FURTHER ACCELERATION OF SECANT-TYPE METHODS FOR SOLVING NONLINEAR EQUATIONS

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ABSTRACT. In this paper, we present new Secant-type methods for finding simple root of nonlinear equations. It is proved that the new Secant-type methods have the convergence order of 2.83 or 2.55 requiring only two function evaluations per full iteration. Finally, the numerical examples are made with several other existing iterative methods to demonstrate the performance of the proposed methods.

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1 Introduction

The development of numerical techniques for solving nonlinear equations is important research subject in numerical analysis [4,5,12]. In this paper, we consider iterative methods to find a simple root of a nonlinear equation $f(x)=0$ where $f: D \subset \Re \to \Re$ is a scalar function on an open interval D and it is sufficiently smooth in a neighbourhood of the root. We present new three-point Secant-type iterative methods to find a simple root of the nonlinear equation. It is well established that the multipoint root-solvers is of great practical importance since it overcomes theoretical limits of one-point methods concerning the convergence order and computational efficiency. Recently, some

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modifications of the Secant-type methods for simple root have been proposed and analysed [7-9,11]. Hence, the purpose of this paper is to show further development of the Secant-type methods. The new Secant-type iterative methods are shown to have a better order of convergence then the methods considered in the previous studies [7-9]. In view of this fact, the proposed methods are significantly better when compared with the established methods [1-14].

We consider two well-known iterative methods for finding simple root of nonlinear equations are namely, the classical Secant method,

$$x_{n+1} = x_n - \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \right] f(x_n), \tag{1}$$

and the classical Newton method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2}$$

and their order of convergence is 1.62 and 2 respectively. However, for the purpose of this paper, we present new three-point secant-type methods for finding simple root of nonlinear equations.

The paper is organized as follows: Some essential definitions relevant to the present work are stated in the section 2. In section 3 we introduce four new three-point Secant-type methods and prove their order of convergence. In section 4, well-established Secant-type methods are stated, which will demonstrate the effectiveness of the new Secant-type iterative methods. Finally, in section 5, numerical comparisons are made to demonstrate the performance of the presented methods.

2 Preliminaries

In order to establish the order of convergence of an iterative method, following definitions are used [4,5,12].

Definition 1 Let $f(x)$ be a real-valued function with a root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α. The order of convergence p is given by

$$\lim_{n\to\infty}\frac{x_{n+1}-\alpha}{(x_n-\alpha)^p}=\zeta\neq 0, \tag{3}$$

where $p\in\Re^+$ and ζ is the asymptotic error constant.

Definition 2 Let $e_k=x_k-\alpha$ be the error in the *k*th iteration, then the relation

$$e_{k+1}=\zeta e_k^p+\mathrm{O}\left(e_k^{p+1}\right), \tag{4}$$

is the error equation. If the error equation exists, then p is the order of convergence of the iterative method.

Definition 3 Let r be the number of function evaluations of the method. The efficiency of the method is measured by the concept of efficiency index and defined as

$$EI(r,p)=\sqrt[r]{p}, \tag{5}$$

where p is the order of convergence of the method [4].

Definition 4 Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of a nonlinear equation. Then the computational order of convergence [15] may be approximated by

$$\mathrm{COC}\approx\frac{\ln\left|(x_{n+1}-\alpha)(x_n-\alpha)^{-1}\right|}{\ln\left|(x_n-\alpha)(x_{n-1}-\alpha)^{-1}\right|}. \tag{6}$$

3 Construction of the methods and convergence analysis

In this section, we define the new three-point Secant-type iterative methods, methods having convergence order of 2.83 or 2.55. To obtain the solution of (1), the new Secant-type methods requires two evaluations of functions and three particular starting points, ideally close to the simple root. The formulae of the new three-point Secant-type iterative methods for determining the simple root of (1) are;

The first of the three-point Secant-type method is of convergence order 2.55 and is expressed as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \left(\frac{1}{2}\right)\left(\frac{f(x_n)}{f'(x_n)}\right)^2\left(\frac{\Delta_7}{f'(x_n)}\right). \tag{7}$$

The next three of the three-point Secant-type methods are of convergence order 2.83 and they are expressed as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + \left[\frac{6f(x_n)f'(x_n)^2 - 3f(x_n)^2\Delta_8}{6f'(x_n)^3 - 6f(x_n)f'(x_n)\Delta_8 + f(x_n)^2\Delta_{10}}\right] + \frac{\Delta_{13}}{6}, \tag{8}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \left(\frac{1}{2}\right)\left(\frac{f(x_n)}{f'(x_n)}\right)^2\left(\frac{\Delta_8}{f'(x_n)}\right) + \left(\frac{\Delta_{13}}{6}\right), \tag{9}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + \left[\frac{6f(x_n)f'(x_n) - 3f(x_n)^2\Delta_8}{6f'(x_n)^3 - 6f(x_n)f'(x_n)\Delta_8 + f(x_n)^2\Delta_{10} + \Delta_{14}}\right], \tag{10}$$

where

$$\Delta_1 = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}, \tag{11}$$

$$\Delta_2 = \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}, \tag{12}$$

$$\Delta_3 = \frac{f(x_n) - f(x_{n-2})}{x_n - x_{n-2}}, \tag{13}$$

$$\Delta_4 = \frac{f'(x_n) - f'(x_{n-1})}{x_n - x_{n-1}}, \tag{14}$$

$$\Delta_5 = \frac{f'(x_{n-1}) - f'(x_{n-2})}{x_{n-1} - x_{n-2}}, \tag{15}$$

$$\Delta_6 = \frac{f'(x_n) - f'(x_{n-2})}{x_n - x_{n-2}}, \tag{16}$$

$$\Delta_7 = \Delta_4 - \Delta_5 + \Delta_6, \tag{17}$$

$$\Delta_8 = \frac{4f'(x_n)+2f'(x_{n-1})-6\Delta_1}{x_n - x_{n-1}}, \tag{18}$$

$$\Delta_9 = \frac{4f'(x_n)+2f'(x_{n-2})-6\Delta_3}{x_n - x_{n-2}}, \tag{19}$$

$$\Delta_{10} = \frac{\Delta_4 - \Delta_8}{x_n - x_{n-1}}, \tag{20}$$

$$\Delta_{11} = \frac{\Delta_6 - \Delta_9}{x_{n-2} - x_n}, \tag{21}$$

$$\Delta_{12} = \frac{\Delta_{11} + \Delta_{10}}{x_{n-1} - x_{n-2}}, \tag{22}$$

$$\Delta_{13} = \left(\frac{\Delta_{12}}{f'(x_n)}\right)\left(\frac{f(x_{n-1})}{f'(x_{n-1})}\right)^2\left(\frac{f(x_n)}{f'(x_n)}\right)^2, \tag{23}$$

$$\Delta_{14} = \left(\Delta_{12} f(x_n) f'(x_n)\right)\left(\frac{f(x_{n-1})}{f'(x_{n-1})}\right)^2. \tag{24}$$

x_{-1}, x_0, x_1 are the initial points and provided that the denominators of (7)-(24) are not equal to zero. It is important to verify our finding and prove the order of convergence of the new three-point Secant-type iterative methods.

Theorem 1

Let $f : D \subset \Re \to \Re$ be a sufficiently differentiable function and let for an open interval D has $\alpha \in D$ be a simple zero of $f(x) = 0$ in an open interval D, with $f'(x) \neq 0$ in D. If the initial points x_{-1}, x_0 and x_1 are sufficiently close to α, then the asymptotic convergence order of the new methods defined by (8)-(10) is 2.83.

Proof

Let α be a simple root of $f(x)$, i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, and the errors at $(k-1)$, k and $(k+1)$ iteration are expressed as $e_{n-2} = x_{n-2} - \alpha$, $e_{n-1} = x_{n-1} - \alpha$, $e_n = x_n - \alpha$ and $e_{n+1} = x_{n+1} - \alpha$, respectively.

Using Taylor expansion and taking into account that $f(\alpha)=0$, we have

$$f(x_n)=c_1e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+\cdots \quad (25)$$

$$f(x_{n-1})=c_1e_{n-1}+c_2e_{n-1}^2+c_3e_{n-1}^3+c_4e_{n-1}^4+\cdots \quad (26)$$

$$f(x_{n-2})=c_1e_{n-2}+c_2e_{n-2}^2+c_3e_{n-2}^3+c_4e_{n-2}^4+\cdots \quad (27)$$

$$f'(x_n)=c_1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+\cdots \quad (28)$$

$$f'(x_{n-1})=c_1+2c_2e_{n-1}+3c_3e_{n-1}^2+4c_4e_{n-1}^3+\cdots \quad (29)$$

$$f'(x_{n-2})=c_1+2c_2e_{n-2}+3c_3e_{n-2}^2+4c_4e_{n-2}^3+\cdots \quad (30)$$

where

$$c_k=\frac{f^{(k)}(\alpha)}{(k!)}, \qquad \text{for } k=1,2,3,4,\dots\ . \quad (31)$$

Using (25)-(30), we obtain

$$\Delta_1=c_1+(e_n+e_{n-1})c_2+\left(e_n^2+e_{n-1}e_n+e_{n-1}^2\right)c_3+\cdots \quad (32)$$

$$\Delta_2=c_1+(e_{n-1}+e_{n-2})c_2+\left(e_{n-1}^2+e_{n-1}e_{n-2}+e_{n-2}^2\right)c_3+\cdots \quad (33)$$

$$\Delta_3=c_1+(e_n+e_{n-2})c_2+\left(e_n^2+e_ne_{n-2}+e_{n-2}^2\right)c_3+\cdots \quad (34)$$

$$\Delta_4=2c_2+3(e_n+e_{n-1})c_3+4\left(e_n^2+e_ne_{n-1}+e_{n-1}^2\right)c_4+\cdots \quad (35)$$

$$\Delta_5=2c_2+3(e_{n-1}+e_{n-2})c_3+4\left(e_{n-1}^2+e_{n-1}e_{n-2}+e_{n-2}^2\right)c_4+\cdots \quad (36)$$

$$\Delta_6=2c_2+3(e_n+e_{n-2})c_3+4\left(e_n^2+e_ne_{n-2}+e_{n-2}^2\right)c_4+\cdots \quad (37)$$

$$\Delta_7=2c_2+3c_3e_n+4\left(8e_n^2+4e_ne_{n-1}+4e_ne_{n-2}-4e_{n-1}e_{n-2}\right)c_4+\cdots \quad (38)$$

$$\Delta_8=2c_2+\left(6c_3+4e_{n-1}c_4+12e_{n-1}^2c_5\right)e_n+\cdots \quad (39)$$

$$\Delta_9=2c_2+\left(6c_3+4e_{n-2}c_4+12e_{n-2}^2c_5\right)e_n+\cdots \quad (40)$$

$$\Delta_{10}=-3c_3-6(e_{n-1}+e_n)c_4+\cdots \quad (41)$$

$$\Delta_{11}=3c_3+6(e_{n-2}+e_n)c_4+\cdots \quad (42)$$

$$\Delta_{12}=-3c_4+6(e_{n-2}+e_{n-1}+3e_n)c_5+\cdots \quad (43)$$

$$\Delta_{13} = -6\frac{c_4}{c_1}e_{n-1}^2 e_n^2 + 6\frac{c_5}{c_1}e_{n-2}e_{n-1}^2 e_n^2 + \cdots \tag{44}$$

$$\Delta_{14} = -6c_4 c_1^2 e_{n-1}^2 e_n + 6c_5 c_1^2 e_{n-2} e_{n-1}^2 e_n + \cdots \tag{45}$$

Substituting the appropriate expressions in (8),

$$e_{n+1} = e_n - \frac{f(x_n)}{f'(x_n)} + \left[\frac{6f(x_n)f'(x_n)^2 - 3f(x_n)^2\Delta_8}{6f'(x_n)^3 - 6f(x_n)f'(x_n)\Delta_8 + f(x_n)^2\Delta_{10}}\right] + \frac{\Delta_{13}}{6}, \tag{46}$$

Simplifying, we obtain the error equation for the new three-point Secant-type iterative method, given by (46) is

$$e_{n+1} = \left(\frac{c_5}{c_1}\right)e_n^2 e_{n-1}^2 e_{n-2} + \cdots. \tag{47}$$

In order to prove the order of convergence of (47) and we defining positive real terms of R_n, R_{n-1} and R_{n-2} as

$$R_n = \frac{|e_{n+1}|}{|e_n^m|}, \quad R_{n-1} = \frac{|e_n|}{|e_{n-1}^m|} \quad R_{n-2} = \frac{|e_{n-1}|}{|e_{n-2}^m|}, \tag{48}$$

The error terms of R_{n-2} are given as

$$|e_{n-1}| = R_{n-2}|e_{n-2}^m| \tag{49}$$

$$|e_n| = R_{n-1}|e_{n-1}^m| = R_{n-1}R_{n-2}^m|e_{n-2}^{m^2}| \tag{50}$$

$$|e_{n+1}| = R_n|e_n^m| = \left(R_n R_{n-1}^m R_{n-2}^{m^2}\right)|e_{n-2}^{m^3}|. \tag{51}$$

It is obtained from (47) that

$$\frac{|e_{n+1}|}{|e_n^2||e_{n-1}^2||e_{n-2}|} = \left|\left(\frac{c_5}{c_1}\right)\right|, \tag{52}$$

$$\frac{|e_{n+1}|}{|e_n^2||e_{n-1}^2||e_{n-2}|} = \left(R_n R_{n-1}^{m-2} R_{n-2}^{m^2-2m-2}\right)|e_{n-2}^{m^3-2m^2-2m-1}| = \left|\left(\frac{c_5}{c_1}\right)\right|. \tag{53}$$

In order to satisfy the asymptotic equation (53), the power of the error term shall approach zero, that is

$$m^3 - 2m^2 - 2m - 1 = 0. \tag{54}$$

The roots of the cubic equation (54) are;

$$m = 2.831177, \quad m = -0.415588 - 0.424848i \quad m = -0.415588 + 0.424848i \tag{55}$$

The order of convergence of the new three-point Secant-type method is determined by the positive root of (54). Hence, the new three-point Secant-type method defined by (8) has a convergence order of 2.83. This completes the proof. □

We repeat the procedure to prove the error equation of the Secant-type methods given by (9) and (10). Consequently, the error equations of the new three-point Secant-type method defined by (9) and (10) is identical to the error equation of (8). Hence, the new three-point Secant-type method defined by (9) and (10) have a convergence order of 2.83.

Theorem 2

Let $f : D \subset \Re \to \Re$ be a sufficiently differentiable function and let for an open interval D has $\alpha \in D$ be a simple zero of $f(x) = 0$ in an open interval D, with $f'(x) \neq 0$ in D. If the initial points x_0 and x_1 are sufficiently close to α, then the asymptotic convergence order of the new Secant-type iterative method defined by (7) is 2.55.

Proof

Using the similar procedure of the previous proof, we substitute the appropriate expressions in (16) and obtain

$$e_{n+1} = e_n - \frac{f(x_n)}{f'(x_n)} - \left(\frac{1}{2}\right)\left(\frac{f(x_n)}{f'(x_n)}\right)^2 \left(\frac{\Delta_7}{f'(x_n)}\right), \tag{56}$$

Simplifying, we obtain the error equation for the new three-point Secant-type iterative method is

$$e_{n+1} = \left(\frac{2c_4}{c_1}\right) e_n^2 e_{n-1} e_{n-2} + \cdots. \tag{57}$$

It is obtained from (57) that

$$\frac{|e_{n+1}|}{|e_n^2||e_{n-1}||e_{n-2}|}=\left|\left(\frac{2c_4}{c_1}\right)\right|, \tag{58}$$

$$\frac{|e_{n+1}|}{|e_n^2||e_{n-1}||e_{n-2}|}=\left(R_nR_{n-1}^{m-2}R_{n-2}^{m^2-2m-1}\right)\left|e_{n-2}^{m^3-2m^2-m-1}\right|=\left|\left(\frac{2c_4}{c_1}\right)\right|. \tag{59}$$

In order to satisfy the asymptotic equation (59), the power of the error term shall approach zero, that is

$$m^3-2m^2-m-1=0. \tag{60}$$

The roots of the cubic equation (60) are;

$$m=2.546818,\quad m=-0.273409-0.563821i\quad m=-0.273409+0.563821i \tag{61}$$

The order of convergence of the new three-point Secant-type methods is determined by the positive root of (60). Hence, the new three-point Secant-type methods defined by (7) has a convergence order of 2.55. □

4 The established methods

For the purpose of comparison, three particular well-established iterative methods [12] are considered and they are given below,

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}-\left(\frac{1}{2}\right)\left(\frac{f(x_n)}{f'(x_n)}\right)^2\left(\frac{\Delta_4}{f'(x_n)}\right), \tag{62}$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}+\left[\frac{f(x_n)^2}{f(x_n)-f(x_{n-1})}\right]\left(\frac{2}{f'(x_n)}+\frac{1}{f'(x_{n-1})}-\frac{3}{\Delta_1}\right), \tag{63}$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}-\left(\frac{1}{2}\right)\left(\frac{f(x_n)}{f'(x_n)}\right)^2\left(\frac{\Delta_8}{f'(x_n)}\right). \tag{64}$$

Remark

The new three-point Secant-type methods requires 2 function evaluations and have the order of convergence 2.83 or 2.55. To determine the efficiency index of the new three-point Secant-type methods, definition 3 will be used. Hence, the efficiency index of the new Secant-type iterative methods given by (8)-(10) is $EI(2.83,2)\approx 1.683,$ the efficiency index of the new 2.55-order Secant-type iterative method given by (7) is $EI(2.55, 2) \approx 1.596$, the efficiency index of the iterative method given by (62) is $EI(2.41, 2) \approx 1.55$, the efficiency index of the iterative methods given by (63) and (64) is $EI(2.732, 2) \approx 1.653$, the efficiency index of the classical Secant method given by (1) is $EI(1.618, 1) \approx 1.618$, and finally the efficiency index of the Newton method given by (2) is $EI(2, 2) \approx 1.414$.

5 Numerical experiments

In this section we check the effectiveness of the new Secant-type iterative methods introduced in this paper and compare them with the established iterative methods. The difference between the simple root α and the approximation x_n for test function with starting points x_{-1}, x_0, x_1 is displayed in tables. Furthermore, the computational order of convergence approximations is displayed in tables and we observe that this perfectly coincides with the theoretical result. The numerical computations listed in the table was performed on an algebraic system called Maple and the errors displayed are of absolute value. We test the different iterative methods using the following smooth functions.

Numerical example 1

We demonstrate the order of convergence of the new three-point Secant-type iterative methods for the following nonlinear equation

$$f(x) = \exp\left(x^2 + 7x - 30\right) - 1, \tag{65}$$

having the exact value of the simple root of (65) is $\alpha = 3$ In Table 1 the errors obtained by the methods described are based on the starting points $x_{-1} = 2.85, \quad x_0 = 2.9, \quad x_1 = 2.95.$

Table 1 Errors occurring in the approximation of the simple root of nonlinear equation (65)

methods	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	$\lvert x_4-\alpha\rvert$	$\lvert x_5-\alpha\rvert$	$f(x_5)$	*COC*
(1)	0.461e-1	0.148e-1	0.414e-2	0.420e-3	0.547e-2	1.1978
(2)	0.206e-1	0.256e-2	0.426e-4	0.120e-7	0.155e-6	2.0000
(62)	0.323e-2	0.163e-4	0.372e-10	0.984e-24	0.128e-22	2.3913
(7)	0.896e-2	0.174e-5	0.220e-12	0.143e-30	0.186e-29	2.5827
(63)	0.191e-2	0.665e-5	0.110e-12	0.301e-33	0.391e-32	2.7569
(64)	0.110e-1	0.684e-4	0.335e-10	0.516e-27	0.670e-26	2.7507
(8)	0.591e-2	0.239e-5	0.138e-14	0.252e-40	0.328e-39	2.8493
(9)	0.224e-1	0.974e-3	0.631e-7	0.122e-19	0.159e-18	2.7001
(10)	0.995e-3	0.113e-6	0.300e-18	0.462e-51	0.601e-50	2.8228

Numerical example 2

We demonstrate the order of convergence of the new three-point Secant-type iterative methods for the following nonlinear equation

$$f(x)=(x-2)\left(x^{10}+x+1\right)\exp(-x-1), \tag{66}$$

having exact value of the simple root of (66) is $\alpha=2.$ In Table 2 the errors obtained by the methods described are based on the starting points $x_{-1}=2.003,\quad x_0=2.002,\quad x_1=2.001..$

Table 2 Errors occurring in the approximation of the simple root of nonlinear equation (66)

methods	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	$\lvert x_4-\alpha\rvert$	$\lvert x_5-\alpha\rvert$	$f(x_5)$	*COC*
(1)	0.792e-5	0.315e-7	0.994e-12	0.125e-18	0.638e-17	1.6522
(2)	0.397e-5	0.628e-10	0.157e-19	0.983e-39	0.503e-37	2.0000
(62)	0.148e-7	0.221e-17	0.733e-42	0.120e-100	0.614e-99	2.4151
(7)	0.248e-7	0.153e-19	0.709e-49	0.232e-124	0.119e-122	2.5504
(63)	0.250e-7	0.117e-18	0.161e-50	0.670e-137	0.342e-135	2.7377
(64)	0.248e-7	0.411e-20	0.631e-55	0.409e-150	0.209e-148	2.7315
(8)	0.101e-7	0.144e-22	0.932e-64	0.805e-181	0.412e-179	2.8303
(9)	0.248e-7	0.406e-21	0.445e-60	0.357e-170	0.182e-168	2.8330
(10)	0.101e-7	0.122e-22	0.932e-64	0.805e-181	0.412e-179	2.8303

Numerical example 3

We demonstrate the order of convergence of the new three-point Secant-type iterative methods for the following nonlinear equation

$$f(x)=\left[\exp(x)\sin(x)+\ln\left(1+x^{2}\right)\right], \tag{67}$$

having exact value of the simple root of (67) is $\alpha=0$. In Table 3 the errors obtained by the methods described are based on the starting points $x_{-1}=\frac{1}{10}, \quad x_0=\frac{1}{11}, \quad x_1=\frac{1}{12}.$

Table 3 Errors occurring in the approximation of the simple root of nonlinear equation (67)

methods	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	$\lvert x_4-\alpha\rvert$	$\lvert x_5-\alpha\rvert$	$f(x_5)$	*COC*
(1)	0.114e-1	0.162e-2	0.363e-4	0.117e-6	0.117e-6	1.6624
(2)	0.106e-1	0.217e-3	0.937e-7	0.176e-13	0.176e-13	2.0000
(62)	0.245e-2	0.856e-7	0.893e-17	0.341e-41	0.341e-41	2.4013
(7)	0.246e-2	0.696e-7	0.990e-18	0.168e-45	0.168e-45	2.5544
(63)	0.246e-2	0.977e-6	0.213e-15	0.157e-41	0.157e-41	2.7394
(64)	0.246e-2	0.924e-7	0.198e-19	0.167e-53	0.167e-53	2.7437
(8)	0.883e-4	0.689e-11	0.407e-35	0.131e-99	0.131e-99	2.8894
(9)	0.247e-2	0.109e-6	0.966e-20	0.140e-57	0.140e-57	2.8053
(10)	0.899e-4	0.715e-11	0.586e-35	0.786e-98	0.786e-98	2.8782

6 Conclusion

In this work, we have developed four new three-point Secant-type methods for solving nonlinear equations with a simple root. The effectiveness of the new iterative methods is examined by showing the accuracy of the simple root of several nonlinear equations. Convergence analysis proves that the new three-point iterative methods preserve their order of convergence. Numerical test examples are provided to support the theoretical results obtained and compared with different methods. The major advantages of the new three-point Secant-type methods are that they are very effective, the new Secant-type iterative methods are shown to have a better order of convergence and efficiency index then the similar iterative methods considered.

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