

SUMMATION FORMULAS FOR THE CONFLUENT HYPERGEOMETRIC FUNCTION $\Psi_2^{(2r)}$ OF SEVERAL VARIABLES

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ABSTRACT. The aim of this paper is to establish a general summation formulas for the confluent hypergeometric function $\Psi_2^{(2r)}$ in several variables by applying the generalized Kummer's summation theorem due to Lavoie *et al.* [5]. Further, a number of generating functions for products of two Laguerre polynomials of two variables are also derived as an applications of our main summation formula.

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1. INTRODUCTION

The generalized hypergeometric function ${}_pF_q$ with p numerator parameters and q denominator parameters is defined by (see [10, p.42 (1)])

$$(1.1) \quad {}_pF_q \left[\begin{matrix} a_1, \dots, a_p ; \\ b_1, \dots, b_q ; \end{matrix} x \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{x^n}{n!},$$

where $(a)_n$ denotes the Pochhammer's symbol defined by

$$(1.2) \quad (a)_n = \begin{cases} 1 & , \text{if } n=0 \\ a(a+1)(a+2)\dots(a+n-1) & , \text{if } n=1,2,3,\dots \end{cases}$$

The special case of (1) when $p=2$ and $q=1$ is usually called Gauss's hypergeometric function.

The Kampé de Fériet function of two variables $F_{l,m;n}^{p,q;k}[x,y]$ is defined by (see[10,p.63(16)])

$$(1.3) \quad F_{l,m;n}^{p,q;k} \left[\begin{matrix} (a_p) : (b_q) ; (c_k) ; \\ (\alpha_l) : (\beta_m) ; (\gamma_n) ; \end{matrix} x, y \right] = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{r+s} \prod_{j=1}^q (b_j)_r \prod_{j=1}^k (c_j)_s}{\prod_{j=1}^l (\alpha_j)_{r+s} \prod_{j=1}^m (\beta_j)_r \prod_{j=1}^n (\gamma_j)_s} \frac{x^r}{r!} \frac{y^s}{s!}.$$

The confluent hypergeometric function of several variables $\Psi_2^{(n)}$ is defined as follows (see [10, p.62 (11)]):

$$(1.4) \quad \Psi_2^{(n)}(a; c_1, \dots, c_n; x_1, \dots, x_n) = \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n}}{(c_1)_{m_1} \dots (c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \dots \frac{x_n^{m_n}}{m_n!}$$

Clearly, we have

$$\Psi_2^{(2)} = \Psi_2,$$

where Ψ_2 is the Humbert's confluent hypergeometric function in two variables [10, p.59(42)]

$$(1.5) \quad \Psi_2[a; b, b'; x, y] = \sum_{m, n=0}^{\infty} \frac{(a)_{m+n}}{(b)_m (b')_n} \frac{x^m}{m!} \frac{y^n}{n!}.$$

Ragab [7] defined Laguerre polynomials of two variables $L_n^{(\alpha, \beta)}(x, y)$ as follows:

$$(1.6) \quad L_n^{(\alpha, \beta)}(x, y) = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!} \sum_{r=0}^n \frac{(-y)^r L_{n-r}^{(\alpha)}(x)}{r! \Gamma(\alpha+n-r+1) \Gamma(\beta+r+1)},$$

where $L_n^{(\alpha)}(x)$ are the generalized Laguerre polynomials of one variable [8, p.200 (1)].

In particular, we note that

$$(1.7) \quad L_n^{(\alpha, \beta)}(x, 0) = \frac{(\beta+1)_n}{n!} L_n^{(\alpha)}(x) \text{ and } L_n^{(\alpha, \beta)}(0, y) = \frac{(\alpha+1)_n}{n!} L_n^{(\beta)}(y).$$

Recently, Choi and Rathie [3] obtained certain summation formula for Humbert's hypergeometric function $\Psi_2[a; c, c+i; x, -x]$ by using the following generalizations of the Kummer's summation theorem due to Lavoie *et al.* [5]:

$$(1.8) \quad {}_2F_1 \left[\begin{matrix} a, b & ; & -1 \\ 1+a-b+i & ; & \end{matrix} \right] = \frac{\Gamma(\frac{1}{2})\Gamma(1+a-b+i)\Gamma(1-b)}{2^a \Gamma(1-b+\frac{1}{2}(i+|i|))}$$

$$\times \left\{ \frac{A_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(1+\frac{1}{2}a-b+\frac{1}{2}i)} + \frac{B_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-[\frac{i}{2}])\Gamma(\frac{1}{2}+\frac{1}{2}a-b+\frac{1}{2}i)} \right\}$$

for $(i=0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5)$,

where $[x]$ denotes the greatest integer less than or equal to x and $|x|$ denotes the usual absolute value of x . The coefficients A_i and B_i are given in the following table:

TABLE 1.

i	A_i	B_i
5	$-4(6+a-b)^2+2b(6+a-b)+b^2+22(6+a-b)-13b-20$	$4(6+a-b)^2+2b(6+a-b)-b^2-34(6+a-b)-b+62$
4	$2(a-b+3)(1+a-b)-(b-1)(b-4)$	$-4(a-b+2)$
3	$3b-2a-5$	$2a-b+1$
2	$1+a-b$	-2
1	-1	1
0	1	0
-1	1	1
-2	$a-b-1$	2
-3	$2a-3b-4$	$2a-b-2$
-4	$2(a-b-3)(a-b-1)-b(b+3)$	$4(a-b-2)$
-5	$4(a-b-4)^2-2b(a-b-4)-b^2+8(a-b-4)-7b$	$4(a-b-4)^2+2b(a-b-4)-b^2+16(a-b-4)-b+12$

Here, in this paper we establish further extension formula for the confluent hypergeometric function $\Psi_2^{(2r)}$ in several variables with the help of (1.8). As an application of our main result, we obtain certain generating functions for products of two Laguerre polynomials of one and two variable.

2. MAIN RESULTS

In this section, the following summation formula will be established:

$$\begin{aligned}
 &\Psi_2^{(2r)}(a; c_1, c_1+i, c_2, c_2+i, \dots, c_r, c_r+i; x_1, -x_1, x_2, -x_2, \dots, x_r, -x_r) \\
 &= \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(a)_{2m_1+\dots+2m_r} x_1^{2m_1} \dots x_r^{2m_r}}{(c_1)_{2m_1} \dots (c_r)_{2m_r} (2m_1)! \dots (2m_r)!} \\
 &\quad \times (A_i^{(1)} E_1 + B_i^{(1)} F_1) (A_i^{(2)} E_2 + B_i^{(2)} F_2) \dots (A_i^{(r)} E_r + B_i^{(r)} F_r) \\
 &+ \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(a)_{2m_1+1+2m_2+\dots+2m_r} x_1^{2m_1+1} x_2^{2m_2} \dots x_r^{2m_r}}{(c_1)_{2m_1+1} (c_2)_{2m_2} \dots (c_r)_{2m_r} (2m_1+1)! (2m_2)! \dots (2m_r)!} \\
 &\quad \times (C_i^{(1)} G_1 + D_i^{(1)} H_1) (A_i^{(2)} E_2 + B_i^{(2)} F_2) \dots (A_i^{(r)} E_r + B_i^{(r)} F_r) + \dots
 \end{aligned}$$

$$\begin{aligned}
& \dots + \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(a)_{2m_1+2m_2+1+\dots+2m_r+1} x_1^{2m_1} x_2^{2m_2+1} \dots x_r^{2m_r+1}}{(c_1)_{2m_1} (c_2)_{2m_2+1} \dots (c_r)_{2m_r+1} (2m_1)! (2m_2+1)! \dots (2m_r+1)!} \\
& \times (A_i^{(1)} E_1 + B_i^{(1)} F_1) (C_i^{(2)} G_2 + D_i^{(2)} H_2) \dots (C_i^{(r)} G_r + D_i^{(r)} H_r) \\
& + \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(a)_{2m_1+1+\dots+2m_r+1} x_1^{2m_1+1} \dots x_r^{2m_r+1}}{(c_1)_{2m_1+1} \dots (c_r)_{2m_r+1} (2m_1+1)! \dots (2m_r+1)!} \\
(2.1) \quad & \times (C_i^{(1)} G_1 + D_i^{(1)} H_1) (C_i^{(2)} G_2 + D_i^{(2)} H_2) \dots (C_i^{(r)} G_r + D_i^{(r)} H_r),
\end{aligned}$$

where

$$\begin{aligned}
E_r &= \frac{2^{2m_r} \Gamma(\frac{1}{2}) \Gamma(c_r+i) \Gamma(c_r+2m_r)}{\Gamma(c_r+2m_r+\frac{1}{2}(i+|i|)) \Gamma(-m_r+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}]) \Gamma(m_r+c_r+\frac{1}{2}i)} \\
F_r &= \frac{2^{2m_r} \Gamma(\frac{1}{2}) \Gamma(c_r+i) \Gamma(c_r+2m_r)}{\Gamma(c_r+2m_r+\frac{1}{2}(i+|i|)) \Gamma(-m_r+\frac{1}{2}i-[\frac{i}{2}]) \Gamma(m_r+c_r-\frac{1}{2}+\frac{1}{2}i)} \\
G_r &= \frac{2^{2m_r+1} \Gamma(\frac{1}{2}) \Gamma(c_r+i) \Gamma(c_r+2m_r+1)}{\Gamma(c_r+2m_r+1+\frac{1}{2}(i+|i|)) \Gamma(-m_r+\frac{1}{2}i-[\frac{1+i}{2}]) \Gamma(m_r+\frac{1}{2}+c_r+\frac{1}{2}i)} \\
H_r &= \frac{2^{2m_r+1} \Gamma(\frac{1}{2}) \Gamma(c_r+i) \Gamma(c_r+2m_r+1)}{\Gamma(c_r+2m_r+1+\frac{1}{2}(i+|i|)) \Gamma(-m_r-\frac{1}{2}+\frac{1}{2}i-[\frac{i}{2}]) \Gamma(m_r+c_r+\frac{1}{2}i)} \\
& \text{for } i=0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5.
\end{aligned}$$

The coefficients $A_i^{(1)}, \dots, A_i^{(r)}$ and $B_i^{(1)}, \dots, B_i^{(r)}$ can be obtained from the tables of A_i and B_i by replacing a and b by $-2m_j$ and $1-c_j-2m_j$, $j=1, 2, \dots, r$. The coefficients $C_i^{(1)}, \dots, C_i^{(r)}$ and $D_i^{(1)}, \dots, D_i^{(r)}$ can be obtained from the tables of A_i and B_i by replacing a and b by $-2m_j-1$ and $-c_j-2m_j$, $j=1, 2, \dots, r$.

Proof. Denoting the left hand side of (2.1) by $\Psi_2^{(2r)}$, then from the definition (1.4), we have

$$\Psi_2^{(2r)} = \sum_{n_1, p_1, \dots, n_r, p_r=0}^{\infty} \frac{(a)_{n_1+p_1+\dots+n_r+p_r} x_1^{n_1} (-x_1)^{p_1} \dots x_r^{n_r} (-x_r)^{p_r}}{(c_1)_{n_1} (c_1+i)_{p_1} \dots (c_r)_{n_r} (c_r+i)_{p_r} n_1! p_1! \dots n_r! p_r!}$$

Using the well-known result [10, p.22(20)]

$$(2.2) \quad (\alpha)_{m+n} = (\alpha)_m (\alpha+m)_n,$$

we have

$$\begin{aligned}
(2.3) \quad \Psi_2^{(2r)} &= \sum_{n_2, p_2, \dots, n_r, p_r=0}^{\infty} \frac{(a)_{n_2+p_2+\dots+n_r+p_r} x_2^{n_2} (-x_2)^{p_2} \dots x_r^{n_r} (-x_r)^{p_r}}{(c_2)_{n_2} (c_2+i)_{p_2} \dots (c_r)_{n_r} (c_r+i)_{p_r} n_2! p_2! \dots n_r! p_r!} \\
& \times \Psi_2 [a+n_2+p_2+\dots+n_r+p_r; c_1, c_1+i; x_1, -x_1]
\end{aligned}$$

Now, applying the following special case of Manako identity [6]

$$(2.4) \quad \Psi_2(a; b, c; x, -x) = \sum_{m=0}^{\infty} \frac{(a)_m x^m}{(b)_m m!} {}_2F_1 \left[\begin{matrix} -m, 1-b-m & ; \\ & -1 \\ & c & ; \end{matrix} \right],$$

we have

$$\begin{aligned}
 \Psi_2^{(2r)} = & \sum_{m_1, n_2, p_2, \dots, n_r, p_r=0}^{\infty} \frac{(a)_{m_1+n_2+p_2+\dots+n_r+p_r} x_1^{m_1} x_2^{n_2} (-x_2)^{p_2} \dots x_r^{n_r} (-x_r)^{p_r}}{(c_1)_{m_1} (c_2)_{n_2} (c_2+i)_{p_2} \dots (c_r)_{n_r} (c_r+i)_{p_r} m_1! n_2! p_2! \dots n_r! p_r!} \\
 (2.5) \quad & \times {}_2F_1 \left[\begin{matrix} -m_1, 1-c_1-m_1 & ; \\ & -1 \\ c_1+i & ; \end{matrix} \right]
 \end{aligned}$$

Then, by repeating the above steps r -times, we have

$$(2.6) \quad \Psi_2^{(2r)} = \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(a)_{m_1+\dots+m_r} x_1^{m_1} \dots x_r^{m_r}}{(c_1)_{m_1} \dots (c_r)_{m_r} m_1! \dots m_r!} f(c_1, i, m_1) \dots f(c_r, i, m_r),$$

where

$$f(c_r, i, m_r) = {}_2F_1 \left[\begin{matrix} -m_r, 1-c_r-m_r & ; \\ & -1 \\ c_r+i & ; \end{matrix} \right]$$

Separating (2.6) into its even and odd terms, we have

$$\begin{aligned}
 \Psi_2^{(2r)} = & \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(a)_{2m_1+\dots+2m_r} x_1^{2m_1} \dots x_r^{2m_r}}{(c_1)_{2m_1} \dots (c_r)_{2m_r} (2m_1)! \dots (2m_r)!} \\
 & \times f(c_1, i, 2m_1) f(c_2, i, 2m_2) \dots f(c_r, i, 2m_r) \\
 & + \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(a)_{2m_1+1+2m_2+\dots+2m_r} x_1^{2m_1+1} x_2^{2m_2} \dots x_r^{2m_r}}{(c_1)_{2m_1+1} (c_2)_{2m_2} \dots (c_r)_{2m_r} (2m_1+1)! (2m_2)! \dots (2m_r)!} \\
 & \times f(c_1, i, 2m_1+1) f(c_2, i, 2m_2) \dots f(c_r, i, 2m_r) + \dots \\
 & \dots + \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(a)_{2m_1+2m_2+1+\dots+2m_r+1} x_1^{2m_1} x_2^{2m_2+1} \dots x_r^{2m_r+1}}{(c_1)_{2m_1} (c_2)_{2m_2+1} \dots (c_r)_{2m_r+1} (2m_1)! (2m_2+1)! \dots (2m_r+1)!} \\
 & \times f(c_1, i, 2m_1) f(c_2, i, 2m_2+1) \dots f(c_r, i, 2m_r+1) \\
 & + \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(a)_{2m_1+1+\dots+2m_r+1} x_1^{2m_1+1} \dots x_r^{2m_r+1}}{(c_1)_{2m_1+1} \dots (c_r)_{2m_r+1} (2m_1+1)! \dots (2m_r+1)!} \\
 (2.7) \quad & \times f(c_1, i, 2m_1+1) f(c_2, i, 2m_2+1) \dots f(c_r, i, 2m_r+1)
 \end{aligned}$$

Now, in (2.7) applying the generalized Kummer’s theorem (1.8) for each ${}_2F_1(-1)$, we get after some simplification the right hand side of (2.1). This completes the proof of (2.1). □

2.1. Special Cases of (2.1). Here we mention some special cases of (2.1) and we will use the following results [10,p.22-23]:

$$(2.8) \quad \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = (\alpha)_n, \quad \frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1-\alpha)_n}$$

$$(2.9) \quad (\alpha)_{2n} = 2^{2n} \left(\frac{1}{2}\alpha\right)_n \left(\frac{1}{2}\alpha + \frac{1}{2}\right)_n$$

$$(2.10) \quad (2n)! = 2^{2n} \left(\frac{1}{2}\right)_n n!, \quad (2n+1)! = 2^{2n} \left(\frac{3}{2}\right)_n n!.$$

1. Taking $i=0$ in (2.1), we have

$$(2.11) \quad \begin{aligned} \Psi_2^{(2r)}(a; c_1, c_1, \dots, c_r, c_r; x_1, -x_1, \dots, x_r, -x_r) &= \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(a)_{2m_1+\dots+2m_r} x_1^{2m_1} \dots x_r^{2m_r}}{(c_1)_{2m_1} \dots (c_r)_{2m_r} (2m_1)! \dots (2m_r)!} \\ &\times \frac{2^{2m_1} \Gamma(\frac{1}{2}) \Gamma(c_1)}{\Gamma(\frac{1}{2}-m_1) \Gamma(c_1+m_1)} \times \dots \times \frac{2^{2m_r} \Gamma(\frac{1}{2}) \Gamma(c_r)}{\Gamma(\frac{1}{2}-m_r) \Gamma(c_r+m_r)} \end{aligned}$$

Now using the results (2.8)–(2.10) in (2.11), then after some simplification we obtain the following transformation formula:

$$(2.12) \quad \begin{aligned} &\Psi_2^{(2r)}(a; c_1, c_1, \dots, c_r, c_r; x_1, -x_1, \dots, x_r, -x_r) \\ &= {}_F \begin{matrix} 2:0; \dots; 0 \\ 0:3; \dots; 3 \end{matrix} \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} : & - & ; \dots ; & - & ; \\ & & & & & -x_1^2, \dots, -x_r^2 \\ & - & : c_1, \frac{1}{2}c_1, \frac{1}{2}c_1+\frac{1}{2} ; & \dots ; & c_r, \frac{1}{2}c_r, \frac{1}{2}c_r+\frac{1}{2} ; & \end{matrix} \right]. \end{aligned}$$

Note that, the formula (2.12) is a generalization of the well-known result [9, p.322(188)]

$$(2.13) \quad \Psi_2(a; c, c; x, -x) = {}_2F_3 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} ; \\ & -x^2 \\ c, \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2} ; \end{matrix} \right].$$

II. Taking $r=1$ in (2.1), we get a known result of Choi and Rathie [3] for Humbert's hypergeometric series $\Psi_2(a; c, c+i; x, -x)$ for $i=0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$.

3. APPLICATIONS TO GENERATING FUNCTIONS

The following double generating function for products of two Laguerre polynomials of two variables is given by Atash [1] :

$$(3.1) \quad \begin{aligned} &\sum_{m,n=0}^{\infty} \frac{(a)_{m+n} m! n! t^m (-t)^n}{(d_1)_m (d_2)_m (c_1)_n (c_2)_n} L_m^{(d_1-1, d_2-1)}(x_1, x_2) L_n^{(c_1-1, c_2-1)}(x_1, x_2) \\ &= \Psi_2^{(4)}[a; c_1, d_1, c_2, d_2; x_1 t, -x_1 t, x_2 t, -x_2 t] \end{aligned}$$

In (3.1), replacing d_1 by c_1+i and d_2 by c_2+i and applying (2.1) with $r=2$, we get the following generating function:

$$\begin{aligned}
 & \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} m! n! t^m (-t)^n}{(c_1+i)_m (c_2+i)_m (c_1)_n (c_2)_n} L_m^{(c_1+i-1, c_2+i-1)}(x_1, x_2) L_n^{(c_1-1, c_2-1)}(x_1, x_2) \\
 &= \sum_{m_1, m_2=0}^{\infty} \frac{(a)_{2m_1+2m_2} (x_1 t)^{2m_1} (x_2 t)^{2m_2}}{(c_1)_{2m_1} (c_2)_{2m_2} (2m_1)! (2m_2)!} (A_i^{(1)} E_1 + B_i^{(1)} F_1) (A_i^{(2)} E_2 + B_i^{(2)} F_2) \\
 &+ \sum_{m_1, m_2=0}^{\infty} \frac{(a)_{2m_1+2m_2+1} (x_1 t)^{2m_1+1} (x_2 t)^{2m_2}}{(c_1)_{2m_1+1} (c_2)_{2m_2} (2m_1+1)! (2m_2)!} (C_i^{(1)} G_1 + D_i^{(1)} H_1) (A_i^{(2)} E_2 + B_i^{(2)} F_2) \\
 &+ \sum_{m_1, m_2=0}^{\infty} \frac{(a)_{2m_1+2m_2+1} (x_1 t)^{2m_1} (x_2 t)^{2m_2+1}}{(c_1)_{2m_1} (c_2)_{2m_2+1} (2m_1)! (2m_2+1)!} (A_i^{(1)} E_1 + B_i^{(1)} F_1) (C_i^{(2)} G_2 + D_i^{(2)} H_2) \\
 (3.2) \quad &+ \sum_{m_1, m_2=0}^{\infty} \frac{(a)_{2m_1+2m_2+2} (x_1 t)^{2m_1+1} (x_2 t)^{2m_2+1}}{(c_1)_{2m_1+1} (c_2)_{2m_2+1} (2m_1+1)! (2m_2+1)!} (C_i^{(1)} G_1 + D_i^{(1)} H_1) (C_i^{(2)} G_2 + D_i^{(2)} H_2)
 \end{aligned}$$

for $i=0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$.

3.1. Special cases of (3.2). Eleven new generating functions for products of two Laguerre polynomials of two variables can be obtained as special cases of (3.2). Here we will mention only the following cases:

I. Taking $i=0$ in (3.2) and using the results (2.8)–(2.10), we get the following generating function:

$$\begin{aligned}
 & \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} m! n! t^m (-t)^n}{(c_1)_m (c_2)_m (c_1)_n (c_2)_n} L_m^{(c_1-1, c_2-1)}(x_1, x_2) L_n^{(c_1-1, c_2-1)}(x_1, x_2) \\
 (3.3) \quad &= {}_2F_3 \left[\begin{matrix} 2:0;0 \\ 0:3;3 \end{matrix} \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} & - & - \\ - & c_1, \frac{1}{2}c_1, \frac{1}{2}c_1+\frac{1}{2} & c_2, \frac{1}{2}c_2, \frac{1}{2}c_2+\frac{1}{2} \\ & & -x_1^2 t^2, -x_2^2 t^2 \end{matrix} \right. \right.
 \end{aligned}$$

Further, putting $x_2=0$ in (3.3) and using (1.7), we get:

$$(3.4) \quad \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} t^m (-t)^n}{(c)_m (c)_n} L_m^{(c-1)}(x) L_n^{(c-1)}(x) = {}_2F_3 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} & - \\ - & c, \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2} \\ & & -x^2 t^2 \end{matrix} \right].$$

Note that, in (3.4) if we take $a=c$, we get a known result of Exton [4, p.406 (4.3)].

II. Taking $i=1$ in (3.2) and using the results (2.8)–(2.10), we get the following generating function:

$$\begin{aligned}
 & \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} m! n! t^m (-t)^n}{(c_1+1)_m (c_2+1)_m (c_1)_n (c_2)_n} L_m^{(c_1, c_2)}(x_1, x_2) L_n^{(c_1-1, c_2-1)}(x_1, x_2) \\
 &= F_{\begin{matrix} 2:0;0 \\ 0:3;3 \end{matrix}} \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} : & - & ; & - & ; \\ & & & & & -x_1^2 t^2, -x_2^2 t^2 \\ - & : c_1, \frac{1}{2}c_1+\frac{1}{2}, \frac{1}{2}c_1+1 ; c_2, \frac{1}{2}c_2+\frac{1}{2}, \frac{1}{2}c_2+1 ; \end{matrix} \right] \\
 &+ \frac{ax_1 t}{c_1(c_1+1)} F_{\begin{matrix} 2:0;0 \\ 0:3;3 \end{matrix}} \left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 : & - & ; & - & ; \\ & & & & & -x_1^2 t^2, -x_2^2 t^2 \\ - & : c_1+1, \frac{1}{2}c_1+1, \frac{1}{2}c_1+\frac{3}{2} ; c_2, \frac{1}{2}c_2+\frac{1}{2}, \frac{1}{2}c_2+1 ; \end{matrix} \right] \\
 &+ \frac{ax_2 t}{c_2(c_2+1)} F_{\begin{matrix} 2:0;0 \\ 0:3;3 \end{matrix}} \left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 : & - & ; & - & ; \\ & & & & & -x_1^2 t^2, -x_2^2 t^2 \\ - & : c_1, \frac{1}{2}c_1+\frac{1}{2}, \frac{1}{2}c_1+1 ; c_2+1, \frac{1}{2}c_2+1, \frac{1}{2}c_2+\frac{3}{2} ; \end{matrix} \right] \\
 &+ \frac{a(a+1)x_1 x_2 t^2}{c_1 c_2 (c_1+1)(c_2+1)} \\
 (3.5) \quad & \times F_{\begin{matrix} 2:0;0 \\ 0:3;3 \end{matrix}} \left[\begin{matrix} \frac{1}{2}a+1, \frac{1}{2}a+\frac{3}{2} : & - & ; & - & ; \\ & & & & & -x_1^2 t^2, -x_2^2 t^2 \\ - & : c_1+1, \frac{1}{2}c_1+1, \frac{1}{2}c_1+\frac{3}{2} ; c_2+1, \frac{1}{2}c_2+1, \frac{1}{2}c_2+\frac{3}{2} ; \end{matrix} \right].
 \end{aligned}$$

Further, putting $x_2=0$ in (3.5) and using (1.7), we get:

$$\begin{aligned}
 & \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} t^m (-t)^n}{(c+1)_m (c)_n} L_m^{(c)}(x) L_n^{(c-1)}(x) \\
 (3.6) \quad &= {}_2F_3 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} ; \\ c, \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}c+1 ; \end{matrix} \right] + \frac{ax t}{c(c+1)} {}_2F_3 \left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 ; \\ c+1, \frac{1}{2}c+1, \frac{1}{2}c+\frac{3}{2} ; \end{matrix} \right] - x^2 t^2.
 \end{aligned}$$

Note that, In (3.6), if we take $a=c+1$, we get a known result of Choi and Rathie [2, p.197 (18)].

III. Taking $i=-1$ in (3.2) and using the results (2.8)–(2.10), we get the following generating function:

$$\begin{aligned}
 & \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} m! n! t^m (-t)^n}{(c_1-1)_m (c_2-1)_m (c_1)_n (c_2)_n} L_m^{(c_1-2, c_2-2)}(x_1, x_2) L_n^{(c_1-1, c_2-1)}(x_1, x_2) \\
 &= F_{\begin{matrix} 2:0;0 \\ 0:3;3 \end{matrix}} \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} : & - & ; & - & ; \\ & & & & & -x_1^2 t^2, -x_2^2 t^2 \\ - & : c_1-1, \frac{1}{2}c_1, \frac{1}{2}c_1+\frac{1}{2} ; c_2-1, \frac{1}{2}c_2, \frac{1}{2}c_2+\frac{1}{2} ; \end{matrix} \right]
 \end{aligned}$$

$$(3.7) \quad \begin{aligned} & -\frac{ax_1t}{c_1(c_1-1)} F_{0:3;3}^{2:0;0} \left[\begin{array}{c} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 : \quad - \quad ; \quad - \quad ; \\ - \quad : c_1, \frac{1}{2}c_1+\frac{1}{2}, \frac{1}{2}c_1+1 ; c_2-1, \frac{1}{2}c_2, \frac{1}{2}c_2+\frac{1}{2} ; \\ \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 : \quad - \quad ; \quad - \quad ; \\ - \quad : c_1-1, \frac{1}{2}c_1, \frac{1}{2}c_1+\frac{1}{2} ; c_2, \frac{1}{2}c_2+\frac{1}{2}, \frac{1}{2}c_2+1 ; \end{array} \right] \\ & -\frac{ax_2t}{c_2(c_2-1)} F_{0:3;3}^{2:0;0} \left[\begin{array}{c} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 : \quad - \quad ; \quad - \quad ; \\ - \quad : c_1-1, \frac{1}{2}c_1, \frac{1}{2}c_1+\frac{1}{2} ; c_2, \frac{1}{2}c_2+\frac{1}{2}, \frac{1}{2}c_2+1 ; \\ \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 : \quad - \quad ; \quad - \quad ; \\ - \quad : c_1, \frac{1}{2}c_1+\frac{1}{2}, \frac{1}{2}c_1+1 ; c_2-1, \frac{1}{2}c_2, \frac{1}{2}c_2+\frac{1}{2} ; \end{array} \right] \\ & +\frac{a(a+1)x_1x_2t^2}{c_1c_2(c_1-1)(c_2-1)} \\ & \times F_{0:3;3}^{2:0;0} \left[\begin{array}{c} \frac{1}{2}a+1, \frac{1}{2}a+\frac{3}{2} : \quad - \quad ; \quad - \quad ; \\ - \quad : c_1, \frac{1}{2}c_1+\frac{1}{2}, \frac{1}{2}c_1+1 ; c_2, \frac{1}{2}c_2+\frac{1}{2}, \frac{1}{2}c_2+1 ; \\ \frac{1}{2}a+1, \frac{1}{2}a+\frac{3}{2} : \quad - \quad ; \quad - \quad ; \\ - \quad : c_1, \frac{1}{2}c_1+\frac{1}{2}, \frac{1}{2}c_1+1 ; c_2-1, \frac{1}{2}c_2, \frac{1}{2}c_2+\frac{1}{2} ; \end{array} \right]. \end{aligned}$$

Further, putting $x_2=0$ in (3.7) and using (1.7), we get:

$$(3.8) \quad \begin{aligned} & \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} t^m (-t)^n}{(c-1)_m (c)_n} L_m^{(c-2)}(x) L_n^{(c-1)}(x) \\ & = {}_2F_3 \left[\begin{array}{c} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} ; \\ -x^2t^2 \\ c-1, \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2} ; \end{array} \right] -\frac{axt}{c(c-1)} {}_2F_3 \left[\begin{array}{c} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 ; \\ -x^2t^2 \\ c, \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}c+1 ; \end{array} \right]. \end{aligned}$$

Note that, in (3.8), if we take $a=c$, we get a known result of Choi and Rathie [2, p.198 (21)].

IV. Taking $i=2$ in (3.2) and using the results (2.8)–(2.10), we get the following generating function:

$$(3.9) \quad \begin{aligned} & \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} m! n! t^m (-t)^n}{(c_1+2)_m (c_2+2)_m (c_1)_n (c_2)_n} L_m^{(c_1+1, c_2+1)}(x_1, x_2) L_n^{(c_1-1, c_2-1)}(x_1, x_2) \\ & = F_{0:3;3}^{2:0;0} \left[\begin{array}{c} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} : \quad - \quad ; \quad - \quad ; \\ - \quad : c_1+1, \frac{1}{2}c_1+\frac{3}{2}, \frac{1}{2}c_1+1 ; c_2+1, \frac{1}{2}c_2+\frac{3}{2}, \frac{1}{2}c_2+1 ; \\ \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} : \quad - \quad ; \quad - \quad ; \\ - \quad : c_1+1, \frac{1}{2}c_1+\frac{3}{2}, \frac{1}{2}c_1+1 ; c_2+1, \frac{1}{2}c_2+\frac{3}{2}, \frac{1}{2}c_2+1 ; \end{array} \right] \\ & +\frac{2ax_1t}{c_1(c_1+2)} F_{0:3;3}^{2:0;0} \left[\begin{array}{c} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 : \quad - \quad ; \quad - \quad ; \\ - \quad : c_1+1, \frac{1}{2}c_1+\frac{3}{2}, \frac{1}{2}c_1+2 ; c_2+1, \frac{1}{2}c_2+\frac{3}{2}, \frac{1}{2}c_2+1 ; \\ \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 : \quad - \quad ; \quad - \quad ; \\ - \quad : c_1+1, \frac{1}{2}c_1+\frac{3}{2}, \frac{1}{2}c_1+1 ; c_2+1, \frac{1}{2}c_2+\frac{3}{2}, \frac{1}{2}c_2+2 ; \end{array} \right] \\ & +\frac{2ax_2t}{c_2(c_2+2)} F_{0:3;3}^{2:0;0} \left[\begin{array}{c} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 : \quad - \quad ; \quad - \quad ; \\ - \quad : c_1+1, \frac{1}{2}c_1+\frac{3}{2}, \frac{1}{2}c_1+1 ; c_2+1, \frac{1}{2}c_2+\frac{3}{2}, \frac{1}{2}c_2+2 ; \\ \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 : \quad - \quad ; \quad - \quad ; \\ - \quad : c_1+1, \frac{1}{2}c_1+\frac{3}{2}, \frac{1}{2}c_1+1 ; c_2+1, \frac{1}{2}c_2+\frac{3}{2}, \frac{1}{2}c_2+2 ; \end{array} \right] \\ & +\frac{4a(a+1)x_1x_2t^2}{c_1c_2(c_1+2)(c_2+2)} \\ & \times F_{0:3;3}^{2:0;0} \left[\begin{array}{c} \frac{1}{2}a+1, \frac{1}{2}a+\frac{3}{2} : \quad - \quad ; \quad - \quad ; \\ - \quad : c_1+1, \frac{1}{2}c_1+\frac{3}{2}, \frac{1}{2}c_1+2 ; c_2+1, \frac{1}{2}c_2+\frac{3}{2}, \frac{1}{2}c_2+2 ; \\ \frac{1}{2}a+1, \frac{1}{2}a+\frac{3}{2} : \quad - \quad ; \quad - \quad ; \\ - \quad : c_1+1, \frac{1}{2}c_1+\frac{3}{2}, \frac{1}{2}c_1+2 ; c_2+1, \frac{1}{2}c_2+\frac{3}{2}, \frac{1}{2}c_2+2 ; \end{array} \right]. \end{aligned}$$

Further, putting $x_2=0$ in (3.9) and using (1.7), we get:

$$(3.10) \quad \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} t^m (-t)^n}{(c+2)_m (c)_n} L_m^{(c+1)}(x) L_n^{(c-1)}(x) \\ = {}_2F_3 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a + \frac{1}{2} & ; \\ & -x^2 t^2 \end{matrix} \right] + \frac{2axt}{c(c+2)} {}_2F_3 \left[\begin{matrix} \frac{1}{2}a + \frac{1}{2}, \frac{1}{2}a + 1 & ; \\ & -x^2 t^2 \end{matrix} \right].$$

The other special cases of (3.2) can also be obtained in the similar manner.

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