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ROMAN EDGE SEMI-TOTAL BLOCK DOMINATION OF A GRAPH

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Abstract. A graph $G = (V, E)$, *semi-total block graph* $T_b(G) = H$, whose set of vertices is the union of the set of vertices and blocks of G in whose two vertices are adjacent if and only if the corresponding vertices of G are adjacent or the corresponding members are incident. A Roman edge dominating function of a graph $G = (V, E)$ is a function $f : E \to \{0, 1, 2\}$ satisfying the condition for which $f(e) = 0$ is adjacent to atleast to one edge h for which $f(h) = 2$. The weight of a roman edge dominating function is the value of $f(E) = \sum_{e \in E} f(e)$. The Roman edge domination number of a graph G denoted by $\gamma_{re}(G)$, equals the minimum weight of a roman edge dominating function of G. A roman edge dominating function of a graph H is a roman semi-total block dominating function if $g : w \to \{0, 1, 2\}$ satisfying the condition for which $g(e) = 0$ is adjacent to atleast to one edge h for which $g(h) = 2$. The weight of a roman edge semi-total block dominating function is the value of $g(w) = \sum_{e \in w} g(e)$. The Roman edge semi-total block domination number of a graph G denoted by $\gamma_{re}(T_b(G))$, equals the minimum weight of a roman edge semi-total block dominating function of G. In this paper we study the graph theoretic properties of this variant of the domination number $H = T_b(G)$ and obtained many bounds of it in terms of its original graph G.

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1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. As usual $|V| = n$ and $|E| = q$ denote the number of vertices and edges of the graph G. The degree of an edge $e = uv$ of G is defined by $deg(e) = degu + degv - 2$ and $\delta^1(\Delta^{'}(G))$ is the minimum (maximum) degree among the edges of G(the degree of the edge is the number of edges adjacent to it). The graph obtained from G by subdividing each edge of G exactly once is called the *subdivision graph* of G and it is denoted by $S(G)$. Any undefined term can refer[4].

For any graph $G = (V, E)$, *semi-total block graph* $T_b(G) = H$, whose set of vertices is the union of the set of vertices and blocks of G in whose two vertices are adjacent if and only if the corresponding vertices of G are adjacent or the corresponding members are incident. The concept of *semi-total block graph* has been studied by Kulli.V.R.[5] and its domination number was studied in[6].

A set of vertices S is said to *dominate* the graph G if for each $v \notin S$, there is a vertex $u \in S$ with v adjacent to u. The minimum cardinality of any *dominating set* is called the *domination number* of G and is denoted by $\gamma(G)$. For a complete review on the topic of domination and its related parameters, see[8].

Mitchell and Hedetniemi in [7] introduced the notion of edge domination as follows. A set F of edges in a graph G is an edge dominating set if every edge in $E-F$ is adjacent to atleast one edge in F. The minimum number of edges in such a set is called the edge domination number of G and is denoted by $\gamma_e(G)$. This concept is also studied by S.Armugam[1].

Let γ_t is a total domination number, γ_{te} is the total edge domination number and $\beta_0, \alpha_0, \beta_1, \alpha_1$ is the vertex independence number, vertex covering number ,edge independence number and edge covering number of the graph G.

The concept of Roman dominating function was introduced by E.J.Cockayne, P.A. Dreyer, S.M.Hedetniemi and S.T. Hedetniemi in [2]. A Roman edge dominating fuction[3] on a graph $G = (V, E)$ is a function $f : E \to \{0, 1, 2\}$ satisfying the condition for which $f(e) = 0$ is adjacent to atleast to one edge h for which $f(h) = 2$. The weight of a roman edge dominating function is the value of $f(E) = \sum_{e \in E} f(e)$. The Roman edge domination number of a graph G denoted by $\gamma_{re}(G)$, equals the minimum weight of a roman edge dominating fuction of G. A roman edge dominating function of a graph H is a roman semi-total block dominating function if $g : w \to \{0, 1, 2\}$ satisfying the condition for which $g(e) = 0$ is adjacent to atleast to one edge h for which $g(h) = 2$. The weight of a roman edge semi-total block dominating function is the value of $g(w) = \sum_{e \in w} g(e)$. The Roman edge semi-total block domination number of a graph G denoted by $\gamma_{re}(T_b(G))$, equals the minimum weight of a roman edge semi-total block dominating function of G. In this paper we study the graph theoretic properties of this variant of the domination number $H = T_b(G)$ and obtained many bounds of it in terms of its original graph G.

2. Main Results

Theorem 2.1. *$\gamma_{re}(T_b(G))$ for standard graphs:*

(i) *For any cycle C_n, $\gamma_{re}(T_b(C_n)) = \lceil \frac{2n+2}{3} \rceil, n \geq 3$.*

(ii) *For any path P_n, $\gamma_{re}(T_b(P_n)) = n, n \geq 2$.*

(iii) *For any complete graph* K_n, $\gamma_{re}(T_b(K_n)) = n, n \geq 3$.

(iv) *For any star graph* $K_{1,n}$, $\gamma_{re}(T_b(K_{1,n})) = n+1, n \geq 1$.

(v) *For any bipartite graph* $K_{m,n}$, $\gamma_{re}(T_b(K_{m,n})) = m+n$.

(vi) *For any wheel graph* W_n, $\gamma_{re}(T_b(W_n)) = n-1, n \geq 4$.

(vii) *For any friendship graph with* n *blocks* F_n, $\gamma_{re}(T_b(F_n)) = 2n+1$.

Theorem 2.2. *For any graph* G, $\gamma_{re}(T_b(G)) > \gamma_{re}(G)$.

Proof. Let D be the minimum roman edge dominating set of the graph G and let $f = (E_0, E_1, E_2)$ be the γ_{re} function of G. We consider the following cases:

Case 1: If $|E_1| \leq 2, |E_2| = \phi$. Then the graph G will be either P_2 or P_3 and the graph $T_b(G)$ will be either cycle C_3 or friendship graph F_2. We have $\gamma_{re}(C_n) = 2$ and $\gamma_{re}(F_2) = 3$. Therefore, $\gamma_{re}(T_b(G)) > \gamma_{re}(G)$.

Case 2: If $|E_1| = \phi$ or $\neq \phi, |E_2| \geq 1$. Let $A = \{(v_i, v_j)/(v_i, v_j) \in E(T_b(G)) - E(G)\}$, then there exists atleast one edge say $(v_i, v_j) \in A$ which is not adjacent to D. Therefore $\gamma_{re}(T_b(G)) > |D| = \gamma_{re}(G)$.

Hence the proof. □

Theorem 2.3. *Let* $G = P_n$ *is a graph of order* n *with* m *blocks. If* n *is even then,* $\gamma_{re}(T_b(G)) = 2\lfloor \frac{m}{2} \rfloor + 2$. *Otherwise,* $\gamma_{re}(T_b(G)) = 2\lceil \frac{m}{2} \rceil + 1$.

Proof. Let $V(G) = v_i, i = 1, 2, 3....n$ such that $v_i \in N(v_{i+1})$ and let $b_j, j = 1, 2, ...n-1$ be the blocks of $T_b(G)$ corresponding to the blocks $B_j \in G, j = 1, 2, 3.........n-1$. We have $V(T_b(G)) = V(G) \cup b_j$. Let $f = (E_0, E_1, E_2)$ be the γ_{re} function of G. We consider the following cases:

Case 1: If G is a graph of even order n,
Then we define the function $g = (w_0, w_1, w_2)$ in $T_b(G)$ by $w_2 = \{(v_i, v_{i+1}), i = 2p, p = 1 \text{ to } n-2\}, w_1 = \{(v_i, b_j), i = j = 1, i = n, j = n-1\}, w_0 = E(T_b(G)) - (w_1 \cup w_2)$. Then $\gamma_{re}(G) = 2|w_2| + |w_1| = 2\lfloor \frac{n-1}{2} \rfloor + 2 = 2\lfloor \frac{m}{2} \rfloor + 2$.

Case 2: If G is a graph of odd order n,
Then we define the function $g = (w_0, w_1, w_2)$ in $T_b(G)$ by $w_2 = \{(v_i, v_{i+1}), i = 2p, p = 2 \text{ to } n-1\}, w_1 = \{(v_i, b_j), i = j = 1\}, w_0 = E(T_b(G)) - (w_1 \cup w_2)$. Then $\gamma_{re}(T_b(G)) = 2|w_2| + |w_1| = 2\lceil \frac{n-1}{2} \rceil + 1 = 2\lceil \frac{m}{2} \rceil + 1$.

The result follows from the above cases. □

Theorem 2.4. *Let* G *is a non-separable graph of order* n. *If* n *is even, then* $\gamma_{re}(G) \leq n$. *Otherwise* $\gamma_{re}(G) < 2\lceil \frac{n}{2} \rceil$.

Proof. Let $\{v_1, v_2, ...v_n\} \in V(G)$ and let $b_i, i = 1, 2, ...n$ be the blocks of $T_b(G)$ corresponding to the blocks $B_i \in G, i = 1, 2, 3.........n$. We have $V(T_b(G)) = V(G) \cup b_i$. Let $f = (E_0, E_1, E_2)$ be the γ_{re} function of G. Let $\alpha_1 = \{(v_i, v_j)/(v_i, v_j) \in E(G)\}$ be the minimum line covering set of the graph G. We consider the following cases:

Case 1: If the graph G is of even order n,

Then we define the roman edge domination function $g = (w_0, w_2, w_3)$ in $T_b(G)$ by $w_2 = \{(v_i, v_j) \in \alpha_1, i = j = 1, 2, 3, , ...n\}, w_1 = \{\emptyset\}, w_0 = E(T_b(G) - (w_1 \cup w_2)$. Then $\gamma_{re}(G) \leq 2|w_2| = 2|\alpha_1| = 2 * \frac{n}{2} = n$.

Case 2: If the graph G is of odd order n,

Then we define the roman edge domination function by $g = (w_0, w_2, w_3)$ in $T_b(G)$ by $w_2 = \{(v_i, v_j) \in \alpha_1, i = j = 1, 2, 3, , ...n\}, w_1 = \{\emptyset\}, w_0 = E(T_b(G) - (w_1 \cup w_2)$. Then $\gamma_{re}(T_b(G)) < 2|w_2| = 2|\alpha_1| = 2\lceil \frac{n}{2} \rceil$.

The result follows from the above cases. □

Proposition 2.5. *For any connected graph G, $\gamma_e(G) \leq \lfloor \frac{n}{2} \rfloor$.*

Theorem 2.6. *For any connected graph G of order $n \geq 2$, $\gamma_{re}(T_b(G)) \leq 2\lfloor \frac{2n-1}{2} \rfloor$.*

Proof. Since the graph $T_b(G)$ can have atmost $n-1$ blocks corresponding to $n \geq 2$ vertices of the graph G. Hence the graph $T_b(G)$ can have atmost $2n - 1$ vertices. By using proposition 2.5, we have $\gamma_{re}(T_b(G)) \leq 2\lfloor \frac{2n-1}{2} \rfloor$. □

Theorem 2.7. *For any graph G of order n, $2\gamma_e(G) \leq \gamma_{re}(T_b(G)) \leq 2\gamma_e(G) + 2m - 1$, where m is the number of blocks of the graph G.*

Proof. let $A = \{v_1, v_2, v_3, ...v_n\}$ are the vertices of the graph G and let $b_i \in T_b(G)$ are the blocks corresponding to the blocks $B_i \in V(G)$ such that $V(G) \cup b_i$. Let $f = (E_0, E_1, E_2)$ be the γ_{re} function of G and D be the edge dominating set of the graph G. If $E(T_b(G)) - E(G) \in N(D)$, then $\gamma_{re}(T_b(G)) = 2\gamma_e(G)$. Otherwise if atleast one edge of $E(T_b(G)) - E(G) \notin N(D)$ then, $\gamma_{re}(T_b(G)) > 2\gamma_e(G)$. For the upper bound, let $f = (E_0, E_1, E_2)$ be the γ_{re} function of G and let $g = (w_0, w_1, w_2)$ be the γ_{re} function of $T_b(G)$. Since each block B_1 of the graph G is K_2 or contains the cycle, then for each edge belongs to $B_i \cap D$ any one of the following case will follow:

Case 1: If $(|E_2| = 1, |E_1| = \emptyset)$ or $(|E_1| = 2, |E_2| = \emptyset)$ in the roman edge domination function of G belonging to different blocks and for that $\gamma_e(G) = 1$. Then in $T_b(G)$ we can define $|w_2| = 1, |w_1| = 1$, so that $\gamma_{re}(T_b(G)) < 2\gamma_e(G) + 2m - 1$.

Case 2: If $|E_2| = 1, |E_1| = \emptyset, |E_0| \geq 2$ in the roman edge domination function of G belonging to same blocks or different blocks and for that $\gamma_e(G) = 1$. Therefore in $T_b(G)$ we define $|w_2| = 1, |w_1| = 1$, so that $\gamma_{re}(T_b(G)) = 2\gamma_e(G) + 2m - 1$.

Therefore $\gamma_{re}(T_b(G)) \leq 2\gamma_e(G) + 2m - 1$. Hence the proof. □

Theorem 2.8. *For any graph G of order n, $\gamma_{re}(T_b(G)) \leq 2q$.*

Proof. let $A = \{v_1, v_2, v_3, ...v_n\}$ are the vertices of the graph G and let $b_i \in T_b(G)$ are the blocks corresponding to the blocks $B_i \in V(G)$ such that $V(G) \cup b_i$. Let $f = (E_0, E_1, E_2)$ be the roman edge dominating dominating function of G. Let $\alpha_1 = \{(v_i, v_j)/(v_i, v_j) \in E(G)\}$ be the minimum line covering set of the graph G. Since α_1 covers all the edges of $T_b(G)$ and also $|\alpha_1| \leq q$ in G. Let $g = (w_0, w_1, w_2)$ be the roman edge dominating function of $T_b(G)$ with $w_2 = \{(v_i, v_j) \in \alpha_1\}, w_1 = \{\emptyset\}, w_0 = E(T_b(G)) - (w_1 \cup w_2)$. Therefore $\gamma_{re}(T_b(G)) \leq 2|w_2| = 2\alpha_1 \leq 2q$. □

Theorem 2.9. *For any tree with n vertices, $\gamma_{re}(T_b(T)) = n$.*

Proof. Let $A = \{v_i/v_i \in V(T), d(v_i) = 1\}$ and $B = V(T) - A$. Let $b_i \in T_b(T)$ are the blocks corresponding to the blocks $B_i \in V(T)$ such that $V(T) \cup b_i$. Let $f = (E_0, E_1, E_2)$ be the γ_{re} function of T. Let S be the set of all edges incident with A with $|A| = |S|$. We consider the following cases:

Case 1: If $< T - A >= K_1$, then the graph $T = K_{1,n}$ and by using Theorem 2.1(iv), we have $\gamma_{re}(T_b(T)) = n$.

Case 2: If $< T - A > \neq K_1$.
Let $\alpha_1 = \{(v_i, v_j/(v_i, v_j) \in E(T)\}$ be the minimum line covering set of the graph $< T - A >$.

subcase 1: If no two edges in α_1 are adjacent, then we define the roman edge dominating function of $T_b(G)$ as $g = (w_0, w_1, w_2)$ as $w_2 = \{(v_i, v_j) \in \alpha_1\}, w_1 = \{(v_i, b_i), v_i \in A\}, w_0 = E(T_b(G)) - (w_1 \cup w_2)$. Therefore $\gamma_{re}(T_b(G)) = 2|w_2| + |w_1| = 2|\alpha_1| + |A| = 2 * (\frac{|B|}{2}) + |A| = |V(T)| - |A| + |A| = n$.

subcase 2: If atleast two edges of α_1 are adjacent say (v_r, v_s) and (v_s, v_n), then choose exactly one edge $(v_n, v_p) \in S$. Let $F = \{(v_n, v_p) \in S\}$. Let $H = (\alpha_1 - \{(v_s, v_n)\}) \cup \{(v_n, v_p)\}$ so that $|H| = \frac{(|B|+|F|)}{2}$. Then we can define the roman edge dominating function of $T_b(G)$ as $g = (w_0, w_1, w_2)$ as $w_2 = H, w_1 = S - F, w_0 = E(T_b(G)) - (w_1 \cup w_2)$. Therefore $\gamma_{re}(T_b(G)) = 2|w_2| + |w_1| = 2\frac{(|B|+|F|)}{2} + |S| - |F| = |V(T)| - |A| + |F| + |A| - |F| = |V(T)| = n$.

□

Corollary 2.10. *For any tree T with n vertices, $\gamma_{re}(T_b(T)) = \alpha_1 + \beta 1$.*

Proof. For any graph G, we have $\alpha_1 + \beta_1 = n$, therefore by using theorem 2.9, the result follows. □

Corollary 2.11. *For any tree T with n vertices, $\gamma_{re}(T_b(T)) = k + p$, where p is the number end vertices of T and k is the number of cutvertices of T.*

Corollary 2.12. *For any tree with m blocks, $\gamma_{re}(T_b(T)) = m + 1$.*

Proof. Since every tree has $n-1$ blocks, therefore by using theorem 2.9, the result follows. □

Theorem 2.13. *If every vertex of the graph G is adjacent to an end vertex, then $\gamma_{re}(T_b(G)) = \gamma_t(G) + p$, where p is the number of end vertices of G.*

Proof. Let $A = \{v_i/v_i \in V(G), d(v_i) = 1\}$ and $B = \{v_j/v_j \in V(G) - A\}$. Let $b_i \in T_b(G)$ be the blocks corresponding to the blocks $B_i \in G$ and let $f = (E_0, E_1, E_3)$ be the roman edge dominating function of the graph G. Let D be the γ_t set of the graph G, since every vertex of the graph G is adjacent to an end vertex $D = B$ and let α_1 be the minimum line covering set of $< V(G) - A >$. Let S be the set of all edges incident with A with $|A| = |S|$. we consider the following cases:

case 1: If no two edges in α_1 are adjacent, then we define the roman edge dominating function of $T_b(G)$ as $g = (w_0, w_1, w_2)$ as $w_2 = \{(v_i, v_j) \in \alpha_1\}, w_1 = \{(v_i, b_i), v_i \in A\}, w_0 = E(T_b(G)) - (w_1 \cup w_2)$. Therefore $\gamma_{re}(T_b(G)) = 2|w_2| + |w_1| = 2|\alpha_1| + |A| = 2 * (\frac{|B|}{2}) + |A| = |D| + |A| = \gamma_t(G) + p$.

Case 2: If atleast two edges of α_1 are adjacent say (v_r, v_s) and (v_s, v_n), then choose exactly one edge $(v_n, v_p) \in S$. Let $F = \{(v_n, v_p) \in S\}$. Let $H = (\alpha_1 - \{(v_s, v_n)\}) \cup \{(v_n, v_p)\}$ so that $|H| = \frac{(|B|+|F|)}{2}$. Then we can define the roman edge dominating function of $T_b(G)$ as $g = (w_0, w_1, w_2)$ as $w_2 = H, w_1 = S - F, w_0 = E(T_b(G)) - (w_1 \cup w_2)$. Therefore $\gamma_{re}(T_b(G)) = 2|w_2| + |w_1| = 2\frac{(|B|+|F|)}{2} + |S| - |F| = |B| + |F| + |A| - |F| = \gamma_t(G) + p$

□

Corollary 2.14. *If every vertex of the graph G is adjacent to an an end vertex, then $\gamma_{re}(T_b(G)) = \gamma(G) + p$, where p is the number of end vertices of G.*

Proof. For the graph G, if every vertex is adjacent with an end vertex the $\gamma_t(G) = \gamma(G)$, therefore by using theorem 2.13, the result follows. □

Theorem 2.15. *For any graph G, $\gamma_{re}(T_b(G)) \geq \gamma_t(G)$.*

Proof. Let D be the minimum total dominating set of the graph G. $F = \{(v_i, v_j)/v_i, v_j \in D\}$. We consider the following cases:

Case 1: Suppose atleast any two edges of F say (v_p, v_q) and $(v_q, v_s), v_s$ are adjacent, then let $C = (F - \{(v_q, v_s)\}) \cup \{(v_s, v_r)\}, (v_s, v_r) \in N((v_q, v_s)), (v_q, v_s) \neq (v_s, v_r)$. For each edge in C, assign the weight 2, since $2|C| > |D|$. Clearly $\gamma_{re}(T_b(G)) > \gamma_t(G)$.

Case 2: Suppose no two edges of F are adjacent, assign each edge of F with weight 2. If F covers all the vertices of $T_b(G)$, then $\gamma_{re}(T_b(G)) = \gamma_t(G)$. Otherwise there exists atleast one edge in $T_b(G)$ say $(v_i, b_j) \notin N(F)$. Therefore $\gamma_{re}(T_b(G)) > \gamma_t(G)$.

□

Theorem 2.16. *For any graph G of order n, $\gamma_{re}(T_b(G)) \leq 2\alpha_1(G)$.*

Proof. Let D be the minimum edge covering set of the graph G. Since D covers every edge of $T_b(G)$, D is the edge dominating set of the graph $T_b(G)$. Let $g = (w_0, w_2, w_3)$ be the roman edge dominating set of $T_b(G)$ by $\{w_2\} = (v_i, v_j) \in D, \{w_1\} = \emptyset, \{w_0\} = E(T_b(G)) - (w_2 \cup w_1)$. Therefore $\gamma_{re}(T_b(G)) \leq 2|w_2| = 2|D| = 2\alpha_1(G)$. □

Theorem 2.17. *For any graph G of order n, $\gamma_{re}(T_b(G)) \geq \beta_0(G) + 1$ and equality holds for $K_{1,n}, n \geq 1$.*

Proof. Let $b_i \in T_b(G)$ be the blocks corresponding to the blocks $B_i \in G$ and let D be the vertex independence set of the graph G. For each vertex in D, choose exactly one edge in G which is incident with D and covers maximum number of vertices of G. Let F be the set of all such edges such that $|F| = |D|$. we consider the following cases:

Case 1: If F covers all the vertices of G, then F is the edge dominating of $T_b(G)$, if no two edges are adjacent, then atleast one edge of F , assign weight 2 in $T_b(G)$, therefore $\gamma_{re}(T_b(G)) > \beta_0(G)$. Otherwise if atleast any two edges say $H = \{(v_i, v_j), (v_j, v_n)\}$ are adjacent, then $K = (F - H) \cup (v_j, b_i)$ assign every edge of K with weight equal to 2 in $T_b(G)$. Therefore $\gamma_{re}(T_b(G)) > \beta_0(G)$.

Case 2: If F does not cover atleast one vertex of G say v_r, then there exists atleast one edge say $(v_r, b_i) \notin N(F)$, assign atleast one edge of F the weight 2 and weight 1 for (v_r, b_i) in $T_b(G)$. Therefore $\gamma_{re}(T_b(G)) > \beta_0(G) + 1$.

For equality: We have $\beta_0(K_{1,n}) = n - 1$. By using theorem 2.1(iv), we have $\gamma_{re}(T_b(G)) = \beta_0(G) + 1$ □

Theorem 2.18. *For any graph G of order n, $\gamma_{re}(T_b(G)) \leq 2\beta_1(G) + m$, where m is the number of blocks of the graph G and equality holds goods for K_n, when n is odd.*

Proof. Let D be the edge independent set of the graph G. If D covers every vertex of G and hence D will covers all the edges of $T_b(G)$. Therefore D is the edge dominating set of $T_b(G)$. Let $g = (w_0, w_1, w_2)$ be the roman edge dominating set of $T_b(G)$ by $w_2 = \{(v_i, v_j) \in D\}, w_1 = \{\emptyset\}$, $w_0 = E(T_b(G)) - (w_2 \cup w_1)$. Therefore $\gamma_{re}(T_b(G)) \leq 2|w_2|=2|D| = 2\beta_1(G)$. Otherwise if D does not cover every vertex of G, then there exists atleast one edge $(v_i, v_j) \in E(T_b(G)) - E(G)$ and $(v_i, v_j) \notin N(D)$. Let F be the set of all such edges, with $|F| \leq m$. Let

$g = (w_0, w_1, w_2)$ be the roman edge dominating set of $T_b(G)$ by $w_2 = \{(v_i, v_j) \in D\}, w_1 = \{(v_r, v_s) \in F\}$, $w_0 = E(T_b(G)) - (w_2 \cup w_1)$. Therefore $\gamma_{re}(T_b(G)) \leq 2|w_2| + |w_1| = 2|D| + |F| \leq 2\beta_1(G) + m$.
For equality: Since for K_n, $\gamma_{re}(T_b(G)) = n$ and $\beta_1(G) = \lfloor \frac{n}{2} \rfloor$ and $m = 1$, the result follows. □

Theorem 2.19. *For any Cycle $G = C_n, n \geq 5$,$\gamma_{re}(T_b(G)) \leq \gamma_{ns}(G) + 1$.*

Proof. We have $\gamma_{ns}(C_n) = n - 2$ and by using Theorem 2.1(i), $\gamma_{re}(T_b(C_n)) = \lceil \frac{2n+2}{3} \rceil$. Since $\lceil \frac{2n+2}{3} \rceil \leq n - 1, n \geq 5$, $\gamma_{re}(T_b(G)) \leq \gamma_{ns}(G) + 1$. □

Corollary 2.20. *For any Cycle C_n , $\gamma_{re}(T_b(C_n)) \leq \gamma_c(C_n), n \geq 5$.*

Proof. For a cycle C_n, $\gamma_c(C_n) = \gamma_{ns}(C_n)$, using theorem 2.19, the result follows. □

Theorem 2.21. *For any grid graph $G_{2,n}$, $\gamma_{re}(T_b(G_{2,n})) \leq 2(2 + \lfloor \frac{2n-4}{2} \rfloor), n \geq 2$.*

Proof. Let $V(G_{2,n}) = \{v_{1,i}, v_{2,i}\}, i = 1$ to n, where $\{v_{i,1}, v_{2,1}\}$ denotes the first and second row vertices of $G_{2,n}$ and let b_j is the block vertices of $T_b(G_{2,n})$ corresponding to the blocks $B_j \in G_{2,n}$. Let $f = (E_0, E_1, E_2)$ be the γ_{re} function of G and $D = A \cup B$, where $A = \{(v_{1,2}, b_i), (v_{2,1}, v_{2,2})\}$ and B is the edge dominating set of $G_{2,n-2}$ with $|B| = \lfloor \frac{2n-4}{2} \rfloor, n \geq 4, |B| = \emptyset$ for $n = 2$ and $B = \{(v_{1,3}, v_{2,3})\}$ for $n = 3$, so that $|B| = 1$. Then $g = (w_0, w_1, w_2)$ be the γ_{re} function of $T_b(G)$ by $w_2 = D, w_1 = \{\emptyset\}, w_0 = (E(T_b(G_{2,n})) - E(G))$. Therefore $\gamma_{re}(T_b(G) = 2|w_2| \leq 2(2 + \lfloor \frac{2n-4}{2} \rfloor)$. □

Theorem 2.22. *For any connected graph G, $\gamma_{re}(T_b(G)) \leq \lceil (\frac{n+q}{2}) \rceil$.*

Proof. Let $\{v_1, v_2, ...v_n\} \in V(G)$ and let $b_i, i = 1, 2, ...n$ be the blocks of $T_b(G)$ corresponding to the blocks $B_i \in G, i = 1, 2, 3.........n$. We have $V(T_b(G)) = V(G) \cup b_i$. Let $f = (E_0, E_1, E_2)$ be the γ_{re} function of G. We consider the following cases:

Case 1: If $q = n$, then $\lceil (\frac{n+q}{2}) \rceil = \lceil n \rceil$.

Case 2: If $q < n$, for the connected graph containing atleast $n-1$ edges, therefore $\lceil (\frac{n+n-1}{2}) \rceil = \lceil \frac{2n-1}{2} \rceil = \lceil n - \frac{1}{2} \rceil = \lceil n \rceil$.

Case 3: If $q > n$, then $\lceil (\frac{n+q}{2}) \rceil > \lceil n \rceil$.

Let F be the minimum edge covering set of the graph G with $|F| \leq n$ and therefore F is the edge dominating set of $T_b(G)$. Let $g = (w_0, w_1, w_2)$ be the γ_{re} function of $T_b(G)$ by $w_2 = \{(v_i, v_j) \in F\}, w_1 = \emptyset, w_0 = (E(T_b(G) - E(G))$. Therefore $\gamma_{re}(T_b(G) \leq 2|w_2| = 2|F| = n$. □

Hence, $\gamma_{re}(G) \leq \lceil (\frac{n+q}{2}) \rceil$.

Theorem 2.23. *For any graph G, $\gamma_{re}(T_b(G)) \leq \gamma_r(G)+m$, where m is the number of blocks of G.*

Proof. let $A = \{v_1, v_2, v_3, ...v_n\}$ are the vertices of the graph G and let $b_i \in T_b(G)$ are the blocks corresponding to the blocks $B_i \in V(G)$ such that $V(G) \cup b_i$. Let $f = (E_0, E_1, E_2)$ be the γ_{re} function of G and let $g = (w_0, w_1, w_2)$ be the γ_{re} function of $T_b(G)$. Since each block $B_i = G_i$ of the graph G is K_2 or contains the cycle, then for each block any one of the following case will follow:

Case 1: If $G_i = K_2$, then $\gamma_r(G_i) = 2, m = 1$ and $\gamma_{re}(T_b(G_i)) = 2$. Therefore $\gamma_{re}(T_b(G_i)) < \gamma_r(G_i) + m$

Case 2: If $G = C_n$, then $\gamma_r(G_i) = \lceil \frac{2n}{3} \rceil, m = 1$ and $\gamma_{re}(T_b(G_i)) = \lceil \frac{2n+2}{3} \rceil$. Since $\lceil \frac{2n}{3} \rceil + 1 \leq \lceil \frac{2n+2}{3} \rceil, n \geq 3$. $\gamma_{re}(T_b(G_i) \leq \gamma_r(Gi) + m$.

Hence, $\gamma_{re}(T_b(G)) \leq \gamma_r(G) + m$ □

Theorem 2.24. *For any graph G of order n, $\gamma_{re}(T_b(G)) \geq \alpha_0(G) + 1$ and equality holds good for any K_n.*

Proof. Let $b_i \in T_b(G)$ be the blocks corresponding to the blocks $B_i \in G$ and let D be the minimum point covering set of the graph G. For each vertex in D, choose exactly one edge in G such that it covers maximum number of vertices of G. Let F be the set of all such edges such that$|F| = |D|$. we consider the following cases:

Case 1: If F covers all the vertices of G, then F is the edge dominating of $T_b(G)$, if no two edges are adjacent, then atleast one edge of F , assign weight 2 in $T_b(G)$, therefore $\gamma_{re}(T_b(G)) > \alpha_0(G)$. Otherwise if atleast any two edges in F say $H = \{(v_i, v_j), (v_j, v_n)\}$ are adjacent, then $K = (F - H) \cup (v_j, b_i)$ assign every edge of K with weight equal to 2 in $T_b(G)$. Therefore $\gamma_{re}(T_b(G)) > \alpha_0(G)$.

Case 2: If F does not cover atleast one vertex of G say v_r, then there exists atleast one edge say $(v_r, b_i) \notin N(F)$, assign atleast one edge of F the weight 2 and weight 1 for (v_r, b_i) in $T_b(G)$. Therefore $\gamma_{re}(T_b(G)) > \alpha_0(G) + 1$.

For equality: we have $\gamma_{re}(T_b(K_n) = n$ and $\alpha_0(K_n) = n - 1$. Therefore $\gamma_{re}(T_b(G)) = \alpha_0(G) + 1$. □

Theorem 2.25. *For any graph G, $\gamma_{re}(T_b(G)) \leq 2(q - \Delta'(G)) + p$, where p is the number of end vertices.*

Proof. let $A = \{v_1, v_2, v_3, ...v_n\}$ are the vertices of the graph G and let $b_i \in T_b(G)$ are the blocks corresponding to the blocks $B_j \in V(G)$ such that $V(G) \cup b_j$. Let (v_i, v_j) be an edge with degree Δ^1 and let S be the set of edges adjacent to $(v_i, v_j$ in G. we consider the following cases:

Case 1: If $< E(G)-S >$ does not contains an isolate in G, then $(E(G)-S)$ cover all the edges of $T_b(G)$. Hence $(E(G)-S)$ is a edge dominating set of $T_b(G)$. Let $g = g(w_0, w_1, w_2)$ be the roman edge dominating function of $T_b(G)$ with $w_2 = \{(v_i, v_j) \in (E(G) - S)\}, w_1 = \{\emptyset\}, w_0 = E(T_b(G)) - w_2$. Therefore $\gamma_{re}(T_b(G)) \leq 2|w_2| = 2(q - \Delta^{'}(G))$

Case 2: If $< E(G) - S >$ does contains an isolate in G. Let F be the set of all such isolates in $< E(G) - S >$ such that $|F| \leq p$. Let $g = (w_0, w_1, w_2)$ be the roman edge dominating function of $T_b(G)$ by $w_2 = \{(v_i, v_j) \in (E(G) - S)\}, w_1 = \{(v_r, b_i), v_r \in F\}, w_0 = E(T_b(G)) - w_2$. Therefore $\gamma_{re}(T_b(G)) \leq 2|w_2| + |w_1| = 2(q - \Delta^{'}(G)) + p$

□

Theorem 2.26. *If every vertex of the graph G is adjacent to an end vertex, then $\gamma_{re}(T_b(G)) \leq 2\gamma_e(G) + p$, where p is the number of end vertices of G and equality holds for $K_{1,n}$.*

Proof. Let $A = \{v_i/v_i \in V(G), d(v_i) = 1\}$. Let $b_i \in T_b(G)$ be the blocks corresponding to the blocks $B_i \in G$ and let $f = f(E_0, E_1, E_2)$ be the roman edge dominating function of the graph G. Let D be the γ_e set of the graph G. Let $g = g(w_0, w_1, w_2)$ be the roman edge dominating function of $T_b(G)$ with $\{w_2\} = (v_i, v_j) \in D, \{w_1\} = (v_i, b_i), v_1 \in A, w_0 = E(T_b(G)) - (w_2 \cup w_1)$, then the weight of g will the minimum roman dominating edge set of $T_b(G)$. Therefore $\gamma_{re}(T_b(G)) \leq 2|w_2| + |w_1| = 2|D| + p = 2\gamma_e(G) + p$, where p is the number of end vertices of G.

For equality: Since $\gamma_e(K_{1,n}) = 1$ and $p = n$, therefore the results follows from theorem 2.1(iv). □

Corollary 2.27. *If every vertex of the graph G is adjacent to an end vertex, then $\gamma_{re}(T_b(G)) \leq 2\gamma_{te}(G) + p$, where p is the number of end vertices of G.*

Proof. we have $\gamma_{te}(G) \geq \gamma_e(G)$ and by using theorem 2.26, the result follows. □

Proposition 2.28. *For any tree T,$\gamma_{re}(T_b(T)) \geq \chi(T)$, where χ is the chromatic number.*

Proof. $\chi(T) = 2$ and $\gamma_{re}(T_b(T) \geq 2$. Hence,$\gamma_t(T_b(T)) \geq \chi(T)$. □

Proposition 2.29. *For any tree T ,$\gamma_{re}(T_b(T)) \geq \omega(T)$, where ω is the clique number.*

Proof. We have $\chi(T) \geq \omega(T)$, the result follows from proposition 2.28. □

Proposition 2.30. *For any tree T, $\gamma_{re}(T_b(T)) \geq \frac{q}{\beta_0(T)}$.*

Proof. We have $\chi(T) \geq \frac{q}{\beta_0(T)}$, the result follows from proposition 2.28. □

Proposition 2.31. *For any cycle $G = C_n$, $\gamma_{re}(S(T_b(G)) = 2n$.*

Proof. Let $A = \{v_i/v_i \in V(G)\}$ and let b_i be the block vertex of $T_b(G)$ corresponding to the block B_i of the graph G. Let $f = (E_0, E_1, E_2)$ be the roman dominating function of G. Let $B = \{v_r/v_r$ is a block vertex $\in b_i\}$ and let $C = \{v_s/v_s \in V(S(T_b(G))) - (A \cup B)\}$. Let $g = (w_0, w_1, w_2)$ be the γ_{re} function of $T_b(G)$ with $w_2 = \{(v_i, v_j), v_i \in A, v_j \in C \cap N(B)\}, w_1 = \{\emptyset\}, w_0 = E(S(T_b(G))) - (w_1 \cup w_2)$ with $|w_2| = n$. Therefore $\gamma_{re}(S(T_b(G))) = 2|w_2| = 2n$ □

Proposition 2.32. *For any wheel graph $G = W_n$, $\gamma_{re}(S(T_b(G)) = 2n$.*

Proof. Let $A = \{v_i/v_i \in (V(G) - v_p)\}$ where v_p is a vertex $\in V(G), d(v_p) = n - 1$ and let b_i be the block vertex of $T_b(G)$ corresponding to the block B_i of the graph G. Let $f = f(E_0, E_1, E_2)$ be the roman dominating function of G. Let $B = \{v_r/v_r$ is a block vertex $\in b_i\}$ and let $C = \{v_s/v_s \in V(S(T_b(G))) - (A \cup B)\}$. Let $D = A \cup (v_r, b_i)$ with $|(v_r, b_i)| = 1, v_r \in C \cap N(v_p)$ with $|D| = n$, where $A = \{(v_i, v_j)/v_i \in A, v_j \in N(v_p) \cap C\}$. Let $g = (w_0, w_1, w_2)$ be the γ_{re} function of $T_b(G)$ with $w_2 = D, w_1 = \{\emptyset\}, w_0 = E(S(T_b(G))) - (w_1 \cup w_2)$ with $|w_2| = n$. Therefore $\gamma_{re}(G) = 2|w_2| = 2n$. □

Theorem 2.33. *For any subdivision graph of a tree T with n vertices, $\gamma_{re}(S(T_b(T)) \leq 4(n - 1)$.*

Proof. Let $A = \{v_i/v_i \in V(T)\}$ and let b_i be the block vertex of $T_b(G)$ corresponding to the block B_i of the graph G. Let $f = f(E_0, E_1, E_2)$ be the roman dominating function of G. The graph $T_b(G)$ will consists of $n - 1$ blocks and each block is of C_6 and let D be the edge dominating set of C_6. Now for each block of $T_b(G)$, we can define the roman dominating function $g = (w_0, w_1, w_2)$ be the γ_{re} function with $w_2 = D, w_1 = \{\emptyset\}, w_0 = E(S(T_b(G))) - (w_1 \cup w_2)$ with $|w_2| = 2$. Therefore $\gamma_{re}(G) \leq 2|w_2| = 4(n - 1)$ □

Proposition 2.34. *For any subdivision graph of $K_{m,n}$, we have $\gamma_{re}(S(T_b(Km, n)) = 2(m + n), m \geq n$.*

Proof. Let $A = \{v_1, v_2, v_3, ...v_n\} \in V(G)$ and let b_i be the block vertex of $T_b(G)$ corresponding to the block B_i of the graph G. Let $f = f(E_0, E_1, E_2)$ be the roman dominating function of G. Let b_1 is a block vertex in $T_b(G)$ and let $C = \{v_s/v_s \in S(T_b(G)) - (A \cup b_1)\}$. Let $g = (w_0, w_1, w_2)$ be the γ_{re} function of $T_b(G)$ with $w_2 = \{(v_i, v_j), v_i \in A, v_j \in (C \cap N(b_i))\}, w_1 = \{\emptyset\}, w_0 = E(S(T_b(G))) - (w_1 \cup w_2)$ with $|w_2| = (m + n)$. Therefore $\gamma_{re}(G) = 2|w_2| = 2(m + n)$ □

Theorem 2.35. *For any connected graph G of order n with m blocks, then*

(i) $\gamma_{re}(T_b(G)) + \gamma_{re}(T_b(\overline{G})) \leq 2(2\lceil \frac{p}{2} \rceil + m)$.
(i) $\gamma_{re}(T_b(G)) * \gamma_{re}(T_b(\overline{G})) \leq (\lceil \frac{p}{2} \rceil + m)^2$.

Proof. we have $\beta_1(G) \leq \lceil \frac{p}{2} \rceil$ and by using theorem 2.18, the result follows. □

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