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HAUSDORFFNESS OF GENERAL COMPACTIFICATIONS

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Abstract. Magill proved that the remainders of two locally compact Hausdorff spaces in their Stone-Čech compactifications are homeomorphic if and only if the lattices of their Hausdorff compactifications are lattice isomorphic. His construction for compactifications are explicitely discussed through the partitions of their Stone-Čech compactifications. Partitions in a Stone-Čech compactification which lead to Hausdorff compactifications are characterized in this article. Embeddings of certain upper semi-lattices of compactifications into lattices of compactifications are constructed.

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1. Introduction

Let X be a completely regular Hausdorff space and $\alpha_1 X$ and $\alpha_2 X$ be two Hausdorff compactifications of X. These two compactifications may be compared by an order relation: $\alpha_1 X \geq \alpha_2 X$ if and only if there is a continuous function $h_1 : \alpha_1 X \to \alpha_2 X$ such that $h_1(x) = x$ for all $x \in X$. The collection $K(X)$ of all Hausdorff compactifications of a Tychonoff space X forms a complete upper semi-lattice under the natural order defined above. It is known that for a Tychonoff space X, $K(X)$ is a lattice if and only if X is locally compact (see: theorem 4.3 (e) in [2]). Magill [1] proved that the remainders $\beta X \setminus X$ and $\beta Y \setminus Y$ of X and Y are homeomorphic if and only if $K(X)$ and $K(Y)$ are lattice isomorphic, where X and Y are locally compact spaces. Rayburn [3] considered non locally compact points and obtained some extensions of Magill's results. These two articles are fundamental articles for studies on lattice structure on compactifications and topological structure of remainders. Magill furnished indirectly a construction for all Hausdorff compactifications of a given Tychonoff space. This construction is based on partitions in their Stone-Čech compactifications and this is explained in the first section. Every partition of a Stone-Čech compactification by compact subsets always leads to a compactification, which is the corresponding quotient space. A characterization for partitions which lead to Hausdorff compactifications is discussed in the second section. If Y is the collection of all locally compact points of a given Tychonoff space

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X and if Y is dense in X, then the upper semi-lattice $K(X)$ can be embedded into the lattice $K(Y)$. This is explained in the third section.

2. Magill's Construction

Let $K(X)$ be the collection of all Hausdorff compactifications of a Tychonoff space X and βX be its Stone-Čech compactification. For every $\alpha X \in K(X)$, there is a continuous map, called Čech map, $f_\alpha : \beta X \to \alpha X$ such that $f_\alpha(x) = x$ for all $x \in X$. Also $\{f_\alpha^{-1}(y) : y \in \alpha X\}$ forms a partiton in βX , where each $f_\alpha^{-1}(y)$ is a compact subset of βX, when $y \in \alpha X$. Moreover $\{x\}$ is in this partition, for every $x \in X$. This is justified by the following lemma.

Lemma 2.1. *Let $\alpha_1 X$, $\alpha_2 X$ be two Hausdorff compactifications of X such that $\alpha_1 X \geq \alpha_2 X$. Let $f : \alpha_1 X \to \alpha_2 X$ be the natural continuous onto mapping such that $f(x) = x$, for every $x \in X$. Then$f^{-1}(x) = \{x\}$, for every $x \in X$.*

Proof. On the contrary assume that, there is an element $y \in f^{-1}(x)$ such that $y \neq x$, for some $x \in X$. Then there are two disjoint open neighbourhoods $U_x,\ U_y$ of $x,\ y$ in $\alpha_1 X$, respectively. For $U_x \cap X$, find an open neighbourhood V_x of x in $\alpha_2 X$ such that $V_x \cap X = U_x \cap X$. Since $f(y) = x \in V_x$, find an open neighbourhood W_y of y in $\alpha_1 X$ such that $y \in W_y \subseteq U_y$ and $f(W_y) \subseteq V_x$. Then $W_y \cap X = f(W_y \cap X) \subseteq V_x \cap X = U_x \cap X$. Thus $U_x \cap W_y \cap X \neq \phi$. This contradicts the fact that $U_x \cap W_y \cap X = \phi$. This proves the lemma. □

On the other hand, consider a partition π of βX such that

(i) Every member of π is a compact subset of βX.

(ii) $\{x\} \in \pi$, for every $x \in X$.

Now consider the quotient space $\beta X/\pi$ with the quotient topology induced by a quotient map $f : \beta X \to \beta X/\pi$. Since the quotient map is continuous and βX is compact, f is surjective and $\beta X/\pi$ is a compact space. Also $\beta X/\pi$ is a compactification of X, because X is dense in $\beta X/\pi$. This is a construction of Magill[1] for compactifications. But this compactification may not be Hausdorff unless π is a Hausdorff partition of βX. That is, $\beta X/\pi$ is made into a Hausdorff space under the quotient topology.

Lemma 2.2. *Let X be a Hausdorff space and $\{K_i\}_{i\in I}$ be a collection of mutually disjoint non empty compact subsets of X and it is locally finite in X. Then, for any fixed K_m, there is an open set U such that U contains K_m and U does not intersect any of the $K_j,\ j \neq m$.*

Proof. Let $x \in K_m$. Since $\{K_i\}_{i\in I}$ is locally finite, there is an open set U of x which intersects only finite number of K_i's. Suppose U intersects only K_{i_1}, K_{i_2}..........K_{i_n} other than K_m. Then $U_x = U \setminus \bigcup_{k=i_1}^{i_n} K_k$ does not intersect none of the K_i's other than K_m. Then $\{U_x : x \in K_m\}$ is an open cover for K_m. Since K_m is compact, there is finite subcover $\{U_{x_1}, U_{x_2}, \ldots, U_{x_p}\}$ for K_m. Let Γ be the collection of all K_i which are intersected by the members of this finite subcollection, except Km. Write $G = \bigcup_{i=1}^{p} U_{x_i}$. Let $U = G \setminus \bigcup_{Ki\in\Gamma} K_i$. Then U is an open set containing K_m which does not intersect $K_i,\ i \neq m$. □

Theorem 2.3. *Let X be a Tychonoff space and αX be any Hausdorff compactification of X. Let $\{K_i\}_{i\in I}$ be a collection of mutually disjoint non empty compact subsets of $\alpha X \setminus X$ such that it is locally finite in αX. Then there is a Hausdorff* *compactification* $\gamma X = (\alpha X \setminus \bigcup_{i\in I} K_i) \cup \{p_i : i \in I\}$ *of* X*, where* p_i *are distinct and* $p_i \notin \alpha X$*, and there is a continuous mapping* $h : \alpha X \to \gamma X$ *such that* $h(x) = x$*, for* $x \notin \bigcup_{i\in I} K_i$ *and* $h(x) = p_i$*, for* $x \in K_i$*.*

Proof. Let $\gamma X = (\alpha X \setminus \bigcup_{i\in I} K_i) \cup \{p_i : i \in I\}$ and $Y = (\alpha X \setminus \bigcup_{i\in I} K_i)$ where p_i are distinct, and $p_i \notin \alpha X$. Define a map $h : \alpha X \to \gamma X$ by $h(x) = x$ if $x \in Y$ and $h(x) = p_i$ if $x \in K_i$. Let γX have the quotient topology under the quotient map h. Since αX is compact, γX is compact. Let U be an open set in γX. Then $h^{-1}(U)$ is an open set in αX which intersects X so that $h(h^{-1}(U)) = U$ intersects $h(X) = X$. Hence X is dense in γX. To prove the Hausdorffness, we have to consider the following three cases for any $x,\ y \in \gamma X$ such that $x \neq y$.

(i) $x \in \gamma X \setminus Y$ and $y \in \gamma X \setminus Y$.

(ii) $x \in Y$ and $y \in \gamma X \setminus Y$.

(iii) $x \in Y$ and $y \in Y$.

Case (i):

Let $x \in \gamma X \setminus Y$ and $y \in \gamma X \setminus Y$. Then $x = p_i$ and $y = p_j$, $i \neq j$. Since αX is normal, we can find open sets V and W in αX such that $K_i \subseteq V$ and $K_j \subseteq W$ and $V \cap W = \phi$. Since $\{K_i : i \in I\}$ is locally finite in αX, we can find open sets V_1 and W_1 such that $K_i \subseteq V_1$ and $K_j \subseteq W_1$ and V_1 and W_1 does not intersect any of the K_s's other than K_i and K_j, respectively. Then $V \cap V_1$ and $W \cap W_1$ are open sets in αX such that $(V \cap V_1) \cap (W \cap W_1) = (V \cap W) \cap (V_1 \cap W_1) = \phi$. Let $V_2 = V \cap V_1$ and $W_2 = W \cap W_1$. Let $V^\star = h(V_2)$ and $W^\star = h(W_2)$. Since $h^{-1}(V^\star) = V_2$ and $h^{-1}(W^\star) = W_2$, $V^\star$ and $W^\star$ are disjoint open sets in γX such that $x \in V^\star$ and $y \in W^\star$.

Case (ii):

Let $x \in Y$ and $y \in \gamma X \setminus Y$. Then $y = p_j$ for some $j \in I$. Since αX is normal, we can find open sets V and W in αX such that $x \in V$, $K_j \subseteq W$ and $V \cap W = \phi$. Since $\{K_i\}_{i\in I}$ is locally finite, there is an open set U of x which intersects only finite number of K_k's. Suppose U intersects $K_{i_1}, K_{i_2}..........K_{i_n}$. Then $U_1 = U \setminus \bigcup_{k=i_1}^{i_n} K_k$ does not intersect none of the K_k's. Similarly, find an open set U_2 containing K_j, but not containing other K_k's. Let $V_1 = h(U_1 \cap V)$ and $W_1 = h(U_2 \cap W)$. Since $h^{-1}(V_1) = U_1 \cap V$ and $h^{-1}(W_1) = U_2 \cap W$, V_1 and W_1 are open sets in γX containing x and p_j, respectively such that their intersection is empty.

Case (iii):

Let $x \in Y$ and $y \in Y$. Since $\{K_i\}_{i\in I}$ is locally finite, there exist disjoint open sets U and V in αX containing x and y, respectively such that they intersect only finite number of K_i's. Suppose U intersects $K_{i_1}, K_{i_2}..........K_{i_n}$ and V intersects $K_{j_1}, K_{j_2}..........K_{j_m}$. Since αX is Hausdorff, there exist disjoint open sets U_1 and V_1 such that $x \in U_1$ and $y \in V_1$. Let $U_2 = (U \setminus \bigcup_{k=i_1}^{i_n} K_k) \cap U_1$ and $V_2 = (V \setminus \bigcup_{k=j_1}^{j_m} K_k) \cap V_1$. Since $h(U_2) = U_2$ and $h(V_2) = V_2$, U_2 and V_2 are disjoint open sets in γX containing x and y, respectively. Hence γX is a Hausdorff compactification of X. □

Remark 2.4. *This theorem 2.3 generalizes the lemma 2 in* [1]*. If* K_i *are selected in* $\alpha X = X \cup (\alpha X \setminus X)$*, then theorem 2.3 is true except the fact that* γX *is just a compact Hausdorff space; but not a compactification of* X*.*

3. Hausdorff partitions

Hausdorff partitons lead to Hausdorff compactifications. A characterization for Hausdorff partitions is obtained in this section.

Let X be a Tychonoff space with its Stone-$\check{C}$ech compactification βX. Let π be a partition of βX such that

(i) Every member of π is a compact subset of βX.

(ii) $\{x\} \in \pi$, for every $x \in X$.

Then we have the following theorem.

Theorem 3.1. *Let X be a Tychonoff space and π be a partition of its Stone-Čech compactification βX. Then $\beta X/\pi$ is a Hausdorff compactification of X under the quotient topology if and only if for every $A \in \pi$ and for every open subset U of βX such that $A \subseteq U$, there is an open subset V of βX such that (i) $A \subseteq V \subseteq U$ (ii) V is a union of members of π.*

Proof. Let $f : \beta X \to \beta X/\pi$ be the quotient map and $\beta X/\pi$ be endowed with the quotient topology.

Suppose $\beta X/\pi$ is Hausdorff. Let $A \in \pi$ and U be an open subset of βX such that $A \subseteq U$. Then $\beta X \setminus U$ is closed and hence is a compact subset of βX. Since f is continuous, $f(\beta X \setminus U)$ is compact. Since $\beta X/\pi$ is Hausdorff, $f(\beta X \setminus U)$ is closed in $\beta X/\pi$. Moreover $f(A)$ is a singleton subset of $\beta X/\pi$ and it is contained in the open set $(\beta X/\pi) \setminus f(\beta X \setminus U)$. Choose $V = f^{-1}((\beta X/\pi) \setminus (f(\beta X \setminus U)))$. Then V is an open subset of βX such that (i) and (ii) are true.

Conversely, assume that for every $A \in \pi$ and for every open subset U of βX such that $A \subseteq U$, there is an open subset V of βX such that (i) and (ii) are true. Let us fix two distinct points u_1 and u_2 in $\beta X/\pi$. Let $A_1 = f^{-1}(u_1)$ and $A_2 = f^{-1}(u_2)$. Since f is continuous, A_1 and A_2 are closed subsets of βX. Since βX is normal, there are disjoint open subsets U_1 and U_2 in βX such that $A_1 \subseteq U_1$ and $A_2 \subseteq U_2$. By assumption, there are disjoint open subsets V_1 and V_2 of βX such that $A_1 \subseteq V_1 \subseteq U_1$ and $A_2 \subseteq V_2 \subseteq U_2$ and V_1 and V_2 are unions of members of π. Now $f(V_1)$ and $f(V_2)$ are disjoint open subsets of $\beta X/\pi$ such that $u_1 \in f(V_1)$ and $u_2 \in f(V_2)$. This proves that $\beta X/\pi$ is Hausdorff. This completes the proof of the theorem. ☐

In the previous theorem βX may be replaced by any other compactification αX of X.

4. Embedding into Lattices

A point in a Tychonoff space X is locally compact in X if it has a compact neighbourhood in X. Let Y be the collection of all locally compact points of a Tychonoff space X. Suppose Y is dense in X. (For example, let X be the closed unit disc without some points on the unit circle and Y be an open unit disc in the Euclidean plane). Then Y is dense in βX and Y is locally compact. So Y is open in βX (see: theorem 4.3 in [2]). The collection $K(X)$ of all Hausdorff compactificaions of X is a complete upper semi-lattice. Since Y is locally compact, $K(Y)$ is a complete lattice. Now $K(X)$ is considered as a subset of $K(Y)$, because every Hausdorff compactification of X is a Hausdorff compactification of Y. This identification is an order preserving map. The construction explained in section 1 reveals that this order preserving map also preserves join. Note that the join of two Hausdorff compactifications given by two partitions π_1, π_2 of βX is given by the partition $\{A \cap B : A \in \pi_1, B \in \pi_2\} \setminus \{\phi\}$. So we have the following theorem.

Theorem 4.1. *If Y is the set of all locally compact points of a Tychonoff space X and if Y is dense in X, then the complete upper semi-lattice $K(X)$ can be embedded into the lattice $K(Y)$ by an order preserving map which also preserves join.*

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