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(δ, γ)-GENERALIZED DUNKL LIPSCHITZ FUNCTIONS IN THE SPACE L^2_Q

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Abstract. Using a generalized dual translation operator, we obtain an analog of Theorem 5.2 in Younis (1986) for the Dunkl transform for functions satisfying the (δ, γ)-generalized Dunkl Lipschitz condition in the space L^2_Q.
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1. Introduction and preliminaries

Younis Theorem 5.2 [5] characterized the set of functions in $\mathrm{L}^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely, we have the following

Theorem 1.1. [5] *Let $f \in \mathrm{L}^2(\mathbb{R})$. Then the following are equivalents*

(1) $\|f(x+h) - f(x)\|_2 = O\left(\frac{h^\alpha}{(\log \frac{1}{h})^\beta}\right)$ *as* $h \longrightarrow 0,\ 0 < \alpha < 1, \beta > 0,$

(2) $\int_{|x|\geq r} |\mathcal{F}(f)(x)|^2 dx = O\left(\frac{r^{-2\alpha}}{(\log r)^{2\beta}}\right)$ *as* $r \longrightarrow +\infty,$

where $\mathcal{F}$ stands for the Fourier transform of f.

In this paper, we prove an analog of this theorem 1.1 for the generalized Dunkl transform in the space L^2_Q.

Consider the first-order singular differential-difference operator on the real line

$$\mathrm{D}f(x) = \frac{df}{dx} + (\alpha + \frac{1}{2})\frac{f(x) - f(-x)}{x} + q(x)f(x),$$

where $\alpha > -\frac{1}{2}$ and q is a C^∞ real-valued odd function on $\mathbb{R}$. For $q = 0$, we obtain the classical Dunkl operator

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$$\mathrm{D}_\alpha f(x) = \frac{df}{dx} + (\alpha + \frac{1}{2})\frac{f(x) - f(-x)}{x}$$

Put

$$Q(x) = \exp\left(-\int_0^x q(t)dt\right), \tag{1}$$

with Q is a even function.

We denote by:

(1) $\mathrm{L}^2_\alpha(\mathbb{R})$ the class of measurable functions f on $\mathbb{R}$ with the norm

$$\|f\|_{2,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^2|x|^{2\alpha+1}dx\right)^{1/2} < +\infty$$

(2) $\mathrm{L}^2_Q = \mathrm{L}^2_Q(\mathbb{R})$ the class of measurable functions f on $\mathbb{R}$ for which

$$\|f\|_{2,Q} = \|Qf\|_{2,\alpha} < +\infty,$$

where Q is given by formula (1).

The following statement is proved in [3]

Lemma 1.2. (1) *For each $\lambda \in \mathbb{C}$, the differential-difference equation*

$$\mathrm{D}u = i\lambda u, \quad u(0) = 1$$

admits a unique C^∞ solution on $\mathbb{R}$, denoted by ψ_λ given by

$$\psi_\lambda(x) = Q(x)e_\alpha(i\lambda x),$$

where e_α denotes the Dunkl kernel on $\mathbb{R}$ defined by

$$e_\alpha(z) = j_\alpha(iz) + \frac{z}{2\alpha+2}j_{\alpha+1}(iz), \quad z \in \mathbb{C}$$

j_α being the normalized spherical Bessel function given by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{k!\Gamma(n+p+1)}(\frac{z}{2})^{2n}, z \in \mathbb{C},$$

(2) *For all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and $n = 0, 1, 2, ...,$ we have*

$$\left|\frac{\partial^n}{\partial\lambda^n}\psi_\lambda(x)\right| \le Q(x)|x|^n e^{|Im\lambda||x}$$

Lemma 1.3. *For $x \in \mathbb{R}$ the following inequalities are fulfilled.*

(1) $|j_\alpha(x)| \le 1$,

(2) $|1 - j_\alpha(x)| \ge c$ *with* $|x| \ge 1$, *where* $c > 0$ *is a certain constant which depends only on* α.

Proof. (analog of Lemma 2.9 in [4]). ■

In the terms of $j_\alpha(x)$, we have (see [1])

$$1 - j_\alpha(x) = O(1), \ x \ge 1, \tag{2}$$

$$1 - j_\alpha(x) = O(x^2), \ 0 \le x \le 1. \tag{3}$$

Definition 1.4. *The generalized Dunkl transform for a function $f \in \mathrm{L}^1_Q$ is defined by*

$$\mathcal{F}_{\mathrm{D}}(f)(\lambda) = \int_{\mathbb{R}} f(x)\psi_{-\lambda}(x)|x|^{2\alpha+1}dx.$$

From [2], we have two following theorems

Theorem 1.5. *Let $f \in \mathrm{L}^1_Q$ such that $\mathcal{F}_{\mathrm{D}}(f) \in \mathrm{L}^1_Q$. Then*

$$f(x)(Q(x))^2 = m_\alpha \int_{\mathbb{R}} \mathcal{F}_{\mathrm{D}}(f)(\lambda)\psi_\lambda(x)|\lambda|^{2\alpha+1}d\lambda,$$

where

$$m_\alpha = \frac{1}{2^{2\alpha+2}(\Gamma(\alpha+1))^2}$$

Theorem 1.6. (1) *For every $f \in \mathrm{L}^2_Q$ we have the Plancherel formula*

$$\int_{\mathbb{R}} |f(x)|^2(Q(x))^2|x|^{2\alpha+1}dx = m_\alpha \int_{\mathbb{R}} |\mathcal{F}_{\mathrm{D}}(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda$$

(2) *The generalized Dunkl transform $\mathcal{F}_{\mathrm{D}}$ extends uniquely to an isometric isomorphism from L^2_Q onto L^2_α.*

From [2], we define the generalized dual translation operators are given by

$$\mathrm{T}_h f(x) = \frac{Q(h)}{Q(x)}\tau_\alpha^{-h}(Qf)(x),$$

where the Dunkl translation operators

$$\tau_\alpha^{-h} f(x) = \int_{\mathbb{R}} f(z)d\mu^\alpha_{h,x}(z),$$

and $\mu^{\alpha}_{h,x}$ is a finite signed measure on $\mathbb{R}$, of total mass 1, with support

$$[-|h|-|x|,||h|-|x||]\cup[||h|-|x||,|h|+|x|]$$

and such that $\|\mu^{\alpha}_{h,x}\|\leq 2$.

By [2], we have the formula

$$\mathcal{F}_{\mathrm{D}}(\mathrm{T}_h f)(\lambda)=\psi_{-\lambda}(h)\mathcal{F}_{\mathrm{D}}(f)(\lambda),\ f\in \mathrm{L}^2_Q \tag{4}$$

2. Main Result

In this section we give the main result of this paper. We need first to define the (δ,γ)-generalized Dunkl Lipschitz class

Definition 2.1. *Let $0<\delta<1$ and $\gamma>0$. A function $f\in \mathrm{L}^2_Q$ is said to be in the (δ,γ)-generalized Dunkl Lipschitz class, denoted by $Lip(Q,\delta,\gamma)$, if*

$$\|\mathrm{T}_h f(.)+\mathrm{T}_{-h}f(.)-2Q(h)f(.)\|_{2,Q}=O\left(\frac{Q(h)h^{\delta}}{(\log\frac{1}{h})^{\gamma}}\right)\ \text{as } h\longrightarrow 0.$$

Theorem 2.2. *Let $f\in \mathrm{L}^2_Q$. Then the following are equivalents*

1. $f\in Lip(Q,\delta,\gamma)$,
2. $\int_{|\lambda|\geq r}|\mathcal{F}_{\mathrm{D}}(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda=O\left(\frac{r^{-2\delta}}{(\log r)^{2\gamma}}\right)$ *as* $r\longrightarrow+\infty$.

Proof.

1) $\Longrightarrow$ 2) Assume that $f\in Lip(Q,\delta,\gamma)$. Then

$$\|\mathrm{T}_h f(.)+\mathrm{T}_{-h}f(.)-Q(h)f(.)\|_{2,\alpha,n}=O\left(\frac{Q(h)h^{\delta}}{(\log\frac{1}{h})^{\gamma}}\right)\ \text{as } h\to 0.$$

From formula (4), we have

$$\begin{aligned}
\mathcal{F}_{\mathrm{D}}\left(\mathrm{T}_h f+\mathrm{T}_{-h}f-2Q(h)f\right)(\lambda) &= \left(\psi_{-\lambda}(h)+\psi_{-\lambda}(-h)-2Q(h)\right)\mathcal{F}_{\mathrm{D}}(f)(\lambda)\\
&= (Q(h)e_{\alpha}(-i\lambda h)+Q(-h)e_{\alpha}(i\lambda h)-2Q(h))\mathcal{F}_{\mathrm{D}}(f)(\lambda)\\
&= \left((Q(h)(e_{\alpha}(-i\lambda h)+e_{\alpha}(i\lambda h))-2Q(h)\right)\mathcal{F}_{\mathrm{D}}(f)(\lambda)\\
&= 2Q(h)(j_{\alpha}(\lambda h)-1)\mathcal{F}_{\mathrm{D}}(f)(\lambda)
\end{aligned}$$

Plancherel identity gives

$$\|\mathrm{T}_h f(.) + \mathrm{T}_{-h} f(.) - 2Q(h) f(.)\|_{2,Q}^2 = m_\alpha \int_{\mathbb{R}} (2Q(h))^2 |1 - j_\alpha(\lambda h)|^2 |\mathcal{F}_{\mathrm{D}}(f)(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda$$

If $|\lambda| \in [\frac{1}{h}, \frac{2}{h}]$, then $|\lambda h| \geq 1$ and (2) of Lemma 1.3 implies that

$$1 \leq \frac{1}{c^2} |1 - j_\alpha(\lambda h)|^2$$

Then

$$\begin{aligned}
\int_{\frac{1}{h} \leq |\lambda| \leq \frac{2}{h}} |\mathcal{F}_{\mathrm{D}}(f)(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda &\leq \frac{1}{c^2} \int_{\frac{1}{h} \leq |\lambda| \leq \frac{2}{h}} |1 - j_\alpha(\lambda h)|^2 |\mathcal{F}_{\mathrm{D}}(f)(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda \\
&\leq \frac{1}{c^2} \int_{\mathbb{R}} |1 - j_\alpha(\lambda h)|^2 |\mathcal{F}_{\mathrm{D}}(f)(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda \\
&\leq \frac{1}{c^2} \frac{1}{4 m_\alpha (Q(h))^2} \|\mathrm{T}_h f(.) + \mathrm{T}_{-h} f(.) - 2Q(h) f(.)\|_{2,Q}^2 \\
&= O\left(\frac{h^{2\delta}}{(\log \frac{1}{h})^{2\gamma}} \right)
\end{aligned}$$

We obtain

$$\int_{r \leq |\lambda| \leq 2r} |\mathcal{F}_{\mathrm{D}}(f)(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda = O\left(\frac{r^{-2\delta}}{(\log r)^{2\gamma}} \right) \ as\ r \to +\infty$$

Thus these exists $C > 0$ such that

$$\int_{r \leq |\lambda| \leq 2r} |\mathcal{F}_{\mathrm{D}}(f)(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda \leq C \frac{r^{-2\delta}}{(\log r)^{2\gamma}}$$

So that

$$\begin{aligned}
\int_{|\lambda| \geq r} |\mathcal{F}_{\mathrm{D}}(f)(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda &= \left[\int_{r \leq |\lambda| \leq 2r} + \int_{2r \leq |\lambda| \leq 4r} + \int_{4r \leq |\lambda| \leq 8r} + \ldots.. \right] |\mathcal{F}_{\mathrm{D}}(f)(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda \\
&\leq C \frac{r^{-2\delta}}{(\log r)^{2\gamma}} + C \frac{(2r)^{-2\delta}}{(\log 2r)^{2\gamma}} + C \frac{(4r)^{-2\delta}}{(\log 4r)^{2\gamma}} + \ldots\ldots \\
&\leq C \frac{r^{-2\delta}}{(\log r)^{2\gamma}} (1 + 2^{-2\delta} + (2^{-2\delta})^2 + (2^{-2\delta})^3 + \ldots..) \\
&\leq C C_\delta \frac{r^{-2\delta}}{(\log r)^{2\gamma}}
\end{aligned}$$

where $C_\delta = (1 - 2^{-2\delta})^{-1}$ since $2^{-2\delta} < 1$.

This proves that

$$\int_{|\lambda|\geq r} |\mathcal{F}_{\mathrm{D}}(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda = O\left(\frac{r^{-2\delta}}{(\log r)^{2\gamma}}\right) \; as \; r \longrightarrow +\infty.$$

2) $\Longrightarrow$ 1) Suppose now that

$$\int_{|\lambda|\geq r} |\mathcal{F}_{\mathrm{D}}(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda = O\left(\frac{r^{-2\delta}}{(\log r)^{2\gamma}}\right) \; as \; r \longrightarrow +\infty.$$

We write

$$\int_{\mathbb{R}} |1 - j_\alpha(\lambda h)|^2 |\mathcal{F}_{\mathrm{D}}(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda = \mathrm{I}_1 + \mathrm{I}_2,$$

where

$$\mathrm{I}_1 = \int_{|\lambda|<\frac{1}{h}} |1 - j_\alpha(\lambda h)|^2 |\mathcal{F}_{\mathrm{D}}(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda$$

and

$$\mathrm{I}_2 = \int_{|\lambda|\geq\frac{1}{h}} |1 - j_\alpha(\lambda h)|^2 |\mathcal{F}_{\mathrm{D}}(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda$$

Estimate the summands I_1 and I_2.

From inequality (1) of Lemma 1.3, we have

$$\begin{aligned} \mathrm{I}_2 &= \int_{|\lambda|\geq\frac{1}{h}} |1 - j_\alpha(\lambda h)|^2 |\mathcal{F}_{\mathrm{D}}(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda \\ &\leq 4\int_{|\lambda|\geq\frac{1}{h}} |\mathcal{F}_{\mathrm{D}}(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda \\ &= O\left(\frac{h^{2\delta}}{(\log \frac{1}{h})^{2\gamma}}\right). \end{aligned}$$

Then

$$4(Q(h))^2\mathrm{I}_2 = O\left(\frac{Q(h)^2 h^{2\delta}}{(\log \frac{1}{h})^{2\gamma}}\right).$$

To estimate I_1, we use the inequality (3). Set

$$\psi(x) = \int_x^{+\infty} |\mathcal{F}_{\mathrm{D}}(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda.$$

An integration by parts, we obtain

$$
\begin{aligned}
\int_0^x |\mathcal{F}_{\rm D}(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda &= \int_0^x -\lambda^2\psi'(x)dx \\
&= -x^2\psi(x)+2\int_0^x \lambda\psi(\lambda)d\lambda \\
&\leq C_1\int_0^x \lambda\lambda^{-2\delta}(\log\lambda)^{-2\gamma}d\lambda \\
&= O(x^{2-2\delta}(\log x)^{-2\gamma}),
\end{aligned}
$$

We use the formula (3)

$$
\begin{aligned}
\int_{\mathbb{R}}|1-j_\alpha(\lambda h)|^2|\mathcal{F}_{\rm D}(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda &= O(h^2\int_{|\lambda|<\frac{1}{h}}\lambda^2|\mathcal{F}_{\rm D}(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda)+O\left(\frac{h^{2\delta}}{(\log\frac{1}{h})^{2\gamma}}\right) \\
&= O\left(h^2h^{2\delta-2}(\log\frac{1}{h})^{-2\gamma}\right)+O\left(\frac{h^{2\delta}}{(\log\frac{1}{h})^{2\gamma}}\right) \\
&= O\left(\frac{h^{2\delta}}{(\log\frac{1}{h})^{2\gamma}}\right)
\end{aligned}
$$

Therefore

$$
m_\alpha\int_{\mathbb{R}}(2Q(h))^2|1-j_\alpha(\lambda h)|^2|\mathcal{F}_{\rm D}(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda = O\left(\frac{Q(h)^2h^{2\delta}}{(\log\frac{1}{h})^{2\gamma}}\right),
$$

and this ends the proof. ■

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