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SOME LAGUERRE-HAHN ORTHOGONAL POLYNOMIALS OF CLASS ONE

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Abstract. Let $u(\beta_0)$ be the regular form fulfilling $(u(\beta_0))_{2n+1} = \beta_0 (u(\beta_0))_{2n}, n \geq 0$ where β_0 is an arbitrary complex number in such a way that for $\beta_0 = 0$ one has the symmetric forms. Recently, the symmetric Laguerre-Hahn forms (when $\beta_0 = 0$) of class $s \leq 1$ are determined. In this paper, we determine all the Laguerre-Hahn forms $u(\beta_0)$ of class $s = 1$, when $\beta_0 \neq 0$, through the resolution of a nonlinear system satisfied by the coefficients of the three-term recurrence relation of their sequences of monic corresponding orthogonal polynomials.

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1. Introduction and preliminary results

Laguerre-Hahn forms were introduced in [8]. These forms are characterized by the fact that their formal Stieltjes function $S(u)$ satisfies a Riccati differential equation [5, 7]

$$\Phi(z)S'\big(u\big)(z) = B(z)S^2\big(u\big)(z) + C(z)S\big(u\big)(z) + D(z).$$

Nowadays, most of the known regular forms belong to this class. The subclass of semi-classical forms [12], corresponds to the case B = 0. Since the system corresponding to the problem of determining all the Laguerre-Hahn forms of class $s \geq 1$ becomes non-linear, the problem was only solved when s = 1 and for the symmetric case [1]. Thus, several authors use different processes in order to obtain Laguerre-Hahn forms of class $s \geq 1$. For instance, we can mention the adjunction of either a Dirac mass or its derivative to Laguerre-Hahn forms [6, 9], the product and the division of a form by a polynomial [6, 16]. So, some examples of Laguerre-Hahn forms are given in terms of classical ones. But, they are just few and isolated examples. Let $u(\beta_0)$ be a regular form verifying the relation

$$(u(\beta_0))_{2n+1} = \beta_0 (u(\beta_0))_{2n}, \quad n \geq 0, \quad \beta_0 \in \mathbb{C},$$

which is equivalent to its corresponding sequence of monic orthogonal polynomials $\{B_n\}_{n\geq 0}$ verifies the recurrence relation [3, 11]

$$B_0(x) = 1, \qquad B_1(x) = x - \beta_0, B_{n+2}(x) = (x - (-1)^{n+1}\beta_0) B_{n+1}(x) - \gamma_{n+1} B_n(x), \quad n \geq 0.$$

These forms are wide to accommodate all the symmetric forms as well as some particular non-symmetric. Indeed, if we take $\beta_0 = 0$, we meet the symmetric form. The form $u(\beta_0)$ has been the subject of some works. In [11], Maroni gives a characterization of the orthogonal sequence $\{B_n\}_{n\geq 0}$ related to quadratic decomposition. When $\beta_0 = 0$, Maroni and Alaya [1] give all the Laguerre-Hahn forms $u(0)$ of class one. Later,when $\beta_0 \neq 0$, in [3], Bouras and Alaya give all the semi-classical forms $u(\beta_0)$ of class one; in [10], P. Maroni and M. Mejri also found these forms by using another approach .
The aim of this work is to approach the problem of determining all the Laguerre-Hahn forms $u(\beta_0)$ of class $s = 1$ when $\beta_0 \neq 0$ through the study of the Riccati differential equation satisfied by their corresponding formal Stieltjes function $S(u(\beta_0))$ and solving a nonlinear system satisfied by the coefficients of the three-term recurrence relation of their sequences of monic corresponding orthogonal polynomials $\{B_n\}_{n\geq 0}$.

The structure of the manuscript is as follows. The first section contains material of preliminary and some results regarding the class of Laguerre-Hahn forms. In the second section, we give some properties of the Laguerre-Hahn forms $u(\beta_0)$ of class $s = 1$ when $\beta_0 \neq 0$. In the last section we give all the forms which we look for by means of these properties and the resolution of the system mentioned above. Let $\mathcal{P}$ be the vector space of polynomials with complex coefficients and let $\mathcal{P}'$ be its dual. The elements of $\mathcal{P}'$ will be called either form or linear functional. We denote by $\langle v, f\rangle$ the action of $v \in \mathcal{P}'$ on $f \in \mathcal{P}$. For $n \geq 0$, $(v)_n = \langle v, x^n\rangle$ are the moments of v. In particular a form is called symmetric if all of its moments of odd order are zero [4]. We define in the space $\mathcal{P}'$ the derivative v' of the form v by $\langle v', f\rangle := -\langle v, f'\rangle$, the left multiplication by a polynomial hv by $\langle hv, f\rangle := \langle v, hf\rangle$, the Dirac form at origin δ_0 by $\langle \delta_0, f\rangle := f(0)$ and the inverse multiplication by a polynomial of degree one $(x-c)^{-1}v$, through

$$\langle (x-c)^{-1}v, f\rangle := \langle v, \theta_c f\rangle \text{ with } \big(\theta_c f\big)(x) := \frac{f(x) - f(c)}{x - c}, \ f \in \mathcal{P}.$$

It is straightforward to prove that for $v \in \mathcal{P}'$ and $f \in \mathcal{P}$, we have [12]

$$x^{-1}(xv) = v - (v)_0 \delta_0 \,, \tag{1}$$

$$x(x^{-1}v) = v \,, \tag{2}$$

We also define the right-multiplication of a form v by a polynomial h with

$$(vh)(x) := \left\langle v, \frac{xh(x) - \xi h(\xi)}{x - \xi} \right\rangle .$$

Next, it is possible to define the product of two forms through

$$\langle uv, f \rangle := \langle u, vf \rangle \,, \;\; u, v \in \mathcal{P}', \; f \in \mathcal{P}.$$

Let us recall that a form u is said to be regular (quasi-definite) if there exists a sequence $\{B_n\}_{n\geq 0}$ of polynomials with $\deg B_n = n, \; n \geq 0$, such that

$$\langle u, B_n B_m \rangle = r_n \delta_{n,m} \quad , \quad r_n \neq 0, \quad n, m \geq 0 \,.$$

We can always assume that each B_n is monic, i.e. $B_n(x) = x^n +$ lower degree terms. Then the sequence $\{B_n\}_{n\geq 0}$ is said to be orthogonal with respect to u (monic orthogonal polynomial sequence (MOPS) in short). It is a very well-known fact that the sequence $\{B_n\}_{n\geq 0}$ satisfies a three-term recurrence relation (see, for instance, the monograph by Chihara [4])

(3)

$$B_{n+2}(x) = (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x) \,, \quad n \geq 0 \,, B_1(x) = x - \beta_0 \,, \qquad B_0(x) = 1 \,.$$

with $\left(\beta_n, \gamma_{n+1}\right) \in \mathbb{C} \times \mathbb{C}\backslash\{0\} \,, \quad n \geq 0$. By convention we set $\gamma_0 = (u)_0$. The form u is said to be normalized if $(u)_0 = 1$. In this paper, we suppose that any form will be normalized. Here, we will be considering a regular form $u(\beta_0)$ verifying the following relation:

(4) $$\left(u\left(\beta_0\right)\right)_{2n+1} = \beta_0 \left(u\left(\beta_0\right)\right)_{2n} \,, \quad n \geq 0, \quad \beta_0 \in \mathbb{C}.$$

It was shown in [3] that the corresponding MOPS $\{B_n\}_{n\geq 0}$ of $u(\beta_0)$ satisfies (3) with $\beta_n = (-1)^n \beta_0, n \geq 0$. It is possible to characterize the regular form $u(\beta_0)$ as following:

Theorem 1.1 (see [14]) *There exists a regular and symmetric form ϑ such that*

(5) $$(x - \beta_0)u(\beta_0) = \lambda x \vartheta, \quad \lambda = \frac{\gamma_1}{(\vartheta)_2}.$$

Remark 1. [11, 13] *The sequence $\{B_n\}_{n\geq 0}$ orthogonal with respect to $u\left(\beta_0\right)$ has the following quadratic decomposition*

(6) $$B_{2n}(x) = P_n(x^2), \quad B_{2n+1}(x) = (x - \beta_0)R_n(x^2), \quad n \geq 0.$$

In addition, the sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are respectively orthogonal with respect to $\sigma\left(u\left(\beta_0\right)\right)$ and $(x - \beta_0^2)\sigma\left(u\left(\beta_0\right)\right)$ where $\langle \sigma u, \; x^n \rangle = \langle u, \; x^{2n} \rangle$, $n \geq 0$. Let w_1 the normalized form defined by $\gamma_1 w_1 = (x - \beta_0^2)\sigma\left(u\left(\beta_0\right)\right)$. The form ϑ is the symmetrized of $w_2 = \frac{\gamma_1}{\lambda} x^{-1} w_1 +$

δ_0 (e.i $(\vartheta)_{2n} = (w_2)_n, (\vartheta)_{2n+1} = 0, n \geq 0)$. Furthermore, the symmetric form ϑ is regular for every $\lambda \neq 0$ such that $\lambda \neq \lambda_n$, $n \geq 0$ where

$$\lambda_n = -\gamma_1 \frac{R^{(1)}_{n-1}(0)}{R_n(0)} \ , \ n \geq 0 \ , \tag{7}$$

with

$$R^{(1)}_{n-1}(x) = (w_1 R_n)(x) := \Big\langle w_1, \frac{R_n(x) - R_n(\xi)}{x - \xi} \Big\rangle, \ n \geq 0 \ . \tag{8}$$

Now, let us recall some features about the Laguerre-Hahn character [1, 2]. **Definition** 1.2. *The regular form u is called a Laguerre-Hahn form if its formal Stieltjes function $S\big(u\big)(z)$ satisfies the Riccati differential equation*

$$\Phi(z) S'\big(u\big)(z) = B(z) S^2\big(u\big)(z) + C_0(z) S\big(u\big)(z) + D_0(z) \ , \tag{9}$$

where Φ monic , B , C_0 and D_0 are polynomials with

$$D_0(z) = -\big(u\theta_0\Phi\big)'(z) + \big(u\theta_0 C_0\big)(z) - \big(u^2\theta_0^2 B\big)(z) \ , \tag{10}$$

and

$$S(u)(z) = -\sum_{n \geq 0} \frac{(u)_n}{z^{n+1}} \ . \tag{11}$$

The corresponding MOPS $\{B_n\}_{n\geq 0}$ is called Laguerre-Hahn.In 1988 Dini [5] (see also [7]) obtained a characterization theorem in terms of the functional equation satisfied by u

$$\big(\Phi u\big)' + \Psi u + B\big(x^{-1}u^2\big) = 0 \ , \tag{12}$$

with

$$\Psi(x) = -\Phi'(x) - C_0(x) \ . \tag{13}$$

Remark 2. When $B = 0$, the form u is semi-classical.

Proposition 1.3. [1] *We define $d = \max\big(\deg(\Phi), \deg(B)\big)$ and $p = \deg(\Psi)$. The Laguerre-Hahn form u satisfying* (12) *is of class $s_u = \max(d-2, p-1)$ if and only if*

$$\prod_{c \in \mathcal{Z}} \big\{ \big|\Phi'(c) + \Psi(c)\big| + \big|B(c)\big| + \big|\langle u, \theta_c^2\Phi + \theta_c\Psi + u\theta_0\theta_c B\rangle\big| \big\} \neq 0 \ ,$$

where $\mathcal{Z}$ denotes the set of roots of Φ.

Corollary 1.4. *The form u satisfying* (9) *is of class s_u if and only if*

$$\prod_{c \in \mathcal{Z}} (|C_0(c)| + |B(c)| + |D_0(c)|) \neq 0. \tag{14}$$

i.e., Φ, B, C_0 and D_0 are coprime.

The Laguerre-Hahn character is invariant by shifting. Indeed, the shifted form $\hat{u} = (h_{a^{-1}} o \tau_{-b})u, a \in \mathbb{C} - \{0\}, b \in \mathbb{C}$ satisfies

$$\hat{\Phi}(z)S'\big(\hat{u}\big)(z) = \hat{B}(z)S^2\big(\hat{u}\big)(z) + \hat{C}_0(z)S\big(\hat{u}\big)(z) + \hat{D}_0(z) \ ,$$

with

$$\begin{aligned} &\hat{\Phi}(z) = a^{-k}\Phi(az+b) \ , \quad , \hat{B}(z) = a^{-k}B(az+b), \\ &\hat{C}_0(z) = a^{1-k}C_0(az+b) \ , \quad \hat{D}_0(z) = a^{2-k}D_0(az+b) \ , \quad k = deg(\Phi) \ . \end{aligned}$$

The forms $h_a u$ (dilation of u) and $\tau_b u$ (translation of u) are defined by

$$\langle h_a u, f\rangle := \langle u, h_a f\rangle = \langle u, f(ax)\rangle \ , \quad \langle \tau_b u, f\rangle := \langle u, \tau_{-b} f\rangle = \langle u, f(x+b)\rangle \ , \quad f \in \mathcal{P}.$$

The sequence $\hat{B}_n(x) = a^{-n}B_n(ax+b)$ is orthogonal with respect $\hat{u}$ and fulfils (3) with

$$\hat{\beta}_n = \frac{\beta_n - b}{a} \ , \quad \hat{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2} \ , \quad n \geq 0 \ . \tag{15}$$

We finish this section by recalling this important result. **Proposition** 1.5. [1] *Let u be a symmetric Laguerre-Hahn form of class s_u satisfying* (12). *The following statements hold.* (i) *When s_u is odd then Φ and B are odd and Ψ is even.* (ii) *When s_u is even then Φ and B are even and Ψ is odd.* **Remark** 3. According to (10), (13) and Proposition 1.5, we have

i) If s_u is even then C_0 is odd and D_0 is even.

ii) If s_u is odd then C_0 is even and D_0 is odd.

2. Laguerre-Hahn forms $u(\beta_0)$ of class $s = 1$

Let $u(\beta_0)$ denotes a Laguerre-Hahn form of class $s_{u(\beta_0)} = 1$ verifying (4). Since the case $\beta_0 = 0$ is treated in [1], so from now on, we always assume that $\beta_0 \neq 0$. Thus, thanks to a suitable homothetic, we can take $\beta_0 = 1$. Indeed $h_{\beta_0^{-1}} u(\beta_0) = u(1)$. In the sequel, we will simply denote $u(1) := u$ every regular form satisfying (4) with $\beta_0 = 1$ and $\{B_n\}_{n\geq 0}$ be the corresponding MOPS witch verifies (3) with $\beta_n = (-1)^n, \quad n \geq 0$. Since u is of class one, this means

(16)

$$\begin{aligned} &\Phi(x) = a_3x^3 + a_2x^2 + a_1x + a_0 \ , \quad \Psi(x) = e_2x^2 + e_1x + e_0 \ , B(x) = b_3x^3 + b_2x^2 + b_1x + b_0 \ , \\ &|a_3| + |b_3| + |e_2| \neq 0 \ , C_0(x) = c_2x^2 + c_1x + c_0 \ , \quad D_0(x) = d_1x + d_0 \ . \end{aligned}$$

Our aim is to characterize the structure of the polynomial elements of the Riccati differential equation (9) satisfied by the formal Stieltjes function $S(u)(z)$. This is possible through the study of the Laguerre-Hahn character of the form ϑ define by (5) where $\beta_0 = 1$.

2.1. Class and Riccati differential equation of the form ϑ. The form ϑ when it is regular, is also Laguerre-Hahn of class s_ϑ such that $s_\vartheta \leq s_u + 2$ and satisfying the Riccati differential equation [16]

$$E(z)S^{'}(\vartheta)(z) = F(z)S^2(\vartheta)(z) + G(z)S(\vartheta)(z) + H(z) \,, \tag{17}$$

with

(18)

$$E(z) = z(z-1)\Phi(z) \,, \quad F(z) = \lambda z^2 B(z) \,, G(z) = \Phi(z) + z\Big((z-1)C_0(z) - 2(1-\lambda)B(z)\Big) \,,$$
$$H(z) = (z-1)^2 D_0(z) + (\lambda-1)\Big((z-1)C_0(z) + \Phi(z)\Big) + (\lambda-1)^2 B(z) \,.$$

Denoting by λ_{-1} and λ_{-2} the solutions of the equation

$$B(0)\lambda^2 - \Big(C_0(0) + 2B(0)\Big)\lambda + D_0(0) + C_0(0) + B(0) = 0 \,,$$

if $B(0) \neq 0$ and $\lambda_{-2} = \lambda_{-1} = \frac{D_0(0)}{C_0(0)} + 1$ otherwise. **Theorem 2.1.** *Let λ be a complex number such that $\lambda \neq \lambda_n$, $n \geq -2$. Then, the form ϑ is Laguerre-Hahn of class s_ϑ satisfying*

$$\tilde{E}(z)S^{'}(\vartheta)(z) = \tilde{F}(z)S^2(\vartheta)(z) + \tilde{G}(z)S(\vartheta)(z) + \tilde{H}(z) \,. \tag{19}$$

Moreover, (i) *If* $(\Phi(1), B(1)) \neq (0,0)$*, then*

$$\tilde{E}(z) = E(z) \,, \quad \tilde{F}(z) = F(z) \,, \quad \tilde{G}(z) = G(z) \,, \quad \tilde{H}(z) = H(z),$$

and $s_\vartheta = 3$. (ii) *If* $(\Phi(1), B(1)) = (0,0)$ *and* $(\Psi(1), B^{'}(1)) \neq (0,0)$*, then*

(20)

$$\tilde{E}(z) = z\Phi(z) \,, \quad \tilde{F}(x) = \lambda z^2 (\theta_1 B)(z) \,, \tilde{G}(x) = (\theta_1 \Phi)(z) + z\Big(C_0 - 2(1-\lambda)(\theta_1 B)\Big)(z) \,,$$
$$\tilde{H}(z) = (z-1)D_0(z) + (\lambda-1)\Big(C_0 + (\theta_1\Phi)\Big)(z) + (\lambda-1)^2(\theta_1 B)(z) \,,$$

and $s_\vartheta = 2$. (iii) *If* $(\Phi(1), B(1)) = (0,0)$ *and* $(\Psi(1), B^{'}(1)) = (0,0)$*, then*

$$\tilde{E}(z) = z(\theta_1\Phi)(z) \,, \quad \tilde{F}(z) = \lambda z^2(\theta_1^2 B)(z) \,, \tilde{G}(z) = (\theta_1^2\Phi)(z) + z\Big((\theta_1 C_0) - 2(1-\lambda)(\theta_1^2 B)\Big)(z) \,,$$
$$\tilde{H}(z) = D_0(z) + (\lambda-1)\Big((\theta_1 C_0) + (\theta_1^2\Phi)\Big)(z) + (\lambda-1)^2(\theta_1^2 B)(z) \,,$$

and $s_\vartheta = 1$.

Proof. By our assumption, on account of Theorem 1.1, and by (17) – (18), the form ϑ is regular and so Laguerre-Hahn of class $s_\vartheta \leq s_u + 2$. Let c be a root of E such that $c \neq 1$. According to (18) we get $c\Phi(c) = 0$. If $c \neq 0$, then c is a root of Φ. We suppose $|F(c)| + |G(c)| = 0$, then according to (18), we obtain $|B(c)| + |C_0(c)| = 0$ and $H(c) = (c-1)^2 D_0(c) \neq 0$, since u is Laguerre-Hahn and so satisfies (14). If $c = 0$, then $H(0) \neq 0$, because $\lambda \neq \lambda_{-1}$ and $\lambda \neq \lambda_{-2}$. Therefore, the equation (17) is not simplified by $x - c$ for $c \neq 1$. Next, for (i) and (ii) see [16]. (iii) If $(\Phi(1), B(1)) = (0,0)$ and $(\Psi(1), B^{'}(1)) = (0,0)$, then $\tilde{F}(1) = \tilde{G}(1) = \tilde{H}(1) = 0$. So, (19) − (20) is simplified by the polynomial $x - 1$ and it becomes

$$\widehat{E}(z)S^{'}(\vartheta)(z) = \widehat{F}(z)S^{2}(\vartheta)(z) + \widehat{G}(z)S(\vartheta)(z) + \widehat{H}(z)\ , \tag{21}$$

with

(22)

$$\widehat{E}(z) = z(\theta_1\Phi)(z)\ , \quad \widehat{F}(z) = \lambda z^2(\theta_1^2 B)(z)\ , \widehat{G}(z) = (\theta_1^2\Phi)(z) + z\Big((\theta_1 C_0) - 2(1-\lambda)(\theta_1^2 B)\Big)(z)\ ,$$
$$\widehat{H}(z) = D_0(z) + (\lambda - 1)\Big((\theta_1 C_0) + (\theta_1^2\Phi)\Big)(z) + (\lambda-1)^2(\theta_1^2 B)(z)\ .$$

If 1 is a root of $\theta_1\Phi$, then $C_0(1) = B(1) = 0$. Assuming that $|\widehat{F}(1)| + |\widehat{G}(1)| = 0$. Thus, (22) gives $\widehat{H}(1) = D_0(1) \neq 0$, since u is a Laguerre-Hahn form of class $s_u = 1$ and it satisfies (14). Hence, equation (21) − (22) is not simplified and so $s_\vartheta = 1$. □

2.2. Structure of the polynomials Φ, B, C_0 and D_0. Let us spilt up each polynomial form Φ, B, C_0, D_0, $\theta_1\Phi$, $\theta_1 B$, $\theta_1 C_0$, $\theta_1^2\Phi$ and $\theta_1^2 B$ according to its odd and even parts that is to say

(23)

$$\Phi(x) = \Phi^e(x^2) + x\Phi^o(x^2)\ ,\ (\theta_1\Phi)(x) = \Phi_1^e(x^2) + x\Phi_1^o(x^2)\ , B(x) = B^e(x^2) + xB^o(x^2)\ ,$$
$$(\theta_1 B)(x) = B_1^e(x^2) + xB_1^o(x^2)\ , C_0(x) = C_0^e(x^2) + xC_0^o(x^2)\ ,\ (\theta_1 C_0)(x) = C_{0,1}^e(x^2) + xC_{0,1}^o(x^2)\ ,$$
$$(\theta_1^2\Phi)(x) = \Phi_2^e(x^2) + x\Phi^o(x^2)\ ,\ (\theta_1^2 B)(x) = B_2^e(x^2) + xB_2^o(x^2)\ , D_0(x) = D_0^e(x^2) + xD_0^o(x^2)\ .$$

PROPOSITION 2.2. *Let u be a Laguerre-Hahn form of class $s_u = 1$ satisfying* (4) *with $\beta_0 = 1$. The following statements hold:* (i) *If $(\Phi(1), B(1)) \neq (0,0)$, then* $\Phi^e(x) = \Phi^o(x) = C_0^e(x) - xC_0^o(x)$, $B^e = 0$ *and* $(x+1)D_0^e(x) - 2xD_0^o(x) = 0$. (ii) *If $(\Phi(1), B(1)) = (0,0)$ and $(\Psi(1), B^{'}(1)) \neq (0,0)$, then* $\Phi^e = B_1^o = 0$, $\Phi_1^e(x) + xC_0^o(x) = 0$, *and* $D_0^e = D_0^o$. (iii) *If $(\Phi(1), B(1)) = (0,0)$ and $(\Psi(1), B^{'}(1)) = (0,0)$, then* $\Phi^e + \Phi^o = 0$, $D_0^e = 0$, $\Phi_2^o + C_{0,1}^e = 0$ *and* $(x+1)B^e(x) + 2xB^o(x) = 0$.

Proof. Writing

$$\begin{aligned}&\tilde{E}(x) = \tilde{E}^e(x) + x\tilde{E}^o(x)\ ,\ \tilde{F}(x) = \tilde{F}^e(x) + x\tilde{F}^o(x)\ , \tilde{G}(x) = \tilde{G}^e(x) + x\tilde{G}^o(x)\ ,\\ &\tilde{H}(x) = \tilde{H}^e(x^2) + x\tilde{H}^o(x^2)\ .\end{aligned} \tag{24}$$

We have to examine the following situations: (i) $(\Phi(1), B(1)) \neq (0,0)$. According to (23), (24) and from the expression of polynomials $\tilde{E}$, $\tilde{F}$, $\tilde{G}$ and $\tilde{H}$ given in Theorem 2.1, we get

$$\tilde{E}^e(x) = x(\Phi^e - \Phi^o)(x)\ , \quad \tilde{E}^o(x) = x\Phi^o(x) - \Phi^e(x)\ , \tilde{F}^e(x) = \lambda xB^e(x)\ , \quad \tilde{F}^o(x) = \lambda xB^o(x),$$
$$\tilde{G}^e(x) = \Phi^e(x) + x\big(C_0^e - C_0^o\big)(x) - 2(1-\lambda)xB^o(x)\ ,$$
$$\tilde{G}^o(x) = \Phi^o(x) - C_0^e(x) + xC_0^o(x) - 2(1-\lambda)xB^e(x)\ ,$$
$$\tilde{H}^e(x) = (x+1)D_0^e(x) - 2xD_0^o(x) + (\lambda-1)xC_0^o(x) - (\lambda-1)(C_0^e - \Phi^e)(x) + (\lambda-1)^2B^e(x)\ ,$$
$$\tilde{H}^o(x) = (x+1)D_0^o(x) - 2D_0^e(x) + (\lambda-1)(C_0^e - C_0^o + \Phi^o)(x) + (\lambda-1)^2B^o(x)\ .$$

Then, $\tilde{E}^e = \tilde{F}^e = \tilde{G}^o = \tilde{H}^e = 0$, from Proposition 1.5 and remark 3, since $s_\vartheta = 3$. This gives (i). (ii) $(\Phi(1), B(1)) = (0,0)$ and $(\Psi(1), B'(1)) \neq (0,0)$. Similar as above,

$$\begin{aligned}
&\tilde{E}^e(x) = x\Phi^o(x)\,, \quad \tilde{E}^o(x) = \Phi^e(x)\,, \tilde{F}^e(x) = \lambda x B_1^e\,,\ \tilde{F}^o(x) = \lambda x B^o(x)\,,\\
&\tilde{G}^e(x) = \Phi_1^e(x) + xC_0^o(x) - 2(1-\lambda)xB_1^o(x)\,,\ \tilde{G}^o(x) = \Phi_1^o(x) + C_0^e(x) - 2(1-\lambda)B_1^e(x)\,,\\
&\tilde{H}^e(x) = xD_0^o(x) - D_0^e(x) + (\lambda-1)(C_0^e + \Phi_1^e)(x) + (\lambda-1)^2 B_1^e(x)\,,\\
&\tilde{H}^o(x) = D_0^e(x) - D_0^o(x) + (\lambda-1)(C_0^o + \Phi_1^o)(x) + (\lambda-1)^2 B_1^o(x)\,.
\end{aligned}$$

Here $s_\vartheta = 2$, then $\tilde{E}^o = \tilde{F}^o = \tilde{G}^e = \tilde{H}^o = 0$. This leads to result (ii), since $B(x) = (x-1)(\theta_1 B)(x)$. (iii) $(\Phi(1), B(1)) = (0,0)$ and $(\Psi(1), B'(1)) = (0,0)$. In this case, we have

$$\begin{aligned}
&\tilde{E}^e(x) = x\Phi_1^o(x)\,, \quad \tilde{E}^o(x) = \Phi_1^e(x)\,, \tilde{F}^e(x) = \lambda x B_2^e(x)\,, \quad \tilde{F}^o(x) = \lambda x B_2^o(x)\,,\\
&\tilde{G}^e(x) = \Phi_2^e(x) + xC_{0,1}^o(x) - 2(1-\lambda)xB_2^o(x)\,, \quad \tilde{G}^o(x) = (\Phi_2^o + C_{0,1}^e)(x) - 2(1-\lambda)B_2^e(x)\,,\\
&\tilde{H}^e(x) = D_0^e(x) + (\lambda-1)(C_{0,1}^e + \Phi_2^e)(x) + (\lambda-1)^2 B_2^e(x)\,,\\
&\tilde{H}^o(x) = D_0^o(x) + (\lambda-1)(C_{0,1}^o + \Phi_2^o)(x) + (\lambda-1)^2 B_2^o(x)\,.
\end{aligned}$$

Since ϑ is of odd class, $\tilde{E}^e = \tilde{F}^e = \tilde{G}^o = \tilde{H}^e = 0$. Therefore $\Phi_1^o = B_2^e = D_0^e = 0$ and $c\Phi_2^o + C_{0,1}^e = 0$. This gives the desired result because $\Phi(x) = (x-1)(\theta_1\Phi)(x)$ and $B(x) = (x-1)^2(\theta_1^2 B)(x)$. □

Theorem 2.3. *If u is a Laguerre-Hahn form of class one satisfying (4) with $\beta_0 = 1$, then*

(25)

$$\Phi(x) = x(x^2-1)\,, \quad B(x) = (x-1)(b_3x^2 + b_1)\,, C_0(x) = c_2x^2 - x + c_0\,, \quad D_0(x) = d_1(x+1)\,,$$

with

$$\begin{aligned}
&|2b_3 - c_2 - 3| + |b_3 + b_1| \neq 0\,, \quad |b_3 + b_1| + |c_0| + |d_1| \neq 0\,,\\
&|c_2 + c_0 - 1| + |d_1| \neq 0\,, \quad |c_2 + c_0 + 1| + |b_3 + b_1| \neq 0\,.
\end{aligned} \tag{26}$$

For the proof, we use the following lemmas. **Lemma 2.4.** *We have the following formulas:*

$$(u)_1 = (u)_0 = 1\,,$$

$$(u)_3 = (u)_2 = \gamma_1 + 1\,,$$

$$(u)_5 = (u)_4 = (1+\gamma_1)^2 + \gamma_1\gamma_2\,,$$

$$(u)_7 = (u)_6 = \gamma_1(\gamma_2 + \gamma_1 + 1)^2 + (\gamma_1 + 1)^2 + \gamma_1\gamma_2(\gamma_3 + 1)\,,$$

$$(u)_8 = \gamma_1\gamma_2\gamma_3(\gamma_4 + \gamma_3 + 2\gamma_2 + 2\gamma_1 + 4) + \gamma_1(\gamma_2 + \gamma_1 + 2)\big((\gamma_2 + \gamma_1 + 1)^2 + 1\big) + 1\,.$$

Lemma 2.5. *We have the following:*

$$a_3 + c_2 - b_3 - d_1 = 0\,, \tag{27}$$

$$2a_3 + a_2 + c_2 + c_1 - 2b_3 - b_2 - d_0 = 0\,, \tag{28}$$

$$(29)\qquad (3a_3+c_2-2b_3)(u)_2+2a_2+a_1+c_1+c_0-b_3-2b_2-b_1=0\,,$$

$$(30)\qquad \begin{aligned}&2a_0-2b_0-b_1+(5a_3+c_2-2b_3)(u)_4\\&+\big(4a_2+3a_1+c_1+c_0-4b_2-2b_1-((u)_2+2)b_3\big)(u)_2=0\,,\end{aligned}$$

$$(31)\qquad \begin{aligned}&(7a_3+c_2-2b_3)(u)_6+(6a_2+5a_1+c_1+c_0-2b_3-4b_2-2b_1-2b_3(u)_2)(u)_4\\&+\big(4a_0-2b_1-4b_0-(b_3+2b_2+b_1)(u)_2\big)(u)_2=0\,,\end{aligned}$$

$$(32)\qquad \begin{aligned}&(9a_3+c_2-2b_3)(u)_8+(8a_2+7a_1+c_1+c_0-2b_3-4b_2-2b_1-2b_3(u)_2)(u)_6\\&+\big(6a_0-2b_1-4b_0-b_3(u)_4-2(b_3+2b_2+b_1)(u)_2\big)(u)_4-(b_1+2b_0)(u)_2^2=0\,.\end{aligned}$$

Proof. (27) − (32) are gotten from (9), (11), (16) and taking into account Lemma 2.4. □

Proof. (of Theorem 2.3) We have to consider two cases: A. $\deg(\Phi)\le 2$. we have $|b_3|+|c_2|\neq 0$. Following Proposition 2.2, three situations to establish. A_1. $(\Phi(1),B(1))\neq(0,0)$, then $\Phi^e(x)=\Phi^o(x)=C_0^e(x)-xC_0^0(x)$, $B^e=0$ and $(x+1)D_0^e(x)-2xD_0^o(x)=0$. So, from (23), $\Phi(x)=(x+1)\Phi^e(x^2)$, $B(x)=xB^o(x^2)$ and $D_0(x)=0$. Since Φ is a monic polynomial of degree at most two, then necessarily $\Phi^e(x)=1$. In addition, we have $C_0^e(x)-xC_0^o(x)=1$. This implies that $c_2=c_1$ and $c_0=1$. Then $\Phi(x)=x+1$, $B(x)=b_3x^3+b_1x$, and $C_0(x)=c_2x^2+c_2x+1$ with $(b_3,c_2)\neq(0,0)$. According to (27), (29), (30) and Lemma 2.4, we have

$$(33)\qquad c_2-b_3=0\,,$$

$$(34)\qquad (c_2-2b_3)(u)_2-b_1+2=0\,,$$

$$(35)\qquad (c_2-b_3)\big((u)_4+(u)_2\big)-(2(u)_2+1)\big(b_3(u)_2+b_1-2\big)-b_3\gamma_1\gamma_2=0\,.$$

From (33) and (34), we obtain

$$(u)_2b_3+b_1-2=0\,.$$

Hence, relation (35) becomes

$$b_3\gamma_1\gamma_2=0\,.$$

Then, from (33) and taking into account the regularity of u, we get $b_3=c_2=0$, that is a contradiction with $(b_3,c_2)\neq(0,0)$. A_2.$(\Phi(1),B(1))=(0,0)$ and $(\Psi(1),B'(1))\neq(0,0)$, then $\Phi^e=0$. So, $\Phi(x)=x$, since Φ is monic polynomial and $\deg(\Phi)\le 2$. This contradicts $\Phi(1)=0$. A_3. $(\Phi(1),B(1))=(0,0)$ and $(\Psi(1),B'(1))=(0,0)$, hence

$$(36)\qquad \Phi(x)=x-1\,,\quad B(x)=b_3x(x-1)^2\,,\quad C_0(x)=c_2x(x-1)-1\,,\quad D_0(x)=d_1x\,.$$

Using (36), (29), (30) and (32) , we deduce

$$(37)\qquad (c_2-2b_3)\gamma_1=0\,,$$

$$(38)\qquad (c_2-2b_3)\gamma_1(\gamma_2+\gamma_1+1)+(2-b_3\gamma_1)\gamma_1=0\,,$$

$$(39)\qquad \begin{aligned}&(c_2-2b_3)\gamma_1\big(\gamma_2\gamma_3(\gamma_4+\gamma_3+2\gamma_2+2\gamma_1+3)+(\gamma_2+\gamma_1+1)^3\big)+2\gamma_1\gamma_2\gamma_3\\ &+(2-\gamma_1 b_3)\gamma_1\big(3(\gamma_2+\gamma_1+1)^2+2\gamma_2\gamma_3\big)=0\,.\end{aligned}$$

From (37) and (38), we get

$$\gamma_1(2-b_3\gamma_1)=0\,.$$

Hence, from the above equation and (39), we can deduce $\gamma_1\gamma_2\gamma_3=0$. It is a contradiction with the orthogonality of the sequence $\{B_n\}_{n\geq 0}$. B. $\deg(\Phi)=3$, In this case, we obtain from (16) $\deg(\Phi^e)\leq 1$ and $\deg(\Phi^o)=1$. By virtue of Proposition 2.2, we have to examine three subcases: B_1. $(\Phi(1),B(1))\neq(0,0)$, then $\Phi^e(x)=\Phi^o(x)$, $C_0^e(x)=xC_0^o(x)+\Phi^e(x)$, $B^e=0$ and $(x+1)D_0^e(x)-2xD_0^o(x)=0$. We get $\Phi(x)=(x+1)\Phi^o(x^2)$, $C_0(x)=(x^2+x)C_0^o(x^2)+\Phi^e(x^2)$, $B(x)=xB^o(x^2)$ and $D_0(x)=0$. Therefore C_0^o is constant polynomial, $\deg(B^o)\leq 1$ and Φ^o is a monic polynomial of degree one since $\deg(C_0)\leq 2$, $\deg(B)\leq 3$ and $\deg(\Phi)=3$. Hence,

(40)

$$\Phi(x)=(x+1)(x^2+a_1)\,,\quad B(x)=b_3x^3+b_1x\,,\quad C_0(x)=(c_1+1)x^2+c_1x+a_1\,,\quad D_0(x)=0\,.$$

From (40), (28), (29), (30), (31) and (32), we obtain

$$(41)\qquad c_1+2-b_3=0\,,$$

$$(42)\qquad (c_1+2-b_3)((u)_2+1)+(2-b_3)(u)_2+2a_1-b_1=0\,,$$

$$(43)\quad (c_1+2-b_3)\big((u)_4+(u)_2\big)+\big((2-b_3)(u)_2+2a_1-b_1\big)(2(u)_2+1)+(4-b_3)\gamma_1\gamma_2=0\,,$$

$$(44)\qquad \begin{aligned}&(c_1+2-b_3)\big((u)_6+(u)_4\big)+(4-b_3)\gamma_1\gamma_2(\gamma_3+\gamma_2+2\gamma_1+4)\\ &+\big((2-b_3)(u)_2+2a_1-b_1\big)\big(2(u)_4+(u)_2^2+2(u)_2\big)+2\gamma_1\gamma_2(\gamma_3+\gamma_2+1+a_1)=0\,,\end{aligned}$$

(45)

$$\begin{aligned}&(c_1+2-b_3)\big((u)_8+(u)_6\big)+(4-b_3)\gamma_1\gamma_2\big(\gamma_3(\gamma_4+\gamma_3+2\gamma_2+2\gamma_1+5)+3\gamma_2+2+3(u)_4\\ &+(u)_2^2+4(u)_2\big)+\big((2-b_3)(u)_2+2a_1-b_1\big)\big(4(u)_2^2+3(u)_4+\gamma_1\gamma_2(6(u)_2+1\\ &+2\gamma_3+2\gamma_2)\big)+2\gamma_1\gamma_2(\gamma_3+\gamma_2+1+a_1)\big(2(u)_2+2\gamma_2+2\gamma_3+3\big)++4\gamma_4\gamma_3\gamma_2\gamma_1=0\,.\end{aligned}$$

Using (41) and (42), we get

$$(2-b_3)(u)_2+2a_1-b_1=0\,.$$

Hence, from the last equation and (43), we obtain

$$(46)\qquad (4-b_3)\gamma_1\gamma_2=0\,.$$

Then, from (44) and (46), we can deduce

$$(\gamma_3+\gamma_2+1+a_1)\gamma_1\gamma_2=0\ .$$

Thus, from the above equation and (45), we have $\gamma_1\gamma_2\gamma_3\gamma_4=0$. It is a contradiction by virtue of the regularity of the form u. B_2. $(\Phi(1),B(1))=(0,0)$ and $(\Psi(1),B^{'}(1))=(0,0)$, then $\Phi^e=-\Phi^o$, $\Phi_2^e+C_{0,1}^o=0$, $D_0^e=0$ and $2xB^o(x)+(x+1)B^e(x)=0$. Therefore

(47) $\Phi(x)=(x-1)(x^2+a_1)\ ,\quad C_0=c_2x^2-x+c_0\ ,\quad D_0(x)=d_1x\ ,\quad B(x)=b_3x(x-1)^2\ .$

From (47), (29), (30) and (31), we have

$$(2-2b_3-c_1)\gamma_1=0\ , \tag{48}$$

$$(2-2b_3-c_1)\gamma_1(\gamma_2+(u)_2)+2\gamma_1(\gamma_2+(u)_2+a_1)-\gamma_1^2b_3=0\ , \tag{49}$$

$$(2-2b_3-c_1)\gamma_1\big((\gamma_2+(u)_2)^2+\gamma_2\gamma_3\big)+2(\gamma_2+(u)_2)\big(2\gamma_1(\gamma_2+(u)_2+a_1)-\gamma_1^2b_3\big)+4\gamma_1\gamma_2\gamma_3=0\ . \tag{50}$$

From (48) and (49), we get

$$2\gamma_1(\gamma_2+(u)_2+a_1)-\gamma_1^2b_3=0\ .$$

Here, from the above equation and (50), we obtain $\gamma_1\gamma_2\gamma_3=0$. It is a contradiction with the orthogonality of the sequence $\{B_n\}_{n\geq 0}$. B_3. $(\Phi(1),B(1))=(0,0)$ and $(\Psi(1),B^{'}(1))\neq(0,0)$, then $\Phi^e=0$, $\Phi_1^o+C_0^o=0$, $D_0^e=D_0^o$ and $B_1^o=0$. So, $\Phi(x)=x(x^2-1)$, $C_0(x)=c_2x^2-x+c_0$, $D_0(x)=d_1(x+1)$ and $B(x)=(x-1)(b_3x^2+b_1)$. If $(2b_3-c_2-3,b_3+b_1)=(0,0)$, then $c_2+c_0+1=0$ and $b_3=-b_1$ since from (29), we have $c_0+b_3-b_1-2=0$. Thus $\Psi(1)=B^{'}(1)=0$ which contradiction $(\Psi(1),B^{'}(1))\neq(0,0)$. Necessarily $(2b_3-c_2-3,b_3+b_1)\neq(0,0)$. Moreover the form u is of class one, we shall have the condition (14) with $\mathcal{Z}_\Phi=\{-1,0,1\}$, which leads to relation (26). □

Remark 4. The form u satisfies (12) where Φ, B are given by (25) and $\Psi(x)=e_2x^2+x+e_0$.

2.3. The recurrence and the structure relation coefficients. Let $\{B_n\}$ be a Laguerre-Hahn sequence of class $s_u=1$, orthogonal with respect to u verifying (4) with $\beta_0=1$, and solution of (12). Such a sequence satisfies the following structure relation [1]

(51)

$$\Phi(x)B_{n+1}'(x)-B(x)B_n^{(1)}(x)=\frac{1}{2}\big(C_{n+1}(x)-C_0(x)\big)B_{n+1}(x)-\gamma_{n+1}D_{n+1}(x)B_n(x)\ ,\ n\geq 0,$$

with for $n\geq 0$

$$\begin{aligned}C_{n+1}(x)&=-C_n(x)+2(x-(-1)^n)D_n(x)\gamma_{n+1}D_{n+1}(x)\\&=-\Phi(x)+\gamma_nD_{n-1}(x)-(x-\beta_n)C_n(x)+(x-(-1)^n)^2D_n(x)\ ,\end{aligned} \tag{52}$$

where $\gamma_0 D_{-1}(x) = B(x)$ and Φ, B, $C_0(x)$ and $D_0(x)$ are the same polynomials as in (9). **Remark** 5. (i) $\deg(C_n) \leq 2$ and $\deg(D_n) \leq 1, n \geq 0$ [5]. (ii) The polynomials C_n and D_n, $n \geq 0$, enable to obtain the coefficients of the fourth-order differential equation satisfied by each B_n See, for instance [5]. From Theorem 2.3, we have

$$\Phi(x) = x(x^2-1)\,, \quad B(x) = (x-1)(b_3x^2+b_1)\,, \quad \Psi(x) = e_2x^2 + x + e_0\,.$$

Furthermore, the sequence $\{B_n\}_{n\geq 0}$ satisfies the recurrence relation

(53)

$$B_{n+2}(x) = \big(x-(-1)^{n+1}\big)B_{n+1}(x) - \gamma_{n+1}B_n(x)\,, \quad n \geq 0\,, B_1(x) = x-1\,, \qquad B_0(x) = 1\,.$$

According to remark 5, we obtain for (51)

(54) $$\Phi(x)B'_{n+1}(x) - B(x)B^{(1)}_n(x) = (G_nx^2+E_nx+F_n)B_{n+1}(x) + (H_n(x)+K_n)B_n(x)\,,\ n \geq 0\,.$$

Differentiating (53) and multiplying by Φ, according to (54) we get successively

$$B'_{n+2}(x) = \big(x-(-1)^{n+1}\big)B'_{n+1}(x) - \gamma_{n+1}B'_n(x) + B_{n+1}(x)\,,$$

$$\begin{aligned}
&B(x)B^{(1)}_{n+1}(x) + (G_{n+1}x^2 + E_{n+1}x + F_{n+1})B_{n+2}(x) + (H_{n+1}x + K_{n+1})B_{n+1}x)\\
&= \big(x-(-1)^{n+1}\big)\Big(B(x)B^{(1)}_n(x) + (G_nx^2+E_nx+F_n)B_{n+1}(x) + (H_nx+K_n)B_n(x)\Big)\\
&-\gamma_{n+1}\Big(B(x)B^{(1)}_{n-1}(x) + (G_{n-1}x^2+E_{n-1}x+F_{n-1})B_n(x) + (H_{n-1}x+K_{n-1})B_{n-1}(x)\Big)\\
&+\Phi(x)B_{n+1}(x)\,,
\end{aligned}$$

or

$$\begin{aligned}
&B(x)\Big(B^{(1)}_{n+1}(x) - \big(x-(-1)^{n+1}\big)B^{(1)}_n(x) + \gamma_{n+1}B^{(1)}_{n-1}(x)\Big) + \Big(G_{n+1}x^2 + E_{n+1}x + F_{n+1}\Big)B_{n+2}(x)\\
&+\Big(H_{n+1}x + K_{n+1} - \big(x-(-1)^{n+1}\big)(G_nx^2+E_nx+F_n) - \Phi(x)\Big)B_{n+1}(x)\\
&= \Big(\big(x-(-1)^{n+1}\big)(H_nx+K_n) - \gamma_{n+1}(G_{n-1}x^2+E_{n-1}x+F_{n-1})\Big)B_n(x)\\
&-\gamma_{n+1}\Big(H_{n-1}x+K_{n-1}\Big)B_{n-1}(x)\,,\ n \geq 0\,.
\end{aligned}$$

But, from the definition $B^{(1)}_{n+1}(x) - \big(x-(-1)^{n+1}\big)B^{(1)}_n(x) + \gamma_{n+1}B^{(1)}_{n-1}(x) = 0$, $n \geq 0$. Then, from (53), we obtain

(55) $$M(n,x)B_{n+1}(x) = N(n,x)B_n(x)\,,\ n \geq 0\,,$$

where

$$\begin{aligned}
M(n,x) &= (G_{n+1}-G_n-1)x^3 + \big(E_{n+1}-E_n+(-1)^n(G_{n+1}-G_n)\big)x^2\\
&\big(F_{n+1}-F_n+H_{n+1}-\tfrac{\gamma_{n+1}}{\gamma_n}H_{n-1}+(-1)^n(E_{n+1}-E_n)+1\big)x\\
&+K_{n+1}-\tfrac{\gamma_{n+1}}{\gamma_n}K_{n-1}+(-1)^n(F_{n+1}-F_n)\,,
\end{aligned}$$

$$\begin{aligned}
N(n,x) &= \big(H_n-\tfrac{\gamma_{n+1}}{\gamma_n}H_{n-1}+\gamma_{n+1}(G_{n+1}-G_{n-1})\big)x^2\\
&+\big(K_n-\tfrac{\gamma_{n+1}}{\gamma_n}K_{n-1}+\gamma_{n+1}(E_{n+1}-E_{n-1})+(-1)^n(H_n+\tfrac{\gamma_{n+1}}{\gamma_n}H_{n-1})\big)x\\
&+(-1)^n(K_n+\tfrac{\gamma_{n+1}}{\gamma_n}K_{n-1})+\gamma_{n+1}(F_{n+1}-F_{n-1})\,.
\end{aligned}$$

We have put $G_{-1} = E_{-1} = F_{-1} = H_{-1} = K_{-1} = 0$. Obviously, we get

(56) $$G_{n+1} - G_n = 1 \ , \ n \geq 0 \ ,$$

(57) $$(-1)^n (G_{n+1} - G_n) + (E_{n+1} - E_n) = 0 \ , \ n \geq 0 \ .$$

Next, for $n \geq 2$, $M(n,x) = N(n,x) = 0$, that is to say

(58) $$(-1)^n (E_{n+1} - E_n) + (F_{n+1} - F_n) + (H_{n+1} - \frac{\gamma_{n+1}}{\gamma_n} H_{n-1}) = -1 \ , \ n \geq 2 \ ,$$

(59) $$(-1)^n (F_{n+1} - F_n) + (K_{n+1} - \frac{\gamma_{n+1}}{\gamma_n} K_{n-1}) = 0 \ , \ n \geq 2 \ ,$$

(60) $$\gamma_{n+1} (G_{n+1} - G_{n-1}) + (H_n - \frac{\gamma_{n+1}}{\gamma_n} H_{n-1}) = 0 \ , \ n \geq 2 \ ,$$

(61) $$\gamma_{n+1} (E_{n+1} - E_{n-1}) + (K_n - \frac{\gamma_{n+1}}{\gamma_n} K_{n-1}) + (-1)^n (H_n + \frac{\gamma_{n+1}}{\gamma_n} H_{n-1}) = 0 \ , \ n \geq 2 \ ,$$

(62) $$\gamma_{n+1} (F_{n+1} - F_{n-1}) + (-1)^n (K_n + \frac{\gamma_{n+1}}{\gamma_n} K_{n-1}) = 0 \ , \ n \geq 2 \ .$$

For $n = 0$ and $n = 1$, equation (55) gives $M(0,x)(x-1) = N(0,x)$ and $(x^2 - 1 - \gamma_1) M(1,x) = (x-1) N(1,x)$, that is to say

(63) $$(F_2 - F_1) + (H_2 - H_1) = \gamma_2 (G_2 - G_0) + (E_2 - E_1) - 1 \ ,$$

(64) $$-(H_2 + H_1) - (K_2 - K_1) + \gamma_2 (E_2 - E_0) + (E_2 - E_1) = 1 \ ,$$

(65) $$\begin{aligned} &(1+\gamma_1)(F_2 - F_1) + (H_2 + H_1) + \gamma_2 (F_2 - F_0) - \gamma_2 (E_2 - E_0) \\ &= 2K_1 + (1+\gamma_1)(E_2 - E_1) - \gamma_1 H_2 + \gamma_2 H_0 - (1+\gamma_1) \ , \end{aligned}$$

(66) $$(1+\gamma_1)(F_2 - F_1) + \gamma_2 (F_2 - F_0) - (1+\gamma_1) K_2 - K_1 + \gamma_2 K_0 = 0 \ ,$$

(67) $$\gamma_1 F_1 = -(\gamma_1 + 1) G_1 - (\gamma_1 + 1) E_1 + 2 - b_3 \ ,$$

(68) $$\gamma_1 H_1 = (\gamma_1 + 1) E_1 + (\gamma_1 + 1)^2 G_1 + (\gamma_1 + 1)(b_3 - 2) - \gamma_1 b_1 \ ,$$

(69) $$\gamma_1 K_1 = (\gamma_1 + 1)^2 G_1 + (\gamma_1 + 1)^2 E_1 + (\gamma_1 + 1)(b_3 - 2) - \gamma_1 b_1 \ .$$

It remains to express (54) for $n = 0$: $\Phi(x) - B(x) = (G_0 x^2 + E_0 x + F_0)(x-1) + H_0 x + K_0$, or

(70) $$G_0 = 1 - b_3 \ ,$$

(71) $$E_0 = 1 \ ,$$

(72) $$H_0 = -F_0 - b_1 \ ,$$

(73) $$K_0 = F_0 + b_1 \ .$$

Finally, the condition $\langle (\Phi u)' + \Psi u + B(x^{-1}u^2), x^n \rangle = 0, \ n \geq 0$ gives for $n = 0, 2$

(74) $$(2b_3 + e_2)\gamma_1 + b_3 + e_2 + b_1 + e_0 + 1 = 0 \ ,$$

(75)
$$b_3\gamma_1^2 + \gamma_1(\gamma_2 + \gamma_1)(2b_3 + e_2 - 2) + (4b_3 + 2b_1 + 2e_2 + e_0 - 1)\gamma_1 + b_3 + b_1 + e_2 + e_0 + 1 = 0 \ .$$

Lemma 3.1. *The system* (56) – (75) *becomes*

(76) $$G_n = n + 1 - b_3 \ , \ n \geq 0 \ ,$$

(77) $$E_n = \frac{1 + (-1)^n}{2} \ , \ n \geq 0 \ ,$$

(78) $$\frac{K_n}{\gamma_{n+1}} = (-1)^{n+1} \frac{H_n}{\gamma_{n+1}} = (-1)^{n+1} \big(\frac{H_1}{\gamma_2} - 2(n-1) \big) \ , \ n \geq 0 \ ,$$

(79) $$F_n = -n + 1 + \frac{1 - (-1)^n}{2} F_1 + \frac{1 + (-1)^n}{2} \big(F_2 + 1 \big) \ , \ n \geq 1 \ ,$$

(80) $$\big(\frac{H_1}{\gamma_2} - 2n \big) \gamma_{n+2} - \big(\frac{H_1}{\gamma_2} - 2(n-2) \big) \gamma_{n+1} = (-1)^n (F_2 - F_1 + 1) + 1 \ , \ n \geq 1 \ ,$$

(81) $$F_0 = K_0 + H_1 - (\gamma_1 + 1)(2 - b_3) \ , \quad H_0 = -K_0 \ ,$$

(82) $$F_1 = b_3 - 2 \ , \ H_1 = K_1 = (\gamma + 1)(2 - b_3) - b_1 \ ,$$

(83) $$F_2 = H_1 + b_3 - 4 - \gamma_1 \big(\frac{H_1}{\gamma_2} + 4 - b_3 \big) \ ,$$

(84)
$$b_1 = (\gamma_1 + 1)(2 - b_3) - H_1 \ , e_2 = \tfrac{H_1}{\gamma_2} + 2(1 - b_3) \ , e_0 = H_1 - 1 + b_3 - (\gamma_1 + 1)(\tfrac{H_1}{\gamma_2} + 4 - b_3) \ .$$

Proof. From (70) and (56) we get (76). The relation (77) results from (71) and (57), (78) proceeds from (60) – (61) and (79) results from (59) and (78). Lastly, (80) follows from (58), (77) and (79). It is also valid for $n = 1$ through (63) and (78). (81) and (82) ensue respectively from (72) – (73) and (67) – (69). With (63) and (65), the relation (81) gives (83). Finally, the conditions (74), (75) and (82) become (84). □

Lemma 3.2. *For $n \geq 1$, we have*

(85)
$$\gamma_{n+1}(2n-2-\tfrac{H_1}{\gamma_2})(2n-4-\tfrac{H_1}{\gamma_2}) = \gamma_1\Big(b_3-4-\tfrac{H_1}{\gamma_2}\Big)\Big(n-1-\tfrac{1+(-1)^n}{2}\big(1+\tfrac{H_1}{\gamma_2}\big)\Big)(-1)^n$$
$$-\Big(n-1+\tfrac{1-(-1)^n}{2}H_1+\tfrac{1+(-1)^n}{2}\Big)\Big(n-2-\tfrac{H_1}{\gamma_2}-\tfrac{1+(-1)^n}{2}H_1-\tfrac{1-(-1)^n}{2}\Big)\,.$$

Proof. Put

$$\eta_n = (2n-2-\frac{H_1}{\gamma_2})(2n-4-\frac{H_1}{\gamma_2})\ ,\ n \geq 1\,.$$

Then, (80) becomes

$$\eta_{n+1}-\eta_n = (2n-2-\frac{H_1}{\gamma_2})\Big((-1)^n(F_1-F_2-1)-1\Big)\ ,\ n \geq 1\,,$$

with $\frac{H_1}{\gamma_2} \neq 2n\,,\ n \geq -1$. Hence,

$$\eta_{n+1}-\eta_1 = \sum_{k=1}^{n}(2k-2-\frac{H_1}{\gamma_2})\Big((-1)^k(F_1-F_2-1)-1\Big)\ ,\ n \geq 1\,.$$

Now, let the so-called Abel's transformation

(86)
$$\sum_{j=1}^{n}\varpi_j\omega_j = \sum_{j=1}^{n-1}(\varpi_j-\varpi_{j+1})\sum_{k=1}^{j}\omega_k+\varpi_n\sum_{j=1}^{n}\omega_j\ ,\ n \geq 1\,.$$

Taking $\varpi_k = 2k-2-\frac{H_1}{\gamma_2}$ and $\omega_k = (-1)^k(F_1-F_2-1)-1$ in (86), we obtain after some calculation

$$\eta_{n+1}-\eta_1 = n(\frac{H_1}{\gamma_2}-n+1)-(F_2-F_1+1)\Big(\frac{1-(-1)^n}{2}\frac{H_1}{\gamma_2}+(n-1)(-1)^n+\frac{1+(-1)^n}{2}\Big)\ ,\ n \geq 0\,.$$

This yields (85) from (82) and (83). □

Proposition 3.3. *The are three families of Laguerre-Hahn forms u of class one, satisfying (4) with $\beta_0 = 1$. They verifie*

(87)
$$\big(x\left(x^2-1\right)u\big)'+\left(e_2x^2+x+e_0\right)u+(x-1)\left(b_3x^2+b_1\right)\left(x^{-1}u^2\right) = 0\,,$$

and their corresponding MOPS $\{B_n\}_{n\geq 0}$ fulfil (53) into the following cases **F 1.**

(88)
$$b_3 = 2(1-\alpha)\,, b_1 = 2\mu\tfrac{\alpha-1}{\alpha-2}-\alpha\nu\,, e_2 = 2(\alpha-2)\,, e_0 = -\alpha-\mu+1\,, \gamma_1 = -\tfrac{\mu+2+2\nu-\alpha-\alpha\nu}{2(2-\alpha)}\,,$$
$$\gamma_{n+1} = -\tfrac{1}{4}\frac{\big(n+(\alpha-\frac{1}{2})(1-(-1)^n)\big)\big(n+\alpha+\mu+(1+2\mu)(\frac{(-1)^n-1}{2})\big)}{(n+\alpha)(n+\alpha-1)}\ ,\ n \geq 1\,.$$

F 2.

(89)

$b_3=2(\frac{1}{\rho}-1)(\alpha+\beta+2\tau+1)$, $b_1=2(1-\frac{1}{\rho})(\alpha+\beta+2\tau+1)-2\rho\frac{(\tau+1)(\tau+\beta+1)}{\alpha+\beta+2\tau+1}+2(1-\alpha)$,
$e_2=2(1-\frac{2}{\rho})(\alpha+\beta+2\tau)-\frac{4}{\rho}$, $e_0=\frac{4}{\rho}(\alpha+\beta+2\tau+1)-(4\tau+2\beta+3)$,
$\gamma_1=-\rho\frac{(\tau+\beta+1)(\tau+1)}{(\alpha+\beta+2\tau+1)(\alpha+\beta+2\tau+2)}$,
$\gamma_{n+1}=-\frac{1}{4}\frac{\left((n+2\tau+2+(\alpha-\frac{1}{2})(1-(-1)^n)\right)\left((n+2\tau+2\beta+2+(\alpha-\frac{1}{2})(1-(-1)^n)\right)}{(n+2\tau+\alpha+\beta+1)(n+2\tau+\alpha+\beta+2)}$, $n\geq 1$.

F 3.

(90)

$b_3=2(\frac{1}{\rho}-1)(\alpha+\beta+2\tau+1)$, $b_1=2(1-\frac{1}{\rho})(\alpha+\beta+2\tau+1)-2\rho\frac{(\tau+\alpha+1)(\tau+\alpha+\beta+1)}{\alpha+\beta+2\tau+1}+2(\alpha+1)$,
$e_2=2(1-\frac{2}{\rho})(\alpha+\beta+2\tau)-\frac{4}{\rho}$, $e_0=2(\frac{2}{\rho}-1)(\alpha+\beta+2\tau+1)-(2\alpha+1)$,
$\gamma_1=-\rho\frac{(\tau+\alpha+\beta+1)(\tau+\alpha+1)}{(\alpha+\beta+2\tau+1)(\alpha+\beta+2\tau+2)}$, $\gamma_{n+1}=-\frac{1}{4}\frac{\left((n+2\tau+1+(\alpha+\frac{1}{2})(1+(-1)^n)\right)\left((n+2\tau+2\beta+1+(\alpha+\frac{1}{2})(1+(-1)^n)\right)}{(n+2\tau+\alpha+\beta+1)(n+2\tau+\alpha+\beta+2)}$,
$n\geq 1$.

Proof. From (85), we have the following: (1) If $b_3-4-\frac{H_1}{\gamma_2}=0$, then (85) becomes

$$\gamma_{n+1}=-\frac{\left(n-1+\frac{1-(-1)^n}{2}H_1+\frac{1+(-1)^n}{2}\right)\left(n-2-\frac{H_1}{\gamma_2}-\frac{1+(-1)^n}{2}H_1-\frac{1-(-1)^n}{2}\right)}{(2n-2-\frac{H_1}{\gamma_2})(2n-4-\frac{H_1}{\gamma_2})}\ ,\ n\geq 1\ .$$

Put $b_3=-2(\alpha-1)$ and $\frac{H_1}{\gamma_2}=-2(\alpha+1)$. Then, we get

$$\gamma_{n+1}=-\frac{\left(n-1+\frac{1-(-1)^n}{2}H_1+\frac{1+(-1)^n}{2}\right)\left(n-2-\frac{H_1}{\gamma_2}-\frac{1+(-1)^n}{2}H_1-\frac{1-(-1)^n}{2}\right)}{4(n+\alpha)(n+\alpha-1)}\ ,\ n\geq 1\ .$$

We can choose $\gamma_2=-\frac{\alpha-\mu}{2(\alpha+1)}$ or $H_1=\alpha-\mu$ and $\gamma_1=-\frac{\mu+2+2\nu-\alpha-\alpha\nu}{2(2-\alpha)}$. Here, we obtain (88) from (84). (2) If $b_3-4-\frac{H_1}{\gamma_2}\neq 0$, then from (85), we can deduce

$$\gamma_{2n+1}=-\frac{(2n+\xi_1)(2n+\xi_2)}{(4n-2-\frac{H_1}{\gamma_2})(4n-4-\frac{H_1}{\gamma_2})}\ ,\ n\geq 1\ ,$$

$$\gamma_{2n+2}=-\frac{(2n+\xi_3)(2n+\xi_4)}{(4n-\frac{H_1}{\gamma_2})(4n-2-\frac{H_1}{\gamma_2})}\ ,\ n\geq 1\ ,$$

with

(91) $$\xi_1+\xi_2=-\gamma_1(b_3-4-\frac{H_1}{\gamma_2})-\frac{H_1}{\gamma_2}-H_1-2\ ,$$

(92) $$\xi_1\xi_2=\gamma_1(b_3-4-\frac{H_1}{\gamma_2})(\frac{H_1}{\gamma_2}+2)\ ,$$

(93) $$\xi_3+\xi_4=\gamma_1(b_3-4-\frac{H_1}{\gamma_2})+H_1-\frac{H_1}{\gamma_2}-2\ ,$$

$$\xi_3\xi_4 = -(\frac{H_1}{\gamma_2}+2)H_1 \; . \tag{94}$$

Adding (91) and (93), we get

$$\xi_1+\xi_2+\xi_3+\xi_4 = -2(\frac{H_1}{\gamma_2}+2) \; . \tag{95}$$

Hence, from (94), we can deduce

$$H_1 = 2\frac{\xi_3\xi_4}{\xi_1+\xi_2+\xi_3+\xi_4} \; . \tag{96}$$

According to (92) and (95), we obtain

$$\gamma_1(b_3-4-\frac{H_1}{\gamma_2}) = -2\frac{\xi_1\xi_2}{\xi_1+\xi_2+\xi_3+\xi_4} \; . \tag{97}$$

Thus, taking into account (95), we get

$$b_3 = -2\gamma_1^{-1}\frac{\xi_1\xi_2}{\xi_1+\xi_2+\xi_3+\xi_4} - \frac{1}{2}(\xi_1+\xi_2+\xi_3+\xi_4)+2 \; . \tag{98}$$

Using (91), (95), (96) and (97), we deduce

$$(\xi_1+\xi_4-\xi_2-\xi_3)(\xi_1+\xi_3-\xi_2-\xi_4) = 0 \; . \tag{99}$$

This leads to the following cases: • If $\xi_1+\xi_4-\xi_2-\xi_3=0$, denoting $\xi_1 = 2(\tau+\beta+1),\ \xi_2 = 2(\tau+1),\ \xi_3 = 2(\tau+\alpha+\beta+1),\ \xi_4 = 2(\tau+\alpha+1)$ and $\gamma_1 = -\rho\frac{(\tau+\beta+1)(\tau+1)}{(\alpha+\beta+2\tau+1)(\alpha+\beta+2\tau+2)}$. Then, from (95), (96) and (97), we get

$$\frac{H_1}{\gamma_2} = -2(2\tau+\alpha+\beta+3) \; ,$$

$$H_1 = 2\frac{(\tau+\alpha+\beta+1)(\tau+\alpha+1)}{2\tau+\alpha+\beta+2} \; ,$$

$$b_3 = 2(\frac{1}{\rho}-1)(2\tau+\alpha+\beta+1) \; .$$

Hence, from (84) we obtain (89). • If $\xi_1+\xi_3-\xi_2-\xi_4=0$, , choosing $\xi_1 = 2(\tau+\alpha+\beta+1),\ \xi_2 = 2(\tau+\alpha+1),\ \xi_3 = 2(\tau+1),\ \xi_4 = 2(\tau+\beta+1)$ and $\gamma_1 = -\rho\frac{(\tau+\alpha+\beta+1)(\tau+\alpha+1)}{(\alpha+\beta+2\tau+1)(\alpha+\beta+2\tau+2)}$. Thus, from (95), (96) and (97), we get

$$\frac{H_1}{\gamma_2} = -2(2\tau+\alpha+\beta+3) \; ,$$

$$H_1 = 2\frac{(\tau+\beta+1)(\tau+1)}{2\tau+\alpha+\beta+2} \; ,$$

$$b_3 = 2(\frac{1}{\rho}-1)(2\tau+\alpha+\beta+1) \; .$$

Here, we can deduce (90) from (84). □

Proposition 3.4. *Under the conditions of Proposition 3.3, the coefficients* C_n *and* D_n, $n \geq 0$, *of the structure relation of* $\{B_n\}_{n\geq 0}$ *in each cases are given by:* **F 1.** $C_0(x) = (1-2\alpha)x^2 - x + \alpha + \mu$, $D_0(x) = 0$, $C_{n+1}(x) = (2n+2\alpha-1)x^2 + (-1)^n x - 2n - \alpha - \mu + (1+(-1)^n)(\mu - \alpha + 1)$, $n \geq 0$, $D_{n+1}(x) = 2(n+\alpha)(x-(-1)^n)$, $n \geq 0$. **F 2.** $C_0(x) = -\big(2(1-\frac{2}{\rho})(\alpha+\beta+2\tau+1)+1\big)x^2 - x - \frac{4}{\rho}(\alpha+\beta+2\tau+1) + 2(2\tau+\beta+2)$, $D_0(x) = \frac{2(2\tau+\alpha+\beta+1)}{\rho}(x+1)$, $C_{n+1}(x) = \big(2n+4\tau+2\alpha+2\beta+3\big)x^2 + (-1)^n x - 2(n+2\tau+\beta+2) + \big((-1)^n - 1\big)(2\alpha-1)$, $D_{n+1}(x) = 2(n+2\tau+\alpha+\beta+2)(x-(-1)^n)$, **F 3.** $C_0(x) = -\big(2(1-\frac{2}{\rho})(\alpha+\beta+2\tau+1)+1\big)x^2 - x - 2(\frac{2}{\rho}-1)(\alpha+\beta+2\tau+1) + 2(\alpha+1)$, $D_0(x) = \frac{2(2\tau+\alpha+\beta+1)}{\rho}(x+1)$, $C_{n+1}(x) = \big(2n+4\tau+2\alpha+2\beta+3\big)x^2 + (-1)^n x - 2(n+2\tau+\beta+1) - \big((-1)^n+1\big)(2\alpha+1)$, $D_{n+1}(x) = 2(n+2\tau+\alpha+\beta+2)(x-(-1)^n)$.

Proof. C_0 and D_0 are calculated respectively from (13) and (10). The elements G_n , E_n , F_n , H_n and K_n representing in (54) are given from (76), (77), (78), (79), (80), (82) and (83) while taking into account that from (52), we get $K_0 = b_1 - c_0 - d_0$. Then, we have

$$C_{n+1}(x) = C_0(x) + 2(n+1-b_3)x^2 + (1+(-1)^n)x - 2n + 2 + (1+(-1)^n)(F_2+1) + (1-(-1)^n)F_1 \ , \ n \geq 1 \ ,$$

$$C_1(x) = C_0(x) + 2(1-b_3)x^2 + 2x + 2F_0 \ ,$$

$$D_{n+1}(x) = (2n - 2 - \frac{H_1}{\gamma_2})(x-(-1)^n) \ , \ n \geq 1 \ , D_1(x) = \gamma_1^{-1} K_0 (x-1) \ .$$

What gives the result in the different cases. □

Remark 6. If we take $\tau = 0$ and $\rho = 1$ in (89), we get the result given in [3, 10, 15]. Indeed, the form u is semiclassical and satisfies the functional equation

$$\Big((x^3 - x)u\Big)' + \Big(-2(\alpha+\beta+2)x^2 + x + 2\beta + 1\Big)u = 0 \ ,$$

and its corresponding MOPS $\{B_n\}_{n\geq 0}$ fulfills (53) with

$$\gamma_{n+1} = -\frac{1}{4}\frac{\big((n+1+(\alpha+\frac{1}{2})(1+(-1)^n)\big)\big((n+2\beta+1+(\alpha+\frac{1}{2})(1+(-1)^n)\big)}{(n+\alpha+\beta+1)(n+\alpha+\beta+2)} \ , \ n \geq 0 \ .$$

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