.3 and Proposition 2.4. □

Proposition 2.7. *Let $(S, \leq)$ be a strictly ordered monoid. Then every S-Armendariz rings are S-quasi-Armendariz.*

An ideal I of R is said to be right s-unital if, for each $a \in I$ there exists an element $e \in I$ such that $ae = a$. Note that if I and J are right s-unital ideals, then so is $I \cap J$ (if $a \in I \cap J$, then $a \in aIJ \subseteq a(I \cap J)$).

The following result follows from Tominaga [11, Theorem 1].

Lemma 2.8. *An ideal I of a ring R is left (resp. right) s-unital if and only if for any finitely many elements $a_1, a_2, \dots, a_n \in I$, there exists an element $e \in I$ such that $a_i = ea_i$(resp. $a_i = a_i e$) for each $i = 1, 2, \dots, n$.*

Clark defined quasi-Baer rings in [22]. A ring R is called quasi-Baer if the left annihilator of every left ideal of R is generated by an idempotent. Note that this definition is left-right symmetric. Some results of a quasi-Baer ring can be found in [16] and [22] and used them to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. As a generalization of quasi-Baer rings, Birkenmeier, Kim and Park in [10] introduced the concept of principally quasi-Baer rings. A ring R is called left principally quasi-Baer (or simply left $p.q.$-Baer) if the left annihilator of a principal left ideal of R is generated by an idempotent. Similarly, right $p.q.$-Baer rings can be defined. A ring is called $p.q.$-Baer if it is both right and left $p.q.$-Baer. Observe that biregular rings and quasi-Baer rings are $p.q.$-Baer. For more details and examples of left $p.q.$-Baer rings, see ([7]-[10]) and [27]. A ring R is called a right (resp., left) PP-ring if every principal right (resp., left) ideal is projective (equivalently, if the right (resp., left) annihilator of an element of R is generated (as a right (resp., left) ideal) by an idempotent of R). A ring R is called a PP-ring (also called a Rickart ring [3, p. 18]) if it is both right and left PP. We say a ring R is a left APP-ring if the left annihilator $l_R(Ra)$ is right s-unital as an ideal of R for any element $a \in R$. This concept is a common generalization of left $p.q.$-Baer rings and right PP-rings

Proposition 2.9. *Let $(S, \leq)$ a strictly totally ordered monoid. If R is left APP-ring, then R is S-quasi-Armendariz.*

Proof. Let $0 \neq f, g \in [[R^{S,\leq}]]$ be such that $f[[R^{S,\leq}]]g = 0$. We use the transfinite induction to show that $f(u)Rg(v) = 0$ for all $u, v \in S$. Assume that $\pi(f) = u_0, \pi(g) = v_0$. Let $(u, v) \in X_{u_0+v_0}(f, g)$. So $u_0 \leq u$ and $v_0 \leq v$. If $u_0 < u$, then $u_0+v_0 < u+v_0 \leq u+v = u_0+v_0$, a contradiction. Thus $u = u_0$. Similarly, $v = v_0$. So $X_{u_0+v_0}(f, g) = \{(u_0, v_0)\}$. Hence for any $r \in R$, from $f[[R^{S,\leq}]]g = 0$ we have,

$$0 = (fc_r g)(u_0 + v_0) = \sum_{(u,v)\in X_{u_0+v_0}(f,c_r g)} f(u)rg(v) = f(u_0)rg(v_0).$$

So $f(u_0)Rg(v_0)=0$. Now, let $\lambda \in S$ with $u_0+v_0 \leq \lambda$ and assume that for any $u \in supp(f)$ and any $v \in supp(g)$, if $u+v<\lambda$, then $f(u)Rg(v)=0$. We claim that $f(u)Rg(v)=0$, for each $u \in supp(f)$ and each $v \in supp(g)$ with $u+v=\lambda$. For convenience, we write $X_\lambda(f,g)=\{(u,v) \mid u+v=\lambda, u \in supp(f), v \in supp(g)\}$ as $\{(u_i,v_i) \mid i=1,2,\ldots,n\}$ such that

$$u_1 < u_2 < \cdots < u_n,$$

where n is a positive integer (Note that if $u_1=u_2$, then from $u_1+v_1=u_2+v_2$ we have $v_1=v_2$, and then $(u_1,v_1)=(u_2,v_2)$). Since $f[[R^{S,\leq}]]g=0$, for any $r \in R$ we have:

$$0=(fc_rg)(\lambda)=\sum_{(u,v)\in X_\lambda(f,c_rg)} f(u)rg(v)=\sum_{i=1}^{n} f(u_i)rg(v_i). \tag{3}$$

Let $e_{u_1} \in r_R(f(u_1)R)$. So $f(u_1)Re_{u_1}=0$ and which implies $f(u_1)Re_{u_1}g(v_1)=0$. Let $r' \in R$ be an arbitrary element. Then we have $f(u_1)r'e_{u_1}g(v_1)=0$. Take $r=r'e_{u_1}$ in Eq. (3). Thus,

$$\sum_{i=2}^{n} f(u_i)r'e_{u_1}g(v_i)=0.$$

Note that $u_1+v_i<u_i+v_i=\lambda$ for any $i \geq 2$. So by compatibility and induction hypothesis, $f(u_1)Rg(v_i)=0$ for each $i \geq 2$. Since R is right APP, $r_R(f(u_1)R)$ is left s-unital. So without lose of generality and using Lemma 2.8, we can assume that $g(v_i)=e_{u_1}g(v_i)$, for each $i \geq 2$. Therefore

$$\sum_{i=2}^{n} f(u_i)r'g(v_i)=0. \tag{4}$$

Let $e_{u_2} \in r_R(f(u_2)R)$. So $f(u_2)Re_{u_2}=0$ and then $f(u_2)Re_{u_2}g(v_2)=0$. This implies $f(u_2)Re_{u_2}g(v_2)=0$.

Let $p \in R$ be an arbitrary element. So $f(u_2)pe_{u_2}g(v_2)=0$. Also note that $u_2+v_i<u_i+v_i=\lambda$ for any $i \geq 3$. So by induction hypothesis, $f(u_2)Rg(v_i)=0$. Therefore $g(v_i) \in r_R(f(u_2)R)$, for each $i \geq 3$. Since $r_R(f(u_2)R)$ is left s-unital, without lose of generality and using Lemma 2.8, again we can assume that $g(v_i)=e_{u_2}g(v_i)$, for each $i \geq 3$. Take $r'=pe_{u_2}$ in Eq. (4), so we have:

$$\sum_{i=2}^{n} f(u_i)pe_{u_2}g(v_i)=0. \tag{5}$$

Continuing in this manner, we have $f(u_n)qg(v_n)=0$, where q is an arbitrary element of R. Thus $f(u_n)Rg(v_n)=0$. Hence $f(u_{n-1})Rg(v_{n-1})=0,\ldots,f(u_2)Rg(v_2)=0, f(u_1)Rg(v_1)=0$. Therefore, by transfinite induction, $f(u)Rg(v)=0$ for any $u,v \in S$, and the proof is complete. □

Corollary 2.10. *Let $(S,\leq)$ a strictly totally ordered monoid. If I is a finitely generated left ideal of R then for all $a \in l_R(I), a \in al_R(I)$. So R is S-quasi-Armendariz.*

Proof. By Proposition 2.9 and [26, Proposition 2.6]. $\square$

Proposition 2.11. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S and R an S-quasi-Armendariz ring. If $f_1, \dots, f_n \in [[R^{S,\leq}]] \triangleq \Lambda$ are such that $f_1 \Lambda f_2 \Lambda \cdots \Lambda f_n = 0$, then $f_1(u_1)Rf_2(u_2)R \cdots Rf_n(u_n) = 0$ for all $u_1, u_2, \dots, u_n \in S$.*

Proof. Assume that $f_1 \Lambda f_2 \Lambda \cdots \Lambda f_n = 0$. Then for any $g_2, g_3, \dots, g_{n-1} \in \Lambda$,

$$f_1 \Lambda (f_2 g_2 \cdots g_{n-1} f_n) = 0.$$

Since R is S-quasi-Armendariz, we have

$$f_1(u_1)R((f_2 g_2 \cdots g_{n-1} f_n)(v)) = 0$$

for any $u_1, v \in S$. Thus

$$(C_{f_1(u_1)r_1}(f_2 g_2 \cdots g_{n-1} f_n))(v) = 0$$

for any $r_1 \in R$ and any $v \in S$. So $C_{f_1(u_1)r_1} f_2 g_2 \cdots g_{n-1} f_n = 0$, therefore $C_{f_1(u_1)r_1} f_2 \Lambda \cdots \Lambda f_n = 0$, for any $r_1 \in R$. Thus

$$(C_{f_1(u_1)r_1} f_2) \Lambda (f_3 g_3 \cdots g_{n-1} f_n) = 0.$$

By the hypothesis, we have

$$(C_{f_1(u_1)r_1} f_2)(u_2) R (f_3 g_3 \cdots g_{n-1} f_n)(z) = 0$$

for any $u_2, z \in S$. Yields

$$f_1(u_1) r_1 f_2(u_2) R (f_3 g_3 \cdots g_{n-1} f_n)(z) = 0.$$

So $f_1(u_1) r_1 f_2(u_2) r_2 (f_3 g_3 \cdots g_{n-1} f_n)(z) = 0$, for any $r_1, r_2, \in R$. Thus

$$(C_{f_1(u_1)r_1 f_2(u_2) r_2} f_3 g_3 \cdots g_{n-1} f_n)(z) = 0,$$

for any $r_1, r_2, \in R$ and any $z \in S$. So $C_{f_1(u_1)r_1 f_2(u_2)r_2} f_3 g_3 \cdots g_{n-1} f_n = 0$ for any $g_3, \dots, g_{n-1} \in \Lambda$. Thus,

$$C_{f_1(u_1)r_1 f_2(u_2) r_2} f_3 \Lambda \cdots \Lambda f_n = 0.$$

Since R is S-quasi-Armendariz. Repeating this process, we can get

$$C_{f_1(u_1)r_1 f_2(u_2) r_2 \cdots r_{n-1} f_n(u_n)} = 0.$$

So $f_1(u_1) r_1 f_2(u_2) r_2 \cdots r_{n-1} f_n(u_n) = 0$ for any $u_1, u_2, \dots, u_n \in S$ and any $r_1, r_2, \dots,$ $r_{n-1} \in R$. Therefore $f_1(u_1)Rf_2(u_2)R \cdots Rf_n(u_n) = 0$ for any $u_1, u_2, \dots, u_n \in S$. $\square$

The following is a generalization of Proposition 2.4.

Corollary 2.12. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S and R a reduced ring. If $f_1, \ldots, f_n \in [[R^{S,\leq}]] \triangleq \Lambda$ are such that $f_1 \Lambda f_2 \Lambda \cdots \Lambda f_n = 0$, then $f_1(u_1)Rf_2(u_2)R \ldots Rf_n(u_n) = 0$ for all $u_1, u_2, \ldots, u_n \in S$.*

Proposition 2.13. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S, and R a reduced ring. Then $fRg = 0$ if and only if $f[[R^{S,\leq}]]g = 0$.*

Proof. ($\Rightarrow$) Assume that $0 \neq f, g \in [[R^{S,\leq}]]$ are such that $fRg = 0$. By Corollary 2.5, R is S-Armendariz, so for any $h \in [[R^{S,\leq}]]$ and any $s \in S$,

$$(fhg)(s) = \sum_{(u,w,v) \in X_s(f,h,g)} f(u)h(w)g(v) = 0.$$

Thus $fhg = 0$. This show that $f[[R^{S,\leq}]]g = 0$. The "only if part" is clear. $\square$

According to [6], a right ideal I is reflexive if $xRy \in I$ implies $yRx \in I$ for $x, y \in R$. Hence we shall call a ring R a reflexive ring if 0 is a reflexive ideal (i.e., $aRb = 0$ implies $bRa = 0$ for $a, b \in R$). Moreover, a right ideal I is called completely reflexive if $xy \in I$ implies $yx \in I$. A ring R is completely reflexive if (0) has the corresponding property. It is clear that every completely reflexive ring is reflexive.

Proposition 2.14. *Let $(S, \leq)$ be a strictly totally ordered monoid and R be an S-quasi-Armendariz ring. Then R is reflexive ring if and only if $[[R^{S,\leq}]]$ is reflexive ring.*

Proof. ($\Rightarrow$) Let R be reflexive ring. Suppose that $f, g \in [[R^{S,\leq}]]$ are such that $f[[R^{S,\leq}]]g = 0$. Since R is S-quasi-Armendariz, we have $f(u)Rg(v) = 0$ for any $u \in supp(f)$ and $v \in supp(g)$. But R is reflexive, so $g(v)Rf(u) = 0$ for all $u, v \in S$. Now for any $h \in [[R^{S,\leq}]]$ and any $s \in S$,

$$(ghf)(s) = \sum_{(v,w,u) \in X_s(g,h,f)} g(v)h(w)f(u) = 0.$$

Thus $ghf = 0$. This show that $g[[R^{S,\leq}]]f = 0$. This means that $[[R^{S,\leq}]]$ is reflexive. ($\Leftarrow$) Let $a, b \in R$ be such that $aRb = 0$. Then $C_a[[R^{S,\leq}]]C_b = 0$. Hence $C_b[[R^{S,\leq}]]C_a = 0$ by reflexive. So $bRa = 0$. Therefore R is reflexive. $\square$

Corollary 2.15. *Let $(S, \leq)$ be a strictly totally ordered monoid and R a reduced ring. Then R is reflexive ring if and only if $[[R^{S,\leq}]]$ is reflexive.*

Due to Hirano [23]. A ring R is called quasi-Armendariz provided that $a_iRb_j = 0$ for all i, j whenever $f(x) = a_0 + a_1x + \cdots + a_nx^n, g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)R[x]g(x) = 0$.

Corollary 2.16. [14, Proposition 3.2] *Let R be a quasi-Armendariz ring, then the following statements are equivalent:*

(1) *R is reflexive.*
(2) *$R[x]$ is reflexive.*
(3) *$R[x; x^{-1}]$ is reflexive.*

A ring R is called semiprime if for any $a \in R$, $aRa = 0$, implies $a = 0$. Let R be a ring and $(S, \leq)$ a strictly totally ordered monoid. A ring R is called S-semiprime if $f[[R^{S,\leq}]]f = 0$, then $f = 0$ for each $f \in [[R^{S,\leq}]]$.

The following result appeared in [25, Lemma 2.7]

Lemma 2.17. *Let R be a ring and $(S, \leq)$ a strictly totally ordered monoid. Then R is a semiprime ring if and only if $[[R^{S,\leq}]]$ is a semiprime ring.*

Proposition 2.18. *Let $(S, \leq)$ be a strictly totally ordered monoid. If R is a semiprime, then R is S-quasi-Armendariz.*

Proof. It follows from Proposition 2.9. □

Corollary 2.19. *If S be a commutative, torsion-free, and cancellative monoid, then every semiprime ring R is S-quasi-Armendariz.*

Corollary 2.20. [23, Corollary 3.8] *A semiprime ring is a quasi-Armendariz ring.*

Corollary 2.21. *Let R be a ring and $(S, \leq)$ a strictly totally ordered monoid. If R is semiprime, then $[[R^{S,\leq}]]$ is S-quasi-Armendariz ring.*

Corollary 2.22. *Let R be a ring and $(S, \leq)$ a strictly totally ordered monoid. Assume that R is semiprime. Then R is reflexive ring if and only if $[[R^{S,\leq}]]$ is reflexive.*

Theorem 2.23. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. Then the following conditions are equivalent:*
(1) *R is semiprime;*
(2) *R is reduced S-quasi-Armendariz.*

Proof. (1) $\Rightarrow$ (2) Is trivial.
(2) $\Rightarrow$ (1) Let R be a reduced S-quasi-Armendariz. In particular for any $0 \neq f \in [[R^{S,\leq}]]$ be such that $f[[R^{S,\leq}]]f = 0$, then $f(u)Rf(u) = 0$. Thus, $(Rf(u))^2 = 0$ since R is reduced. Therefore $f(u) = 0$. □

Let I be an index set and R_i be a ring for each $i \in I$. Let $(S, \leq)$ be a strictly ordered monoid, if there is an injective homomorphism $f : R \to \prod_{i \in I} R_i$ such that, for each $j \in I, \pi_j f : R \to R_j$ is a surjective homomorphism, where $\pi_j : \prod_{i \in I} R_i \to R_j$ is the jth projection. We have the following.

Proposition 2.24. *Let R_i be a ring, $(S, \leq)$ a strictly totally ordered monoid, for each i in a finite index set I. If R_i is S-quasi-Armendariz for each i, then $R = \prod_{i \in I} R_i$ is S-quasi-Armendariz.*

Proof. Let $R = \prod\limits_{i \in I} R_i$ be the direct product of rings $(R_i)_{i \in I}$ and R_i is S-quasi-Armendariz for each $i \in I$. Denote the projection $R \to R_i$ as Π_i. Suppose that $f, g \in [[R^{S,\leq}]]$ are such that $f[[R^{S,\leq}]]g = 0$. Set $f_i = \prod_i f$, $g_i = \prod_i g$ and $h_i = \prod_i h$. Then $f_i, g_i \in [[R_i^{S,\leq}]]$. For any $u, v \in S$, assume $f(u) = (a_i^u)_{i \in I}$, $g(v) = (b_i^v)_{i \in I}$. Now, for any $h \in [[R^{S,\leq}]]$, any $r \in R$ and any $s \in S$,

$$\begin{aligned}
(fc_r g)(s) &= \sum_{(u,v) \in X_s(f, c_r g)} f(u) r g(v) \\
&= \sum_{(u,v) \in X_s(f, c_r g)} (a_i^u)_{i \in I} (r_i)_{i \in I} (b_i^v)_{i \in I} \\
&= \sum_{(u,v) \in X_s(f, c_r g)} ((a_i^u) r_i (b_i^v))_{i \in I} \\
&= \sum_{(u,v) \in X_s(f, c_r g)} (f_i(u) r_i g_i(v))_{i \in I} \\
&= \Bigg(\sum_{(u,v) \in X_s(f_i, c_{r_i} g_i)} f_i(u) r_i g_i(v) \Bigg)_{i \in I} \\
&= ((f_i h_i g_i)(s))_{i \in I}.
\end{aligned}$$

Since $(fc_r g)(s) = 0$ we have

$$(f_i c_{r_i} g_i)(s) = 0.$$

Thus, $f_i h_i g_i = 0$. Now it follows $f_i(u) r_i g_i(v) = 0$ for any $r \in R$, any $u, v \in S$ and any $i \in I$, since R_i is S-quasi-Armendariz. Hence, for any $u, v \in S$,

$$f(u) r g(v) = (f_i(u)(r_i) g_i(v))_{i \in I} = 0$$

since I is finite. Thus, $f(u) R g(v) = 0$. This means that R is S-quasi-Armendariz. □

3. Characterizations generalized power series quasi-Armendariz rings via annihilators

In this section we give a lattice structure to the right (left) annihilators of a ring and characterize S-quasi-Armendariz rings as those rings R for which an analogue of the Hirano [23] map is a lattice isomorphism from the right (left) annihilators of R to the right (left) annihilators of $[[R^{S,\leq}]]$.

Let $\gamma = C(f)$ be the content of f, i.e., $C(f) = \{f(u) | u \in supp(f)\} \subseteq R$. Since, $R \simeq c_R$ we can identify, the content of f with

$$c_{C(f)} = \{c_{f(u_i)} | u_i \in supp(f)\} \subseteq [[R^{S,\leq}]].$$

Lemma 3.1. [21, Lemma 2.1] *Let R be a ring, S a strictly ordered monoid, $[[R^{S,\leq}]]$ the generalized power series ring and $U \subseteq R$. Then*

$$[[R^{S,\leq}]]\ell_R(U) = \ell_{[[R^{S,\leq}]]}(U), (r_R(U)[[R^{S,\leq}]] = r_{[[R^{S,\leq}]]}(U)).$$

By Lemma 3.1 we have two maps $\phi : rAnn_R(id(R)) \to rAnn_{[[R^{S,\leq}]]}(id([[R^{S,\leq}]]))$ and $\psi : lAnn_R(id(R)) \to lAnn_{[[R^{S,\leq}]]}(id([[R^{S,\leq}]]))$ defined by $\phi(I) = I[[R^{S,\leq}]]$ and $\psi(J) = [[R^{S,\leq}]]J$ for every $I \in rAnn_R(id(R)) = \{r_R(U)|U$ is an ideal of $R\}$ and $J \in lAnn_R(id(R)) = \{l_R(U)|U$ is an ideal of $R\}$, respectively. Obviously, ϕ is injective. In the following Theorem we show that ϕ and ψ are bijective maps if and only if R is S-quasi-Armendariz. This Theorem is a generalization of a result of Hashemi ([4, Proposition 2.1]) that generalizes a result of Hirano ([23, Proposition 3.4]).

Theorem 3.2. *Let R be a ring, S a strictly ordered monoid and $[[R^{S,\leq}]]$ the generalized power series. Then the following are equivalent:*
(1) *R is generalized power series quasi-Armendariz ring.*
(2) *The function $\phi : rAnn_R(id(R)) \to rAnn_{[[R^{S,\leq}]]}(id([[R^{S,\leq}]]))$ is bijective, where $\phi(I) = I[[R^{S,\leq}]]$.*
(3) *The function $\psi : lAnn_R(id(R)) \to lAnn_{[[R^{S,\leq}]]}(id([[R^{S,\leq}]]))$ is bijective, where $\psi(J) = [[R^{S,\leq}]]J$.*

Proof. (1)⇒(2) Let $Y \subseteq [[R^{S,\leq}]]$ and $\gamma = \cup_{f\in Y} C(f)$. From Lemma 3.1 it is sufficient to show that $r_{[[R^{S,\leq}]]}(f) = r_R C(f)[[R^{S,\leq}]]$ for all $f \in Y$. In fact, let $g \in r_{[[R^{S,\leq}]]}(f)$ and for any $h \in [[R^{S,\leq}]]$. Then $fhg = 0$ and by assumption $f(u_i)tg(v_j) = 0$ for each $u_i \in supp(f), t \in R$ and each $v_j \in supp(g)$. Then for a fixed $u_i \in supp(f), t \in R$ and each $v_j \in supp(g), 0 = f(u_i)tg(v_j) = (c_{f(u_i)}c_t g)(v_j)$ and it follows that $g \in r_R \cup_{u_i \in supp(f)} c_{f(u_i)}c_t[[R^{S,\leq}]] = r_R C(f)[[R^{S,\leq}]]$. So $r_{[[R^{S,\leq}]]}(f) \subseteq r_R C(f)[[R^{S,\leq}]]$.

Conversely, let $g \in r_R C(f)[[R^{S,\leq}]]$, then $c_{f(u_i)}c_t g = 0$ for each $u_i \in supp(f), t \in R$. Hence, $0 = (c_{f(u_i)}c_t g)(v_j) = f(u_i)tg(v_j)$ for each $u_i \in supp(f), t \in R$ and $v_j \in supp(g)$. Thus,

$$(fhg)(s) = \sum_{(u_i,v_j)\in X_s(f,c_t g)} f(u_i)tg(v_j) = 0$$

and it follows that $g \in r_{[[R^{S,\leq}]]}(f)$. Hence $r_R C(f)[[R^{S,\leq}]] \subseteq r_{[[R^{S,\leq}]]}(f)$ and it follows that $r_R C(f)[[R^{S,\leq}]] = r_{[[R^{S,\leq}]]}(f)$. So

$$r_{[[R^{S,\leq}]]}(Y) = \cap_{f\in Y} r_{[[R^{S,\leq}]]}(f) = \cap_{f\in Y}\, r_R C(f)[[R^{S,\leq}]] = r_R(\gamma)[[R^{S,\leq}]].$$

(2)⇒(1) Suppose that $f, g \in [[R^{S,\leq}]]$ be such that $f[[R^{S,\leq}]]g = 0$. Then $g \in r_{[[R^{S,\leq}]]}(f)$ and by assumption $r_{[[R^{S,\leq}]]}(f) = \gamma[[R^{S,\leq}]]$ for some right ideal γ of R. Consequently, $0 = fc_t c_{g(v_j)}$ and for any $u_i \in supp(f), 0 = (fc_t c_{g(v_j)})(u_i) = f(u_i)tg(v_j)$ for each $u_i \in supp(f), t \in R$ and

$v_j \in supp(g)$. Hence, R is a generalized power series quasi-Armendariz ring. The proof of (1)⇔(3) is similar to the proof of (1)⇔(2). □

Definition 3.3. *A submodule N of a left R-module M is called a pure submodule if $L \otimes_R N \to L \otimes_R M$ is a monomorphism for every right R-module L. By* [1, Proposition 11.3.13]*, for an ideal I , the following conditions are equivalent:*
(1) *I is right s-unital;*
(2) *R/I is flat as a left R-module;*
(3) *I is pure as a left ideal of R.*

Theorem 3.4. *Let R be a ring, $(S, \leq)$ a strictly totally ordered monoid. Then the following statements are equivalent:*
(1) *$r_R(a)R)$ is pure as a right ideal in R for any element $a \in R$;*
(2) *$r_{[[R^{S,\leq}]]}(f[[R^{S,\leq}]])$ is pure as a right ideal in $[[R^{S,\leq}]]$ for any element $f \in [[R^{S,\leq}]]$.*
In this case R is an S-quasi-Armendariz ring.

Proof. Assume that the condition (1) holds. Firstly, by using the same method of the proof of Proposition 2.9 we can proved that R is an S-quasi-Armendariz. Finally, by using Lemma 2.8 we can see that the condition (2) holds.

Conversely, suppose that the condition (2) holds. Let a be an element of R. Then $r_{[[R^{S,\leq}]]}(a[[R^{S,\leq}]])$ is left s-unital. Hence, for any $b \in r_R(aR)$, there exists an element $f \in [[R^{S,\leq}]]$ such that $bf = b$. Let $f(0)$ be the constant term of f. Then $f(0) \in r_R(aR)$ and $f(0)b = b$. This implies that $r_R(aR)$ is left s-unital. Therefore condition (1) holds. □

Let R be a quasi-Baer ring and let $a \in R$. Then $l_R(Ra) = Re$ for some idempotent $e \in R$, and so $R/l_R(Ra) \cong R(1-e)$ is projective. Therefore a quasi-Baer ring satisfies the hypothesis of Theorem 3.4. Hence we have the following:

Corollary 3.5. *Let R be a ring, $(S, \leq)$ a strictly totally ordered monoid. Then a ring R is a quasi-Baer ring if and only if $[[R^{S,\leq}]]$ is quasi-Baer ring.*

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