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EXISTENCE OF POSITIVE SOLUTIONS FOR $2n^{\text{th}}$ ORDER TWO-POINT p-LAPLACIAN BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we establish the existence of positive solutions for $2n^{\text{th}}$ order two-point p-Laplacian boundary value problems of the form

$$(-1)^n\{[\phi_p(u^{(2n-2)}(t)]'' - \lambda^2[\phi_p(u^{(2n-2)}(t)]\} = f(t, u(t), \ t \in [0, 1],$$

$$u^{(2i)}(0) = 0, u^{(2i)}(1) = 0,$$

$$\phi_p(u^{(2n-2)}(0)) = 0, \phi_p(u^{(2n-2)}(1)) = 0.$$

for $0 \leq i \leq n-2$, where $n \geq 2$, by an application of Guo–Krasnosel'skii fixed point theorem.

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1. Introduction

The theory of differential equations are intrudes in different areas of applied mathematics, physics, mechanics and engineering etc. This theory also serve as an excellent tool for the description of hereditary properties of various materials and processes. In consequence, differential equations have been of great interest and powerful standard of coordinating ordinary differential equations, integro differential equations, differential equations with finite or infinite delay, Volterra integral equations, and neutral equations. Overall the paper much attention has been focused on the study of equations with the classical one dimensional p-Laplacian operator and is defined by $\phi_p(s) = |s|^{p-2}s$, where $p > 1$, $\phi_p^{-1} = \phi_q$ and $\frac{1}{p} + \frac{1}{q} = 1$. At

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the same time, the equation with p-Laplacian operator intrudes in the modelling of different physical and natural phenomena, non- Newtonian mechanics , combustion theory , population biology , nonlinear flow laws and the system of Monge–Kantorovich partial differential equations. Recently, most of the investigators much attention has been paid to the study the equation with p-Laplacian operator, and there exist a very large number of papers devoted to see [2,3,12–14,19,21,23,32,34,35]. For more details on applications, we refer [9].

In this paper, we establish the existence of positive solutions for $2n^{\text{th}}$ order p-Laplacian boundary value problem of the form

$$(-1)^n\{[\phi_p(u^{(2n-2)}(t)]'' - \lambda^2[\phi_p(u^{(2n-2)}(t)]\} = f(t, u(t), \ t \in [0, 1], \tag{1.1}$$

$$u^{(2i)}(0) = 0, \ u^{(2i)}(1) = 0, \phi_p[u^{(2n-2)}(0)] = 0, \ \phi_p[u^{(2n-2)}(1)] = 0, \tag{1.2}$$

for $0 \le i \le n-2$, where $n \ge 2$ and $f : [0, 1] \times R^+ \to R^+$ is a continuous function, by applying Guo–Krasnosel'skii fixed point theorem. For $\lambda = 0$ and $p = 2$, a lot of works have been done on study the existence of positive solutions of higher and $2n^{\text{th}}$ order boundary value problems by applying various methods, see [10,22,24–26,33,38] and for $\lambda = 0$ and $p = 2$ most of the researchers concentration on the existence of positive solutions of higher order differential equations satisfying lidstone and Neumann boundary conditions, see [4–6,16,17,20,27–31,37, 39]. However, to the best of our knowledge, few papers can be done on study of fourth and $2n^{\text{th}}$ order p-Laplacian boundary value problem with parameter λ, see [26,40]. Motivated by above papers, we extend the results to $2n^{\text{th}}$ order p-Laplacian boundary value problem of the form (1.1)-(1.2).

This paper is organized as follows .In Section 2, we express the solution of the boundary value problem (1.1)-(1.2) as a solution of an equivalent integral equation involving Green functions and establish some inequalities for these Green functions. In Section 3, we establish the criteria for the existence of at least one positive solution of the boundary value problem (1.1)-(1.2) by an application of Guo–Krasnosel'skii fixed point theorem. Finally as an application, we provide examples to demonstrate our results.

2. Green's Function and Bounds

In this section, we express the solution of the boundary value problem (1.1)-(1.2) as a solution of an equivalent integral equation involving Green functions and then establish some inequalities for these Green functions, which are useful to prove our main results.

First, we establish the Green's function $G_1(t,s)$ for the second order homogeneous boundary value problem of the following,

$$-u''(t) = 0,\ t \in [0,1], \tag{2.1}$$

$$u(0) = 0,\ u(1) = 0. \tag{2.2}$$

and then we derive the Green's function $H(t,s)$ for the second order homogeneous problem of the form

$$-y'' + \lambda^2 y = 0,\ 0 \le t \le 1, \tag{2.3}$$

$$y(0) = 0,\ y(1) = 0, \tag{2.4}$$

by taking $y(t) = (-1)^{n-2}[\phi_p(x)^{(2n-4)}]$ and $x(t) = -u''$, then the boundary value problem (1.1)-(1.2) is divided into the following two parts:

$$-y'' + \lambda^2 y = f(t,u(t)),\ 0 \le t \le 1, \tag{2.5}$$

$$y(0) = 0,\ y(1) = 0, \tag{2.6}$$

$$(-1)^{n-2}u^{(2n-2)} + \phi_q(y) = 0,\ 0 \le t \le 1, \tag{2.7}$$

$$u^{(2i)}(0) = 0,\ u^{(2i)}(1) = 0. \tag{2.8}$$

Using this Green's function $G_1(t,s)$, we establish the Green's function $G_{n-2}(t,s)$, recursively for the $(2n-4)^{\text{th}}$ order boundary value problem

$$(-1)^{n-2}x^{(2n-4)}(t) = 0,\ t \in [0,1], \tag{2.9}$$

$$x^{(2i)}(0) = 0,\ x^{(2i)}(1) = 0, \tag{2.10}$$

for $0 \le i \le n-3$, where $n \ge 3$.

Lemma 2.1. [2] *The Green's function $G_1(t,s)$ for the homogeneous boundary value problem (2.1)-(2.2) is given by*

$$G_1(t,s) = \begin{cases} t(1-s),\ 0 \le t \le s \le 1, \\ \\ s(1-t),\ 0 \le s \le t \le 1. \end{cases} \tag{2.11}$$

Lemma 2.2. *The Green's function $H(t,s)$ for the homogeneous boundary value problem (2.3)-(2.4) is given by*

$$H(t,s) = \begin{cases} \dfrac{\sin h(\lambda t)\sin h(\lambda(1-s))}{\lambda \sin h(\lambda)},\ t \le s, \\ \\ \dfrac{\sin h(\lambda s)\sin h(\lambda(1-t))}{\lambda \sin h(\lambda)},\ s \le t. \end{cases} \tag{2.12}$$

Proof. By algebraic calculations, we can establish the result. □

Lemma 2.3. [2,36] *The Green's function for the homogeneous boundary value problem (2.9)-(2.11) is $G_{n-2}(t,s)$, where $G_{n-2}(t,s)$ is defined recursively as*

$$G_i(t,s) = \int_0^1 G_{i-1}(t,r)G_1(r,s)dr,\ \textit{for}\ 2 \le i \le n-2. \tag{2.13}$$

Therefore, the solution of the boundary value problem (1.1)-(1.2) is given by

$$u(t) = \int_0^1 G_{n-1}(t,s)\phi_q\left[\int_0^1 H(s,r)f(r,u(r))dr\right]ds. \tag{2.14}$$

Lemma 2.4. [36] *The Green's function $G_1(t,s)$ satisfies the following inequalities:*

(i) $G_1(t,s) \ge 0,\ \textit{for all}\ \ t,s \in [0,1]$,

(ii) $G_1(t,s) \le G_1(s,s),\ \textit{for all}\ \ t,s \in [0,1]$,

(iii) $G_1(t,s) \geq \frac{1}{4} G_1(s,s)$, *for all* $t \in I$ *and* $s \in [0,1]$,

where $I = \left[\frac{1}{4}, \frac{3}{4}\right]$.

Lemma 2.5. *The Green's function* $H(t,s)$ *in* (2.12) *satisfies the following inequalities:*

(i) $H(t,s) \geq 0$, *for all* $t, s \in [0,1]$,

(ii) $H(t,s) \leq H(s,s)$, *for all* $t, s \in [0,1]$,

(iii) $H(t,s) \geq \eta H(s,s)$, *for all* $t \in I$ *and* $s \in [0,1]$,

where $\eta = \dfrac{\sinh(\frac{\lambda}{4})}{\sinh(\lambda)}$ *and* $I = \left[\frac{1}{4}, \frac{3}{4}\right]$.

Proof. By algebraic calculations, one can establish the inequalities. □

Lemma 2.6. [36] *The Green's function* $G_{n-2}(t,s)$ *in* (2.9) *satisfies the following inequalities:*

(i) $G_{n-2}(t,s) \geq 0$, *for all* $t, s \in [0,1]$,

(ii) $G_{n-2}(t,s) \leq \dfrac{1}{6^{n-3}} G_1(s,s)$, *for all* $t, s \in [0,1]$,

(iii) $G_{n-2}(t,s) \geq \dfrac{1}{4^{n-2}} \left(\dfrac{11}{96}\right)^{n-3} G_1(s,s)$, *for all* $t \in I$ *and* $s \in [0,1]$,

where $I = \left[\frac{1}{4}, \frac{3}{4}\right]$.

To establish the existence of positive solutions of the boundary value problem (1.1)-(1.2), we will employ the following Guo–Krasnosel'skii fixed point theorem will be the fundamental tool.

Theorem 2.1. [9, 15, 18] *Let* X *be a Banach Space,* $\kappa \subseteq X$ *be a cone and suppose that* Ω_1, Ω_2 *are open subsets of* X *with* $0 \in \Omega_1$ *and* $\overline{\Omega}_1 \subset \Omega_2$. *Suppose further that* $T : \kappa \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \kappa$ *is completely continuous operator such that either*

(i) $\|Tu\| \leq \|u\|$, $u \in \kappa \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$, $u \in \kappa \cap \partial\Omega_2$, *or*

(ii) $\|Tu\| \geq \|u\|$, $u \in \kappa \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$, $u \in \kappa \cap \partial\Omega_2$ *holds.*

Then T *has a fixed point in* $\kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Existence Positive Solutions

In this section, we establish the existence of at least one positive solution for the nonlinear p-Laplacian boundary value problem (1.1)-(1.2) by using Guo–Krasnosel'skii fixed point theorem.

Let $X = \{u | u \in C[0,1]\}$ be a Banach space with the norm

$$\|u\| = \max_{t \in [0,1]} |u|$$

and let

$$P = \{u \in X | u(t) > 0, t \in [0,1] \text{ and } \min_{t \in I} |u(t)| \geq \mathcal{M}\|u\|\},$$

where $\mathcal{M} = \left(\frac{11^{n-3}}{2^{6n-16}}\right)$. We note that P is a cone in X.
Let the operator $F : P \to X$ be defined as

$$Fu(t) = \int_0^1 G_{n-1}(t,s)\phi_q\left[\int_0^1 H(s,r)f(r,u(r))dr\right]ds. \tag{3.1}$$

To obtain a positive solution of (1.1)-(1.2), we shall seek a fixed point of the operator F in the cone P.

We assume the following conditions hold throughout this paper:

$(A1)$ $0 < \int_0^1 H(t,s)ds < \infty$,
$(A2)$ $f(t,u)$ is a nondecreasing function with respect to u.

Define the nonnegative extended real numbers f_0, f^0, f_∞ and f^∞ by

$$f_0 = \lim_{u \to 0^+} \min_{t \in [0,1]} \frac{f(t,u)}{\phi_p(u)}, \ f^0 = \lim_{u \to 0^+} \max_{t \in [0,1]} \frac{f(t,u)}{\phi_p(u)},$$

$$f_\infty = \lim_{u \to \infty} \min_{t \in [0,1]} \frac{f(t,u)}{\phi_p(u)} \text{ and } f^\infty = \lim_{u \to \infty} \max_{t \in [0,1]} \frac{f(t,u)}{\phi_p(u)},$$

and assume that they will exist. The case $f^0 = 0$ and $f_\infty = \infty$ represents superlinear and the case $f_0 = \infty$ and $f^\infty = 0$ represents the sublinear.

Lemma 3.1. *The operator $F : P \to X$ defined by (3.1) is a self map on P.*

Proof. From $(A1)$ and the positivity of the Green's function $K(t,s)$ and $H_{n-1}(t,s)$ in Lemmas 2.5 and 2.6 that for $u \in P$, $Fu(t) \geq 0$ on $t \in [0,1]$. Now, for $u \in P$ and by Lemma 2.6, we have

$$\begin{aligned} Fu(t) &= \int_0^1 G_{n-1}(t,s)\phi_q\bigg(\int_0^1 H(s,r)f\big(r,u(r)\big)dr\bigg)ds \\ &\leq \frac{1}{6^{n-3}}\int_0^1 G_1(s,s)\phi_q\bigg(\int_0^1 H(s,r)f\big(r,u(r)\big)dr\bigg)ds \end{aligned}$$

so that

$$\|Fu\| \leq \frac{1}{6^{n-3}}\int_0^1 G_1(s,s)\phi_q\bigg(\int_0^1 H(s,r)f\big(r,u(r)\big)dr\bigg)ds. \tag{3.2}$$

Then by Lemma 2.6, for $u \in P$ that

$$\min_{t\in I} Fu(t) = \min_{t\in I}\left\{\int_0^1 G_{n-1}(t,s)\phi_q\left(\int_0^1 H(s,r)f\big(r,u(r)\big)dr\right)ds\right\}$$
$$\geq \frac{1}{4^{n-2}}\left(\frac{11}{96}\right)^{n-3}\int_0^1 G_1(s,s)\phi_q\left(\int_0^1 H(s,r)f\big(r,u(r)\big)dr\right)ds$$
$$= \left(\frac{11^{n-3}}{2^{6n-16}}\right)\frac{1}{6^{n-3}}\int_0^1 G_1(s,s)\phi_q\left(\int_0^1 H(s,r)f\big(r,u(r)\big)dr\right)ds$$
$$\geq \left(\frac{11^{n-3}}{2^{6n-16}}\right)\|Fu\|$$
$$= \mathcal{M}\|Fu\|.$$

Therefore, $F : P \to P$, and hence the proof is complete. □

Further, the operator F is completely continuous by an application of the Arzela–Ascoli theorem.

Now, we establish the existence of at least one positive solution of the boundary value problem (1.1)-(1.2) for superlinear case.

Theorem 3.1. *Assume that the conditions* $(A1)$-$(A2)$ *are satisfied. If* $f^0 = 0$ *and* $f_\infty = \infty$ *then the boundary value problem* (1.1)-(1.2) *has at least one positive solution that lies in* P.

Proof. Let F be the cone preserving, completely continuous operator that was defined by (3.1). From the definition of $f^0 = 0$, there exist $\xi_1 > 0$ and $\mathcal{H}_1 > 0$ such that

$$f(t,u) \leq \xi_1\phi_p(u), \text{ for } 0 < u \leq \mathcal{H}_1,$$

where ξ_1 satisfies

$$(\xi_1)^{q-1}\frac{1}{6^{n-3}}\int_0^1 G_{n-1}(s,s)\phi_q\left(\int_0^1 H(r,r)dr\right)ds \leq 1. \tag{3.3}$$

Now, let $u \in P$ with $\|u\| = \mathcal{H}_1$. Then, by Lemmas 2.5 , 2.6 and for $t \in [0,1]$, we have

$$Fu(t) = \int_0^1 G_{n-1}(t,s)\phi_q\left(\int_0^1 H(s,r)f\big(r,u(r)\big)dr\right)ds$$
$$\leq \frac{1}{6^{n-3}}\int_0^1 G_1(s,s)\phi_q\left(\int_0^1 H(r,r)\xi_1\phi_p(u)dr\right)ds$$
$$\leq (\xi_1)^{q-1}\frac{1}{6^{n-3}}\int_0^1 G_1(s,s)\phi_q\left(\int_0^1 H(r,r)dr\right)ds\|u\|$$
$$\leq \|u\|.$$

Therefore, $\|Fu\| \leq \|u\|$. If we set

$$\Omega_1 = \{u \in X : \|u\| < \mathcal{H}_1\}$$

then

(3.4) $$\|Fu\| \leq \|u\|, \text{ for } u \in P \cap \partial\Omega_1.$$

Further, since $f_\infty = \infty$, there exist $\xi_2 > 0$ and $\bar{\mathcal{H}}_2 > 0$ such that

$$f(t, u(t)) \geq \xi_2 \phi_p(u), \text{ for } u \geq \bar{\mathcal{H}}_2,$$

where ξ_2 satisfies

(3.5) $$(\xi_2)^{q-1}\mathcal{M}^2 \int_{s\in I} G_{n-1}(s,s)\phi_q\Big(\eta \int_{r\in I} H(r,r)dr\Big)ds \geq 1.$$

Let $\mathcal{H}_2 = \max\left\{2\mathcal{H}_1, \frac{\bar{\mathcal{H}}_2}{\mathcal{M}}\right\}$. Choose $u \in P$ and $\|u\| = \mathcal{H}_2$. Then

$$\min_{t\in I} u(t) \geq \mathcal{M}\|u\| \geq \bar{\mathcal{H}}_2.$$

From Lemmas 2.5, 2.6 and for $t \in [0,1]$, we have

$$\begin{aligned}
Fu(t) &= \int_0^1 G_{n-1}(t,s)\phi_q\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)ds \\
&\geq \min_{t\in I} \int_0^1 G_{n-1}(t,s)\phi_q\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)ds \\
&\geq \frac{1}{4^{n-2}}\left(\frac{11}{96}\right)^{n-3} \int_0^1 G_1(s,s)\phi_q\Big(\int_0^1 H(s,r)f\big(r,u(r)\big)dr\Big)ds \\
&= \left(\frac{11^{n-3}}{2^{6n-16}}\right)\frac{1}{6^{n-3}} \int_0^1 G_1(s,s)\phi_q\Big(\int_0^1 H(s,r)f\big(r,u(r)\big)dr\Big)ds \\
&\geq \left(\frac{11^{n-3}}{2^{6n-16}}\right)\frac{1}{6^{n-3}} \int_0^1 G_1(s,s)\phi_q\Big(\eta\int_0^1 H(r,r)f\big(r,u(r)\big)dr\Big)ds \\
&\geq (\xi_2)^{q-1}\mathcal{M}^2 \int_{s\in I} G_1(s,s)\phi_q\Big(\eta\int_{r\in I} H(r,r)dr\Big)\|u\|ds \\
&\geq \|u\|.
\end{aligned}$$

Therefore, $\|Fu\| \geq \|u\|$. So, if we set

$$\Omega_2 = \{u \in X : \|u\| < \mathcal{H}_2\}$$

then

(3.6) $$\|Fu\| \geq \|u\| \text{ for } u \in P \cap \partial\Omega_2.$$

Applying Theorem 2.1 to (3.4) and (3.6), it follows that F has a fixed point $u \in P \cap (\Omega_2 \setminus \bar{\Omega}_1)$ and that u is the positive solution of the boundary value problem (1.1)-(1.2). □

We now establish the existence of at least one positive solution of the boundary value problem (1.1), (1.2) for sub linear case.

Theorem 3.2. *Assume that the conditions* $(A1)$ - $(A2)$ *are satisfied. If* $f_0 = \infty$ *and* $f^\infty = 0$ *then the boundary value problem* (1.1)-(1.2) *has at least one positive solution that lies in* P.

Proof. Let F be the cone preserving, completely continuous operator defined by (3.1). Since $f_0 = \infty$ there exist $\bar{\xi}_1 > 0$ and $J_1 > 0$ such that

$$f(t,u) \geq \bar{\xi}_1 \phi_p(u), \text{ for } 0 < u \leq J_1,$$

where $\bar{\xi}_1 \geq \xi_2$ and ξ_2 is given in (3.5).

Let $u \in P$ and $\|u\| = J_1$. Then from Lemmas 2.5, 2.6, and for $t \in [0,1]$, we have

$$\begin{aligned}
Fu(t) &= \int_0^1 G_{n-1}(t,s)\phi_q\bigg(\int_0^1 H(s,r)f\big(r,u(r)\big)dr\bigg)ds. \\
&\geq \min_{t\in I}\int_0^1 G_{n-1}(t,s)\phi_q\bigg(\int_0^1 H(s,r)f\big(r,u(r)\big)dr\bigg)ds \\
&\geq \frac{1}{4^{n-2}}\left(\frac{11}{96}\right)^{n-3}\int_0^1 G_1(s,s)\phi_q\bigg(\int_0^1 H(s,r)f(r,u(r)))dr\bigg)ds \\
&\geq \frac{1}{4^{n-2}}\left(\frac{11}{96}\right)^{n-3}\int_{s\in I} G_1(s,s)\phi_q\bigg(\eta\int_{r\in I} H(r,r)\bar{\xi}_1\phi_p(u)\big)dr\bigg)ds \\
&\geq \frac{1}{4^{n-2}}\left(\frac{11}{96}\right)^{n-3}(\bar{\xi}_1)^{q-1}\int_{s\in I} G_1(s,s)\phi_q\bigg(\eta\int_{r\in I} H(r,r)\big)dr\bigg)\mathcal{M}\|u\|ds \\
&\geq \|u\|.
\end{aligned}$$

Therefore, $\|Fu\| \geq \|u\|$. Now, if we set

$$\Omega_3 = \{u \in X : \|u\| < J_1\}$$

then

$$\|Fu\| \geq \|u\|, \text{ for } u \in P \cap \partial\Omega_3. \tag{3.7}$$

Next, since $f^\infty = 0$, there exist $\bar{\xi}_2 > 0$ and $\bar{J}_2 > 0$ such that

$$f(t,u(t)) \leq \bar{\xi}_2 \phi_p(u), \text{ for } u \geq \bar{J}_2,$$

where $\bar{\xi}_2 \leq \xi_1$ and ξ_1 is given in (3.3).

Set

$$f^*(t,u) = \sup_{0\le s\le u} f(t,s).$$

Then, it is straightforward that f^* is a non decreasing real-valued function, $f \le f^*$ and

$$\lim_{u\to\infty} \frac{f^*(t,u)}{u} = 0.$$

It follows that there exists $J_2 > \max\{2J_1, \bar{J}_2\}$ such that

$$f^*(t,u) \le f^*(t,J_2), \text{ for } 0 < x \le J_2.$$

Choose $u \in P$ with $\|u\| = J_2$. Then

$$\begin{aligned}
Fu(t) &= \int_0^1 G_{n-1}(t,s)\phi_q\bigg(\int_0^1 H(s,r)f(r,u(r))dr\bigg)ds \\
&\le \frac{1}{6^{n-3}}\int_0^1 G_1(s,s)\phi_q\bigg(\int_0^1 H(s,r)f(r,J_2)dr\bigg)ds \\
&\le \frac{1}{6^{n-3}}\int_0^1 G_1(s,s)\phi_q\bigg(\int_0^1 H(s,r)\bar{\xi}_2\phi_p(J_2)dr\bigg)ds \\
&\le \frac{1}{6^{n-3}}(\bar{\xi}_2)^{q-1}\int_0^1 G_1(s,s)\phi_q\bigg(\int_0^1 H(r,r)dr\bigg)dsJ_2 \\
&\le J_2 = \|u\|.
\end{aligned}$$

Hence, $\|Fu\| \le \|u\|$. So, if we set

$$\Omega_4 = \{u \in P : \|u\| < J_2\}$$

then

$$\|Fu\| \le \|u\|, \text{ for } u \in P \cap \partial\Omega_4. \tag{3.8}$$

Applying by Theorem 2.1 to (3.7) and (3.8), we obtain that F has a fixed point $u \in P\cap(\Omega_4\setminus\bar{\Omega}_3)$ and that u is the positive solution of the boundary value problem (1.1)-(1.2). □

4. Example

The results are demonstrated with example.

Example 4.1. *Consider the boundary value problem*

$$(-1)^4\{[\phi_p(u^{(6)}(t)]'' - \lambda^2[\phi_p(u^{(6)}(t)]\} = f(t,u(t)),\ t \in [0,1], \tag{4.1}$$

$$(4.2)\qquad \left.\begin{array}{l} u(0)=0,\ u(1)=0,\ u^{(2)}(0)=0,\ u^{(2)}(1)=0, \\ u^{(4)}(0)=0,\ u^{(4)}(1)=0,\ \phi_p(u^{(6)}(0))=0,\ \phi_p(u^{(6)}(1))=0. \end{array}\right\}$$

For simplicity, we take $p=2$ *and* $\lambda=3$. *By algebraic computations, we get* $\eta=0.082085$ *and* $\mathcal{M}=0.04297$.

(a) *If* $f(t,u(t))=u^2(1+e^{t(1-2t)})$, *then* $f^0=0$ *and* $f_\infty=\infty$. *So, all the conditions of Theorem 3.1 are satisfied and hence, the boundary value problem* (4.1)-(4.2) *has at least one positive solution.*

(b) *If* $f(t,u(t))=(t^2+1)^{1/2}u^{1/2}$, *then* $f_0=\infty$ *and* $f^\infty=0$. *So, all the conditions of Theorem 3.2 are satisfied and hence, the boundary value problem* (4.1)-(4.2) *has at least one positive solution.*

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