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SUMMATION THEOREMS INVOLVING APPELL'S HYPERGEOMETRIC FUNCTIONS F_2

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Abstract. The objective of this paper is to find the closed form of summation theorem for Appell's double hypergeometric function second kind with suitable convergence conditions. In this paper, we find summation theorems of $F_2[A;B,C;A,A;x,y]$, where y takes form $\frac{1-x}{1+x}$, $\frac{x-1}{2x-1}$, $\frac{8(1-x)}{8+x}$, $\frac{1-x}{1+8x}$, $\frac{x-1}{9x-1}$, $\frac{1-x}{1+3x}$, $\frac{x-1}{4x-1}$, $\frac{3(1-x)}{3+x}$, $\frac{x-1}{3x-1}$, $\frac{1-x}{1+2x}$ and other rational functions of x.

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1. Introduction, Definitions and Preliminaries

In the usual notation, let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. For definitions of Pochhammer symbol, generalized hypergeometric function ${}_pF_q$, we refer monumental work of Srivastava and Manocha [28], Rainville [22] and other notation have their usual meanings.

The Appell's function: The Appell's function of second kind is defined as

$$F_2[A;B,C;D,G;x,y]=\sum_{r,s=0}^{\infty}\frac{(A)_{r+s}(B)_r(C)_s}{(D)_r(G)_s}\frac{x^r y^s}{r!s!}$$

$$(1.1)\qquad =\sum_{s=0}^{\infty}\frac{(A)_s(C)_s}{(G)_s}\frac{y^s}{s!}\,{}_2F_1\left[\begin{array}{r}A+s,B;\\ D;\end{array}\ x\right]$$

$$\left(|x|+|y|<1,\ D,\ G\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

Classical Gauss summation theorem [28, p.30, Equation 1.2(6)]

$$(1.2)\qquad {}_2F_1\left[\begin{array}{r}a,b;\\ d;\end{array}\ 1\right]=\frac{\Gamma(d)\Gamma(d-a-b)}{\Gamma(d-a)\Gamma(d-b)}\ ,\ \left(\Re(d-a-b)>0,\ d\in\mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

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Kummer's first summation theorem [9, p.852, Equation (1.3)]:

$$ {}_2F_1\left[\begin{matrix} a, b; \\ a-b+1; \end{matrix} \; -1\right] = \frac{2^{-a}\sqrt{\pi}\,\Gamma(a-b+1)}{\Gamma(\frac{1+a}{2})\Gamma(\frac{a}{2}-b+1)} = \frac{\Gamma(1+a-b)\Gamma(1+\frac{a}{2})}{\Gamma(1+\frac{a}{2}-b)\Gamma(1+a)}. \tag{1.3} $$

$$ \left(a-b \in \mathbb{C}\backslash\mathbb{Z}^-;\ \Re(b) < 1\right). $$

In the year 2007, some generalization of the Kummer's first summation theorem were given by Choi-Rathie and Malani [8, pp. 1523–1524, Equations (2.2), (2.3)], recently some more generalization of the Kummer's first summation theorem were derived by the Qureshi-Baboo [19, p.14, Equations (3.1), (3.2), (3.3) and (3.4)].

Kummer's second summation theorem [9, p.852, Equation (1.4)]

$$ {}_2F_1\left[\begin{matrix} a, \quad b; \\ \frac{1+a+b}{2}; \end{matrix} \; \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1+a+b}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})}; \qquad \left(\frac{1+a+b}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{1.4} $$

A generalization of the Kummer's second summation theorem is recorded by Prudnikov *et al.* [18, p.(491), Entry (7.3.7.2)]. In the year 2011, a generalization of the Kummer's second summation theorem was given by Rakha-Rathie [23, p.827, Theorems (1)], recently some more generalization of the Kummer's second summation theorem were derived by the Qureshi-Baboo [20, p.48, Equations (3.1) and (3.3)].

Kummer's third summation theorem [9, p.852, Equation (1.5)]

$$ {}_2F_1\left[\begin{matrix} a, 1-a; \\ c; \end{matrix} \; \frac{1}{2}\right] = \frac{2^{1-c}\sqrt{\pi}\,\Gamma(c)}{\Gamma(\frac{c+a}{2})\Gamma(\frac{c+1-a}{2})} = \frac{\Gamma(\frac{c}{2})\Gamma(\frac{c+1}{2})}{\Gamma(\frac{c+a}{2})\Gamma(\frac{c+1-a}{2})}; \qquad \left(c \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{1.5} $$

In the year 2011, some generalization of the Kummer's third summation theorem were given by Rakha-Rathie [23, p.828, Theorems (6, 5)], recently some more generalization of the Kummer's third summation theorem were derived by the Qureshi-Baboo [21, Equations (3.3) and (3.5)].

Along with these summation theorems, there are so many summation theorems for Gauss hypergeometric function ${}_2F_1$ with different argument. Some summation theorems are recoded in the monographs of Abramowitz [1], Andrews *et al.* [2], Brychkov [5], Erdélyi *et al.* [10], results conjectured by Gosper and given by Geesel and Stanton [11], results derived by Heymann [12] and [13], Per W. karlsson [14], Kummer [15], Lavoie and Trottier [16], Luke [17], Prudnikov *et al.* [18] and Spiegel [26].

2. Summation Theorems Derived by the Reduction Formula 2.1

Any values of parameters and variables leading to the result which do not make sense, are tacitly excluded, then we can write the Appell's function of second kind in the following form

$$ F_2[A; B, C; A, A; x, y] = (1-x)^{-B}(1-y)^{-C}\,{}_2F_1\left[\begin{matrix} B, C; \\ A; \end{matrix} \; \frac{xy}{(1-x)(1-y)}\right] \tag{2.1} $$

$$\left(|x|<1,|y|<1,|\frac{xy}{(1-x)(1-y)}|<1;\ A\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A=a, B=b, C=c, x=\frac{1}{2}$ and $y=\frac{1}{2}$, use the classical Gauss summation theorem, we get

$$F_2\left[a;b,c;a,a;\frac{1}{2},\frac{1}{2}\right]=2^{b+c}\frac{\Gamma(a)\Gamma(a-b-c)}{\Gamma(a-b)\Gamma(a-c)} \tag{2.2}$$

$$\left(Re(a-b-c)>0;\ a\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A=\frac{1+b+c}{2}, B=b, C=c$ and $y=\frac{1-x}{1+x}$, use the classical Gauss summation theorem, we get

$$F_2\left[\frac{1+b+c}{2};b,c;\frac{1+b+c}{2},\frac{1+b+c}{2};x,\frac{1-x}{1+x}\right]=\frac{(1+x)^c\Gamma(\frac{1}{2})\Gamma(\frac{1+b+c}{2})}{(2x)^c(1-x)^b\Gamma(\frac{1+b}{2})\Gamma(\frac{1+c}{2})} \tag{2.3}$$

$$\left(0<x<1;\ \tfrac{1+b+c}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A=\frac{1+b+c-m}{2}, B=b, C=c$ and $y=\frac{1-x}{1+x}$, using the generalization of Kummer's second summation theorem recorded by Prudnikov *et al.* [18, p.491, Entry (7.3.7.2)], we get

$$F_2\left[\frac{1+b+c-m}{2};b,c;\frac{1+b+c-m}{2},\frac{1+b+c-m}{2};x,\frac{1-x}{1+x}\right]$$

$$=\frac{2^{b-1}(1+x)^c\Gamma(\frac{b+c+1-p}{2})}{(1-x)^b(2x)^c\Gamma(b)}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{\Gamma(\frac{b+r}{2})}{\Gamma(\frac{c+1+r-m}{2})}\right\}, \tag{2.4}$$

$$\left(0<x<1;\ b,\ \tfrac{1+c+b-m}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m\in\mathbb{N}_0\right).$$

In the equation (2.1) put $A=\frac{1+b+c+m}{2}, B=b, C=c$ and $y=\frac{1-x}{1+x}$, using the generalization of Kummer's second summation theorem given by Rakha-Rathie [23, p.827, Theorems (1)], we get

$$F_2\left[\frac{1+b+c+m}{2};b,c;\frac{1+b+c+m}{2},\frac{1+b+c+m}{2};x,\frac{1-x}{1+x}\right]$$

$$=\frac{2^{b-1}(1+x)^c\Gamma(\frac{b+c+1+m}{2})\Gamma(\frac{c-b+1-m}{2})}{(1-x)^b(2x)^c\Gamma(b)\Gamma(\frac{c-b+1+m}{2})}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(-1)^r\Gamma(\frac{b+r}{2})}{\Gamma(\frac{c+1+r-m}{2})}\right\}, \tag{2.5}$$

$$\left(0<x<1;\ b,\ \tfrac{c+b+1+m}{2},\ \tfrac{c-b+1-m}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m\in\mathbb{N}_0\right).$$

In the equation (2.1) put $A=\frac{b+c-m}{2}, B=b, C=c$ and $y=\frac{1-x}{1+x}$, using the generalization of Kummer's second summation theorem given by Qureshi-Baboo [20, p.48, Equation (3.1)], we get

$$F_2\left[\frac{b+c-m}{2};b,c;\frac{b+c-m}{2},\frac{b+c-m}{2};x,\frac{1-x}{1+x}\right]$$

$$=\frac{2^{c-1}(1+x)^c\Gamma(\frac{c+b-m}{2})}{(1-x)^b(2x)^c\Gamma(c)}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{\Gamma(\frac{r+c}{2})}{\Gamma(\frac{b+r-m}{2})}+\frac{\Gamma(\frac{r+c+1}{2})}{\Gamma(\frac{b+r-m+1}{2})}\right]\right\}, \tag{2.6}$$

$$\left(0<x<1;\ c, \tfrac{c+b-m}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m\in\mathbb{N}_0\right),$$

In the equation (2.1) put $A=\frac{b+c+m}{2}, B=b, C=c$ and $y=\frac{1-x}{1+x}$, using the generalization of Kummer's second summation theorem given by Qureshi-Baboo [20, p.48, Equation (3.3)], we get

$$F_2\left[\frac{b+c+m}{2};b,c;\frac{b+c+m}{2},\frac{b+c+m}{2};x,\frac{1-x}{1+x}\right]$$

$$=\frac{2^{b-1}(1+x)^c\Gamma(\frac{c+b-m}{2})\Gamma(\frac{c-b-m}{2})}{(1-x)^b(2x)^c\Gamma(b)\Gamma(\frac{c-b+m}{2})}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{(-1)^r\Gamma(\frac{r+b}{2})}{\Gamma(\frac{c+r-m}{2})}+\frac{(-1)^r\Gamma(\frac{r+b+1}{2})}{\Gamma(\frac{c+r-m+1}{2})}\right]\right\}, \tag{2.7}$$

$$\left(0<x<1;\ b, \tfrac{c+b+m}{2}, \tfrac{c-b-m}{2}, \tfrac{c-b+m}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m\in\mathbb{N}_0\right),$$

In the equation (2.1) put $A=a, B=b, C=1-b$ and $y=\frac{1-x}{1+x}$, use the Kummer's third summation theorem, we get

$$F_2\left[a;b,1-b;a,a;x,\frac{1-x}{1+x}\right]=\frac{(1+x)^{1-b}\Gamma(\frac{a}{2})\Gamma(\frac{a+1}{2})}{(1-x)^b(2x)^{1-b}\Gamma(\frac{a+b}{2})\Gamma(\frac{a-b+1}{2})} \tag{2.8}$$

$$\left(0<x<1;\ a\in\mathbb{C}\backslash\mathbb{Z}_0^-\right),$$

In the equation (2.1) put $A=a, B=b, C=1-b-m$ and $y=\frac{1-x}{1+x}$, using the generalization of Kummer's third summation theorem given by Rakha-Rathie [23, p.828, Theorem (6)], we get

$$F_2\left[a;b,1-b-m;a,a;x,\frac{1-x}{1+x}\right]$$

$$=\frac{2^{1-m-a}(1+x)^{1-b-m}\Gamma(\frac{1}{2})\Gamma(a)}{(1-x)^b(2x)^{1-b-m}\Gamma(\frac{a-b}{2})\Gamma(\frac{a-b+1}{2})}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{\Gamma(\frac{a-b+r}{2})}{\Gamma(\frac{a+b+r}{2})}\right\} \tag{2.9}$$

$$\left(0<x<1;\ a,\ a-b\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m\in\mathbb{N}_0\right);$$

In the equation (2.1) put $A=a, B=b, C=1-b+m$ and $y=\frac{1-x}{1+x}$, using the generalization of Kummer's third summation theorem given by Rakha-Rathie [23, p.828, Theorem (5)], we get

$$F_2\left[a;b,1-b+m;a,a;x,\frac{1-x}{1+x}\right]$$

$$=\frac{2^{1+m-a}(1+x)^{1-b+m}\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b-m)}{(2x)^{1-b+m}(1-x)^b\Gamma(b)\Gamma(\frac{a-b}{2})\Gamma(\frac{a-b+1}{2})}\sum_{r=0}^{m}\left\{(-1)^r\binom{m}{r}\frac{\Gamma(\frac{a-b+r}{2})}{\Gamma(\frac{a+b+r}{2}-m)}\right\} \tag{2.10}$$

$$\left(0<x<1;\ a,\ b,\ b-m,\ a-b\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m\in\mathbb{N}_0\right);$$

In the equation (2.1) put $A=a, B=b, C=-b-m$ and $y=\frac{1-x}{1+x}$, using the generalization of Kummer's third summation theorem given by Qureshi-Baboo [21, p.144, Equation (3.3)], we get

$$F_2\left[a;b,-b-m;a,a;x,\frac{1-x}{1+x}\right]$$

$$= \frac{2^{-1-m-b}(2x)^{b+m}\Gamma(a)}{(1-x)^b(1+x)^{b+m}\Gamma(a-b)}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{\Gamma(\frac{a-b+r}{2})}{\Gamma(\frac{a+b+r}{2})}+\frac{\Gamma(\frac{a-b+r+1}{2})}{\Gamma(\frac{a+b+r+1}{2})}\right]\right\} \tag{2.11}$$

$$\left(0<x<1;\ a,\ a-b\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m\in\mathbb{N}_0\right);$$

In the equation (2.1) put $A=a, B=b, C=-b+m$ and $y=\frac{1-x}{1+x}$, using the generalization of Kummer's third summation theorem given by Qureshi-Baboo [21, p.145, Equation (3.5)], we get

$$F_2\left[a;b,-b+m;a,a;x,\frac{1-x}{1+x}\right]$$

$$= \frac{(x)^{b-m}\Gamma(a)\Gamma(b-m)}{2(1-x)^b(1+x)^{b-m}\Gamma(b)\Gamma(a-b)}\sum_{r=0}^{m}\left\{\binom{m}{r}(-1)^r\left[\frac{\Gamma(\frac{a-b+r}{2})}{\Gamma(\frac{a+b+r-2m}{2})}+\frac{\Gamma(\frac{a-b+r+1}{2})}{\Gamma(\frac{a+b+r-2m+1}{2})}\right]\right\} \tag{2.12}$$

$$\left(0<x<1;\ a,\ a-b,\ b-m\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m\in\mathbb{N}_0\right);$$

In the equation (2.1) put $A=1+b-c, B=b, C=c$ and $y=\frac{x-1}{2x-1}$, use the Kummer's first summation theorem, we get

$$F_2\left[1+b-c;b,c;1+b-c,1+b-c;x,\frac{x-1}{2x-1}\right]=\frac{(2x-1)^c\Gamma(1+b-c)\Gamma(1+\frac{b}{2})}{x^c(1-x)^b\Gamma(1+b)\Gamma(1+\frac{b}{2}-c)} \tag{2.13}$$

$$\left(\Re(c)<1\,;\frac{2}{3}<x<1;\ 1+b-c\in\mathbb{C}\backslash\mathbb{Z}_0^-\right);$$

In the equation (2.1) put $A=1+b-c-m, B=b, C=c$ and $y=\frac{x-1}{2x-1}$, using the generalization of Kummer's first summation theorem given by Choi, Rathie and Malani [8, p.1524, Equation (2.2)], we get

$$F_2\left[1+b-c-m;b,c;1+b-c-m,1+b-c-m;x,\frac{x-1}{2x-1}\right]$$

$$= \frac{(2x-1)^c\Gamma(1+b-c-m)}{2(x)^c(1-x)^b\Gamma(b)}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{\Gamma(\frac{r+b}{2})}{\Gamma(\frac{r+b}{2}+1-c-m)}\right\} \tag{2.14}$$

$$\left(\Re(c)<\left(\tfrac{2-m}{2}\right);\ \frac{2}{3}<x<1;\ b,\ 1+b-c-m\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m\in\mathbb{N}_0\right),$$

In the equation (2.1) put $A=1+b-c+m, B=b, C=c$ and $y=\frac{x-1}{2x-1}$, using the generalization of Kummer's first summation theorem given by Choi, Rathie and Malani [8, p.1524, Equation (2.3)], we get

$$F_2\left[1+b-c+m;b,c;1+b-c+m,1+b-c+m;x,\frac{x-1}{2x-1}\right]$$

$$= \frac{(2x-1)^c\Gamma(1+b-c+m)}{2\,(x)^c(1-x)^b\Gamma(b)(1-c)_m}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(-1)^r\Gamma(\frac{r+b}{2})}{\Gamma(\frac{r+b}{2}+1-c)}\right\} \tag{2.15}$$

$$\left(\Re(c)<\left(\tfrac{m+2}{2}\right);\ \frac{2}{3}<x<1;\ b,\ 1-c,\ 1+b-c+m\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ m\in\mathbb{N}_0\right).$$

In the equation (2.1) put $A = b - c - m, B = b, C = c$ and $y = \frac{x-1}{2x-1}$, use the summation theorem given by Qureshi-Baboo [19, p.14, Equation (3.1)], we get

$$F_2\left[b-c-m; b, c; b-c-m, b-c-m; x, \frac{x-1}{2x-1}\right]$$

$$= \frac{(2x-1)^c\Gamma(b-c-m)}{2\,(x)^c(1-x)^b\Gamma(b)}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{\Gamma(\frac{b+r}{2})}{\Gamma(\frac{b+r-2c-2m}{2})}+\frac{\Gamma(\frac{b+r+1}{2})}{\Gamma(\frac{b+r+1-2c-2m}{2})}\right]\right\}, \tag{2.16}$$

$$\left(\Re(c) < \left(\tfrac{1-m}{2}\right);\ \tfrac{2}{3} < x < 1;\ b,\ b-c-m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\right),$$

In the equation (2.1) put $A = b - c + m, B = b, C = c$ and $y = \frac{x-1}{2x-1}$, use the summation theorem given by Qureshi-Baboo [19, p.14, Equation (3.2)], we get

$$F_2\left[b-c+m; b, c; b-c+m, b-c+m; x, \frac{x-1}{2x-1}\right]$$

$$= \frac{(2x-1)^c\Gamma(b-c+m)}{2\,(x)^c(1-x)^b\Gamma(b)(-c)_m}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{(-1)^r\Gamma(\frac{b+r}{2})}{\Gamma(\frac{b+r-2c}{2})}+\frac{(-1)^r\Gamma(\frac{b+r+1}{2})}{\Gamma(\frac{b+r+1-2c}{2})}\right]\right\}, \tag{2.17}$$

$$\left(\Re(c) < \left(\tfrac{m+1}{2}\right);\ \tfrac{2}{3} < x < 1;\ b,\ -c,\ b-c+m \in \mathbb{C}\backslash\mathbb{Z}_0^-\ ; m \in \mathbb{N}_0\right).$$

In the equation (2.1) put $A = \frac{a+4-\sqrt{2-a}}{2}, B = a, C = \frac{a-2-\sqrt{2-a}}{2}$ and $y = \frac{x-1}{2x-1}$ and using a result of Brychkov [5, p.579, Equation (100)], we get

$$F_2\left[\frac{a+4-\sqrt{2-a}}{2}; a, \frac{a-2-\sqrt{2-a}}{2}; \frac{a+4-\sqrt{2-a}}{2}, \frac{a+4-\sqrt{2-a}}{2}; x, \frac{x-1}{2x-1}\right] =$$

$$= \left(\frac{x}{2x-1}\right)^{\frac{2-a+\sqrt{2-a}}{2}} \frac{2+a(3+\sqrt{2-a})}{(1-x)^a 2^{a+1}} \tag{2.18}$$

$$\left(\frac{2}{3} < x < 1\ ;\ 4a + \frac{2}{3},\ 2a + \frac{5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A = \frac{a+5-\sqrt{7-3a}}{2}, B = a, C = \frac{a-3-\sqrt{7-3a}}{2}$ and $y = \frac{x-1}{2x-1}$ and using a result of Brychkov [5, p.579, Equation (101)], we get

$$F_2\left[\frac{a+5-\sqrt{7-3a}}{2}; a, \frac{a-3-\sqrt{7-3a}}{2}; \frac{a+5-\sqrt{7-3a}}{2}, \frac{a+5-\sqrt{7-3a}}{2}; x, \frac{x-1}{2x-1}\right] =$$

$$= \left(\frac{x}{2x-1}\right)^{\frac{3-a+\sqrt{7-3a}}{2}} \frac{6+a(15-a+4\sqrt{7-3a})}{6(1-x)^a 2^{a+1}} \tag{2.19}$$

$$\left(\frac{2}{3} < x < 1;\ 4a + \frac{2}{3},\ 2a + \frac{5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A=\frac{2a+2}{3}, B=\frac{a}{2}, C=\frac{a+1}{2}$ and $y=\frac{8(1-x)}{x+8}$, using a result of Andrews *et al.* [2, p.131, Entry 3.1.20], Kummer [15, p.136, Article 25(6)] see also Prudnikov *et al.* [18, p.495, Equation (38)], we get

$$F_2\left[\frac{2a+2}{3};\frac{a}{2},\frac{a+1}{2};\frac{2a+2}{3},\frac{2a+2}{3};x,\frac{8(1-x)}{x+8}\right]=\left(\frac{3}{2}\right)^a\frac{(x+8)^{\frac{a+1}{2}}\sqrt{(\pi)}\Gamma(\frac{2a+2}{3})}{(1-x)^{\frac{a}{2}}(9x)^{\frac{a+1}{2}}\Gamma(\frac{a+4}{6})\Gamma(\frac{a+1}{2})} \tag{2.20}$$

$$\left(0<x<1;\ \frac{2a+2}{3}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A=\frac{2}{3}, B=-a, C=2a+1$ and $y=\frac{8(1-x)}{x+8}$ and using a result of Heymann, see Per W. Karlsson [14, p.335, Equation (3.2)], see also Brychkov [5, p.584, Equation (144)], we get

$$F_2\left[\frac{2}{3};-a,2a+1;\frac{2}{3},\frac{2}{3};x,\frac{8(1-x)}{x+8}\right]=\frac{2\times 3^a(1-x)^a(x+8)^{2a+1}\sin(\pi a+\frac{5\pi}{6})}{(9x)^{2a+1}} \tag{2.21}$$

$$\left(0<x<1\right)$$

In the equation (2.1) put $A=\frac{4}{3}, B=-a, C=2a+2$ and $y=\frac{8(1-x)}{x+8}$ and using a result of Heymann, see Per W. Karlsson [14, p.335, Equation (3.3)], see also Brychkov [5, p.584, Equation (145)], we get

$$F_2\left[\frac{4}{3};-a,2a+2;\frac{4}{3},\frac{4}{3};x,\frac{8(1-x)}{x+8}\right]=\frac{3^a(1-x)^a(x+8)^{2a+2}\Gamma(\frac{3}{2})\Gamma(\frac{1}{6})}{(9x)^{2a+2}\Gamma(\frac{1}{6}-a)\Gamma(a+\frac{3}{2})} \tag{2.22}$$

$$\left(0<x<1\right)$$

In the equation (2.1) put $A=4a+\frac{1}{3}, B=3a, C=3a+\frac{1}{4}$ and $y=\frac{8(1-x)}{x+8}$ and using a summation formula conjectured by Gosper, see Per W. Karlsson [14, p.335, Equation (3.4)], see also Brychkov [5, p.584, Equation (146)], we get

$$F_2\left[4a+\frac{1}{3};3a,3a+\frac{1}{4};4a+\frac{1}{3},4a+\frac{1}{3};x,\frac{8(1-x)}{x+8}\right]=$$

$$=\frac{(108)^a(x+8)^{3a+\frac{1}{4}}\Gamma(a+\frac{7}{12})\Gamma(a+\frac{5}{6})\Gamma(\frac{3}{4})\Gamma(\frac{2}{3})}{(1-x)^{3a}(9x)^{3a+\frac{1}{4}}\Gamma(a+\frac{3}{4})\Gamma(a+\frac{2}{3})\Gamma(\frac{7}{12})\Gamma(\frac{5}{6})} \tag{2.23}$$

$$\left(0<x<1;\ 4a+\frac{1}{3},\ a+\frac{7}{12},\ a+\frac{5}{6}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A=4a+\frac{1}{3}, B=3a, C=3a-\frac{1}{4}$ and $y=\frac{8(1-x)}{x+8}$ and using a summation formula conjectured by Gosper, see Per W. Karlsson [14, p.335, Equation (3.5)], see also Brychkov [5, p.584, Equation (147)], we get

$$F_2\left[4a+\frac{1}{3};3a,3a-\frac{1}{4};4a+\frac{1}{3},4a+\frac{1}{3};x,\frac{8(1-x)}{x+8}\right]=$$

$$=\frac{(108)^a(x+8)^{3a-\frac{1}{4}}\Gamma(a+\frac{1}{12})\Gamma(a+\frac{5}{6})\Gamma(\frac{1}{4})\Gamma(\frac{2}{3})}{(1-x)^{3a}(9x)^{3a-\frac{1}{4}}\Gamma(a+\frac{1}{4})\Gamma(a+\frac{2}{3})\Gamma(\frac{1}{12})\Gamma(\frac{5}{6})} \tag{2.24}$$

$$\left(0 < x < 1;\ 4a + \frac{1}{3},\ a + \frac{1}{12},\ a + \frac{5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A = 4a+\frac{2}{3}, B = 3a, C = 3a+\frac{1}{2}$ and $y = \frac{8(1-x)}{x+8}$ and using a result of Kummer [15, p.136, Article 25(6)], see also Per W. Karlsson [14, p.335, Equation (3.1)], see also Brychkov [5, p.584, Equation (143)], we get

$$F_2\left[4a+\frac{2}{3}; 3a, 3a+\frac{1}{2}; 4a+\frac{2}{3}, 4a+\frac{2}{3}; x, \frac{8(1-x)}{x+8}\right] =$$

$$= \frac{(27)^a (9x)^{3a+\frac{1}{2}}\Gamma(2a+\frac{5}{6})\Gamma(\frac{1}{2})}{(x+8)^{3a+\frac{1}{2}}(1-x)^{3a}\Gamma(a+\frac{5}{6})\Gamma(a+\frac{1}{2})} \tag{2.25}$$

$$\left(0 < x < 1;\ 4a + \frac{2}{3},\ 2a + \frac{5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A = 4a, B = a, C = a + \frac{1}{2}$ and $y = \frac{8(1-x)}{x+8}$ and using a result of Prudnikov *et al.* [18, p.496, Equation (41)], we get

$$F_2\left[4a; a, a+\frac{1}{2}; 4a, 4a; x, \frac{8(1-x)}{x+8}\right] = \frac{(3)^{2a}(x+8)^{a+\frac{1}{2}}\Gamma(\frac{1}{2})\Gamma(4a)}{(9x)^{a+\frac{1}{2}}(1-x)^a\Gamma(3a)\Gamma(a+\frac{1}{2})2^{6a-1}} \tag{2.26}$$

$$\left(0 < x < 1\ ;\ 4a + \frac{2}{3},\ 2a + \frac{5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A = \frac{2a+5}{6}, B = \frac{a}{2}, C = \frac{a+1}{2}$ and $y = \frac{1-x}{1+8x}$, using a result of Andrews *et al.* [2, p.131, Entry 3.1.17]; Abramowitz [1, p.557, Entry 15.1.30]; Kummer [15, p.135, Article 25(4)] ; Per W. karlsson [14, p.330, Equation (1.1)] and Brychkov [5, p.580, Equation (115)], we get

$$F_2\left[\frac{2a+5}{6}; \frac{a}{2}, \frac{a+1}{2}; \frac{2a+5}{6}, \frac{2a+5}{6}; x, \frac{1-x}{1+8x}\right] = \left(\frac{3}{4}\right)^a \frac{\sqrt(\pi)(1+8x)^{\frac{a+1}{2}}\Gamma(\frac{2a+2}{3})}{(1-x)^{\frac{a}{2}}(9x)^{\frac{a+1}{2}}\Gamma(\frac{a+4}{6})\Gamma(\frac{a+1}{2})} \tag{2.27}$$

$$\left(0 < x < 1;\ \frac{2a+5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A = a + \frac{5}{6}, B = \frac{1-2a}{2}, C = 2a$ and $y = \frac{1-x}{1+8x}$ and using a result of Heymann, see Per.W. Karlsson [14, p.330, Equation (1.2)] and Brychkov [5, p.580, Equation 116], we get

$$F_2\left[a+\frac{5}{6}; \frac{1-2a}{2}, 2a; a+\frac{5}{6}, a+\frac{5}{6}; x, \frac{1-x}{1+8x}\right] = \frac{3^a(1+8x)^{2a}(1-x)^{\frac{2a-1}{2}}\Gamma(a+\frac{5}{6})\Gamma(\frac{2}{3})}{4^a(9x)^{2a}\Gamma(a+\frac{2}{3})\Gamma(\frac{5}{6})} \tag{2.28}$$

$$\left(0 < x < 1;\ a + \frac{5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A = a + \frac{2}{3}, B = 1 - a, C = 2a$ and $y = \frac{1-x}{1+8x}$ and using a result of Heymann, see Per W. Karlsson [14, p.330, Equation (1.3)], see also Brychkov [5, p.581, Equation (117)], we get

$$F_2\left[a+\frac{2}{3}; 1-a, 2a; a+\frac{2}{3}, a+\frac{2}{3}; x, \frac{1-x}{1+8x}\right] = \frac{3^a(1-x)^{a-1}(1+8x)^{2a}\Gamma(a+\frac{2}{3})\Gamma(\frac{1}{2})}{4^a(9x)^{2a}\Gamma(a+\frac{1}{2})\Gamma(\frac{2}{3})} \tag{2.29}$$

$$\left(0 < x < 1;\ a + \frac{2}{3} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A = 2a + \frac{5}{4}, B = \frac{1}{4} - a, C = -a$ and $y = \frac{1-x}{1+8x}$ and using a summation formula conjectured by Gosper, see Per W. Karlsson [14, p.330, Equation (1.4)], see also Brychkov [5, p.581, Equation (118)], we get

$$F_2\left[2a + \frac{5}{4}; \frac{1}{4} - a, -a; 2a + \frac{5}{4}, 2a + \frac{5}{4}; x, \frac{1-x}{1+8x}\right] = \frac{2^{6a}(1-x)^{a-\frac{1}{4}}(9x)^a\Gamma(2a+\frac{5}{4})\Gamma(\frac{2}{3})\Gamma(\frac{13}{12})}{3^{5a}(1+8x)^a\Gamma(a+\frac{2}{3})\Gamma(a+\frac{13}{12})\Gamma(\frac{5}{4})} \tag{2.30}$$

$$\left(0 < x < 1;\ 2a + \frac{5}{4} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A = 2a + \frac{9}{4}, B = \frac{1}{4} - a, C = -a$ and $y = \frac{1-x}{1+8x}$ and using a summation formula conjectured by Gosper, see Per W. Karlsson [14, p.330, Equation (1.5)], see also Brychkov [5, p.581, Equation (119)], we get

$$F_2\left[2a + \frac{9}{4}; \frac{1}{4} - a, -a; 2a + \frac{9}{4}, 2a + \frac{9}{4}; x, \frac{1-x}{1+8x}\right] = \frac{2^{6a}(1-x)^{a-\frac{1}{4}}(9x)^a\Gamma(2a+\frac{9}{4})\Gamma(\frac{4}{3})\Gamma(\frac{17}{12})}{3^{5a}(1+8x)^a\Gamma(a+\frac{4}{3})\Gamma(a+\frac{17}{12})\Gamma(\frac{9}{4})} \tag{2.31}$$

$$\left(0 < x < 1;\ 2a + \frac{9}{4} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A = \frac{4}{3} - a, B = a, C = 1 - 2a$ and $y = \frac{1-x}{8x+1}$ and using a result of Prudnikov *et al.* [18, p.495, Equation (27)], we get

$$F_2\left[\frac{4}{3} - a; a, 1-2a; \frac{4}{3} - a, \frac{4}{3} - a; x, \frac{1-x}{8x+1}\right] = \left(\frac{9x}{8x+1}\right)^{2a-1} \frac{\Gamma(\frac{2}{3}-a)\Gamma(\frac{4}{3}-a)}{(1-x)^a 3^a \Gamma(\frac{2}{3})\Gamma(\frac{4}{3}-2a)} \tag{2.32}$$

$$\left(0 < x < 1\ ;\ 4a + \frac{2}{3},\ 2a + \frac{5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A = \frac{2a+5}{6}, B = \frac{a}{2}, C = \frac{2-a}{6}$ and $y = \frac{(x-1)}{9x-1}$, using a result of Andrews *et al.* [2, p.177, Question 3(a)], see also Prudnikov *et al.* [18, p.494, Equation (19)], we get

$$F_2\left[\frac{2a+5}{6}; \frac{a}{2}, \frac{2-a}{6}; \frac{2a+5}{6}, \frac{2a+5}{6}; x, \frac{x-1}{9x-1}\right] = \frac{\sqrt(\pi)(9x-1)^{\frac{2-a}{6}}\Gamma(\frac{2a+2}{3})}{2^{\frac{a}{2}}(1-x)^{\frac{a}{2}}(8x)^{\frac{2-a}{6}}\Gamma(\frac{a+4}{6})\Gamma(\frac{a+1}{2})} \tag{2.33}$$

$$\left(0 < x < 1;\ \frac{2a+5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A = -a + \frac{4}{3}, B = a, C = a + \frac{1}{3}$ and $y = \frac{x-1}{9x-1}$, using a result of Lavoie and Trottier [16, p.45, Equation (8)], see also Prudnikov *et al.* [18, p.494, Equation (18)], we get

$$F_2\left[-a + \frac{4}{3}; a, a + \frac{1}{3}; -a + \frac{4}{3}, -a + \frac{4}{3}; x, \frac{x-1}{9x-1}\right] = \left(\frac{2}{3}\right)^{3a} \frac{(9x-1)^{a+\frac{1}{3}}\Gamma(\frac{2}{3}-a)\Gamma(\frac{4}{3}-a)}{(8x)^{a+\frac{1}{3}}(1-x)^a\Gamma(\frac{2}{3})\Gamma(\frac{4}{3}-2a)} \tag{2.34}$$

$$\left(0 < x < 1;\ -a + \frac{4}{3},\ -a + \frac{2}{3} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A = 2b + \frac{1}{2}, B = \frac{1}{2}, C = 1 - b$ and $y = \frac{1-x}{3x+1}$, using a result of Spiegel [26, p.894]; Luke [17, p.273, Equation 6.8(20)], we get

$$F_2\left[2b + \frac{1}{2}; \frac{1}{2}, 1 - b; 2b + \frac{1}{2}, 2b + \frac{1}{2}; x, \frac{1-x}{3x+1}\right] = \frac{2(4x)^{b-1}\Gamma(b)\Gamma(2b + \frac{1}{2})}{3(1-x)^{\frac{1}{2}}(3x-1)^{b-1}\Gamma(2b)\Gamma(b + \frac{1}{2})} \tag{2.35}$$

$$\left(0 < x < 1;\ b,\ 2b + \frac{1}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A = \frac{5}{2} - 2a, B = \frac{1}{2}, C = a$ and $y = \frac{1-x}{3x+1}$ and using a result of Prudnikov *et al.* [18, p.495, Equation (31)], we get

$$F_2\left[\frac{5}{2} - 2a; \frac{1}{2}, a; \frac{5}{2} - 2a, \frac{5}{2} - 2a; x, \frac{1-x}{3x+1}\right] = \left(\frac{3x+1}{4x}\right)^a \frac{2^{2a}\Gamma(\frac{1}{2})\Gamma(\frac{5}{2} - 2a)}{3(1-x)^{\frac{1}{2}}\Gamma(\frac{3}{2} - a)\Gamma(\frac{3}{2} - a)} \tag{2.36}$$

$$\left(0 < x < 1\ ;\ 4a + \frac{2}{3},\ 2a + \frac{5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A = -2a + \frac{3}{2}, B = a, C = a + \frac{1}{2}$ and $y = \frac{x-1}{4x-1}$, using a result of Abramowitz [1, p.557, Entry 15.1.29]; Erdélyi *et al.* [10, p.104, Entry 2.8.53], see also Per W. Karlsson [14, p.334, Equation (2.12)], Prudnikov *et al.* [18, p.494, Equation (15)], we get

$$F_2\left[-2a + \frac{3}{2}; a, a + \frac{1}{2}; -2a + \frac{3}{2}, -2a + \frac{3}{2}; x, \frac{x-1}{4x-1}\right] =$$

$$= \left(\frac{9}{8}\right)^{2a} \frac{(4x-1)^{a+\frac{1}{2}}\Gamma(\frac{4}{3})\Gamma(-2a + \frac{3}{2})}{(1-x)^a(3x)^{a+\frac{1}{2}}\Gamma(\frac{3}{2})\Gamma(-2a + \frac{4}{3})} \tag{2.37}$$

$$\left(0 < x < 1;\ -2a + \frac{3}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A = a + \frac{1}{2}, B = a, C = 1 - \frac{a}{2}$ and $y = \frac{x-1}{4x-1}$ and using a result of Prudnikov *et al.* [18, p.494, Equation (14)], we get

$$F_2\left[a + \frac{1}{2}; a, 1 - \frac{a}{2}; a + \frac{1}{2}, a + \frac{1}{2}; x, \frac{x-1}{4x-1}\right] = \left(\frac{x}{4x-1}\right)^{\frac{a}{2}-1} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{2a+1}{2})}{3(1-x)^a\Gamma(\frac{a+1}{2})\Gamma(\frac{a+1}{2})} \tag{2.38}$$

$$\left(0 < x < 1\ ;\ 4a + \frac{2}{3},\ 2a + \frac{5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A = 3a, B = b, C = \frac{1}{2}$ and $y = \frac{3(1-x)}{x+3}$ and using a result of Luke [17, p.273, Equation 6.8(18)], see also Prudnikov *et al.* [18, p.495, Equation (35)], we get

$$F_2\left[3a; b, \frac{1}{2}; 3a, 3a; x, \frac{3(1-x)}{x+3}\right] = \left(\frac{16}{27}\right)^a \frac{(x+3)^{\frac{1}{2}}\Gamma(a)\Gamma(3a)}{(4x)^{\frac{1}{2}}(1-x)^a(\Gamma(2a))^2} \tag{2.39}$$

$$\left(0 < x < 1\ ;\ a \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A=\frac{5-a}{2}, B=2, C=a$ and $y=\frac{x-1}{3x-1}$ and using a result of Prudnikov *et al.* [18, p.494, Equation (13)], we get

$$F_2\left[\frac{5-a}{2};2,a;\frac{5-a}{2},\frac{5-a}{2};x,\frac{x-1}{3x-1}\right]=\left(\frac{3x-1}{2x}\right)^a\frac{3-a}{3(1-x)^2} \tag{2.40}$$

$$\left(0<x<1\,;\;4a+\frac{2}{3},\;2a+\frac{5}{6}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A=a+2, B=a, C=1-2a$ and $y=\frac{1-x}{1+2x}$ and using a result of Prudnikov *et al.* [18, p.495, Equation (34)], we get

$$F_2\left[a+2;a,1-2a;a+2,a+2;x,\frac{1-x}{1+2x}\right]=\left(\frac{2x}{1+2x}\right)^{2a-1}\frac{2(a+1)}{3(1-x)^a} \tag{2.41}$$

$$\left(0<x<1\,;\;4a+\frac{2}{3},\;2a+\frac{5}{6}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A=a+\frac{3}{4}, B=2a, C=a+\frac{1}{4}$ and $y=\left(\frac{(\sqrt{2}-1)(1-x)}{2x+\sqrt{2}-1}\right)$, using a result of Andrews *et al.* [2, p.177, Question 3(b)], we get

$$F_2\left[a+\frac{3}{4};2a,a+\frac{1}{4};a+\frac{3}{4},a+\frac{3}{4};x,\frac{(\sqrt{2}-1)(1-x)}{2x+\sqrt{2}-1}\right]$$

$$=\frac{\sqrt(\pi)\Gamma(a+\frac{3}{4})(2x+\sqrt{2}-1)^{a+\frac{1}{4}}}{[x(1+\sqrt{2})]^{a+\frac{1}{4}}(1-x)^{2a}(4-2\sqrt{2})^{2a}\Gamma(\frac{2a+3}{4})\Gamma(\frac{a+1}{2})} \tag{2.42}$$

$$\left(0<x<1;\;a+\frac{3}{4}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A=\frac{2a+3}{4}, B=\frac{1}{2}, C=a$ and $y=\frac{(1-\sqrt{2})(1-x)}{2x+(1-\sqrt{2})(1-x)}$ and using a result of Prudnikov *et al.* [18, p.494, Equation (16)], we get

$$F_2\left[\frac{2a+3}{4};\frac{1}{2},a;\frac{2a+3}{4},\frac{2a+3}{4};x,\frac{(1-\sqrt{2})(1-x)}{2x+(1-\sqrt{2})(1-x)}\right]=$$

$$=\left(\frac{2x+(1-\sqrt{2})(1-x)}{2x}\right)^a\frac{\Gamma(\frac{1}{2})\Gamma(\frac{2a+3}{4})}{2^{\frac{a}{2}}(1-x)^{\frac{1}{2}}\Gamma(\frac{a+2}{4})\Gamma(\frac{a+3}{4})} \tag{2.43}$$

$$\left(0<x<1\,;\;4a+\frac{2}{3},\;2a+\frac{5}{6}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A=\frac{2a+5}{6}, B=a, C=\frac{2-a}{3}$ and $y=\frac{(4-3\sqrt{2})(1-x)}{8x+(4-3\sqrt{2})(1-x)}$ and using a result of Prudnikov *et al.* [18, p.495, Equation (22)], we get

$$F_2\left[\frac{2a+5}{6};a,\frac{2-a}{3};\frac{2a+5}{6},\frac{2a+5}{6};x,\frac{(4-3\sqrt{2})(1-x)}{8x+(4-3\sqrt{2})(1-x)}\right]=$$

$$=\left(\frac{8x}{8x+(4-3\sqrt{2})(1-x)}\right)^{\frac{a-2}{3}}\left(\frac{2}{3}\right)^{\frac{a}{2}}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{2a+5}{6})}{(1-x)^a\Gamma(\frac{a+3}{6})\Gamma(\frac{a+5}{6})} \tag{2.44}$$

$$\left(0<x<1\,;\ 4a+\frac{2}{3},\ 2a+\frac{5}{6}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A=\frac{3-2a}{2}, B=a, C=2-3a$ and $y=\frac{(2-\sqrt{3})(1-x)}{4x+(2-\sqrt{3})(1-x)}$ and using a result of Prudnikov *et al.* [18, p.495, Equation (25)], we get

$$F_2\left[\frac{3-2a}{2};a,2-3a;\frac{3-2a}{2},\frac{3-2a}{2};x,\frac{(2-\sqrt{3})(1-x)}{4x+(2-\sqrt{3})(1-x)}\right]=$$

$$=\left(\frac{4x}{4x+(2-\sqrt{3})(1-x)}\right)^{3a-2}\frac{3^{\frac{3a}{2}}\Gamma(\frac{4}{3})\Gamma(\frac{3-2a}{2})}{(1-x)^a2^{2a-1}\Gamma(\frac{1}{2})\Gamma(\frac{4-3a}{3})} \tag{2.45}$$

$$\left(0<x<1\,;\ 4a+\frac{2}{3},\ 2a+\frac{5}{6}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A=\frac{2a+1}{2}, B=a, C=\frac{2a+1}{4}$ and $y=\frac{(2\sqrt{2}-2)(1-x)}{x+(2\sqrt{2}-2)(1-x)}$ and using a result of Prudnikov *et al.* [18, p.495, Equation (36)], we get

$$F_2\left[\frac{2a+1}{2};a,\frac{2a+1}{4};\frac{2a+1}{2},\frac{2a+1}{2};x,\frac{(2\sqrt{2}-2)(1-x)}{x+(2\sqrt{2}-2)(1-x)}\right]=$$

$$=\left(\frac{x+(2\sqrt{2}-2)(1-x)}{x}\right)^{\frac{2a+1}{4}}\left(\frac{2+\sqrt{2}}{2}\right)^{a}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{3+2a}{4})}{(1-x)^a\Gamma(\frac{a+2}{4})\Gamma(\frac{a+3}{4})} \tag{2.46}$$

$$\left(0<x<1\,;\ 4a+\frac{2}{3},\ 2a+\frac{5}{6}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A=\frac{4a+1}{3}, B=a, C=\frac{4a+1}{6}$ and $y=\frac{(12\sqrt{2}-16)(1-x)}{x+(12\sqrt{2}-16)(1-x)}$ and using a result of Prudnikov *et al.* [18, p.496, Equation (43)], see also Brychkov [5, p.587, Equation (172)], we get

$$F_2\left[\frac{4a+1}{3};a,\frac{4a+1}{6};\frac{4a+1}{3},\frac{4a+1}{3};x,\frac{(12\sqrt{2}-16)(1-x)}{x+(12\sqrt{2}-16)(1-x)}\right]=$$

$$=\left(\frac{x+(12\sqrt{2}-16)(1-x)}{x}\right)^{\frac{4a+1}{6}}\left(\frac{2+\sqrt{2}}{2}\right)^{2a}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{2a+2}{3})}{(1-x)^a\Gamma(\frac{a+1}{2})\Gamma(\frac{a+4}{6})} \tag{2.47}$$

$$\left(0<x<1\,;\ 4a+\frac{2}{3},\ 2a+\frac{5}{6}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

In the equation (2.1) put $A=4a-1, B=a, C=\frac{4a-1}{2}$ and $y=\frac{(12\sqrt{2}-16)(1-x)}{x+(12\sqrt{2}-16)(1-x)}$ and using a result of Prudnikov *et al.* [18, p.496, Equation (46)], we get

$$F_2\left[4a-1;a,\frac{4a-1}{2};4a-1,4a-1;x,\frac{(12\sqrt{2}-16)(1-x)}{x+(12\sqrt{2}-16)(1-x)}\right]=$$

$$=\left(\frac{x+(12\sqrt{2}-16)(1-x)}{x}\right)^{\frac{4a-1}{2}}\left(\frac{2+\sqrt{2}}{2}\right)^{2a}\frac{\Gamma(a)}{(1-x)^a\Gamma(\frac{3a}{2})} \tag{2.48}$$

$$\left(0<x<1\,;\ 4a+\frac{2}{3},\ 2a+\frac{5}{6}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right)$$

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$$1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1).\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2).\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\ldots,$$

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