

QUASI-INTERIOR IDEALS AND WEAK-INTERIOR IDEALS

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ABSTRACT. In this paper, as a further generalization of ideals, we introduce the notion of quasi-interior ideals and weak -interior ideals as a generalization of quasi ideal, bi-ideal and interior ideal of semigroups. We study the properties of weak-interior ideals of semigroup and characterize the weak-interior simple semigroup, regular semigroup using quasi-interior ideals weak-interior ideal ideals of semigroups.

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1. INTRODUCTION

Semigroup, as the basic algebraic structure was used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata, coding theory and formal languages.

The notion of ideals was introduced by Dedekind for the theory of algebraic numbers, was generalized by Noether for associative rings. The one and two sided ideals introduced by her, are still central concepts in ring theory and the notion of a one sided ideal of any algebraic structure is a generalization of notion of an ideal. The notion of the bi-ideal in semigroups is a special case of (m,n) ideals. In 1952, the concept of bi-ideals for semigroup was introduced by Good and Hughes [1] and the notion of bi-ideals in associative rings was introduced by Lazos and Szasz [8,9,10]. We know that the notion of a one sided ideal of any algebraic structure is a generalization of notion of an ideal. The quasi ideals are generalization of left ideals and right

ideals whereas the bi-ideals are generalization of quasi ideals. Steinfeld [30] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki [3.4.5] introduced the concept of quasi ideal for a semigroup. Quasi ideals in Γ - semigroups were studied by Jagtap and Pawar [7]. Murali Krishna Rao et al. [11-29] studied ideals, derivations and properties of Γ - incline and Γ - semiring.

In this paper, as a further generalization of ideals, we introduce the notion of quasi-interior ideal, weak-interior ideal and study the properties of quasi-interior ideals and weak-interior ideal of semigroup, characterization of simple semigroup and regular semigroup using quasi-interior ideals and weak-interior ideal of semigroup.

2. PRELIMINARIES

In this section, we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. A semigroup is an algebraic system (S, \cdot) consisting of a non-empty set S together with an associative binary operation " \cdot ".

Definition 2.2. A subsemigroup T of S is a non-empty subset T of S such that $TT \subseteq T$.

Definition 2.3. A non-empty subset T of S is called a left (right) ideal of S if $ST \subseteq T$ ($TS \subseteq T$).

Definition 2.4. A non-empty subset T of S is called an ideal of S if it is both a left ideal and a right ideal of S .

Definition 2.5. A non-empty subset Q of S is called a quasi ideal of S if $QS \cap SQ \subseteq Q$.

Definition 2.6. A subsemigroup T of S is called a bi-ideal of S if $TST \subseteq T$.

Definition 2.7. A subsemigroup T of S is called an interior ideal of S if $STS \subseteq T$.

Definition 2.8. An element a of a semigroup S is called a regular element if there exists an element b of S such that $a = aba$.

Definition 2.9. A semigroup S is called a regular semigroup if every element of S is a regular element.

Definition 2.10. Let S be a semigroup. An element $1 \in S$ is said to be unity if $x1 = 1x = x$ for all $x \in S$.

Definition 2.11. A subsemigroup A of S is called a left(right) bi- quasi ideal of M if $SA \cap ASA(AS \cap ASA) \subseteq A$.

Definition 2.12. A subsemigroup A of S is called a bi- quasi ideal of M if A is a left bi- quasi ideal and a right bi- quasi ideal of M

Definition 2.13. A semigroup M is a left (right) simple semigroup if M has no proper left (right) ideal of M

Definition 2.14. A semigroup M is a bi-quasi simple semigroup if M has no proper bi-quasi ideal of M

Definition 2.15. A semigroup M is said to be simple semigroup if M has no proper ideals.

3. QUASI-INTERIOR IDEALS OF SEMIGROUPS

In this section, we introduce the notion of left quasi-interior ideal as a generalization of bi-ideal,quasi-ideal and interior ideal of semigroup and study the properties of quasi- interior ideal of semigroups.

Definition 3.1. A non-empty subset B of a semigroup M is said to be left quasi- interior ideal of M if B is a subsemigroup of M and $MBMB \subseteq B$

Definition 3.2. A non-empty subset B of M is said to be right quasi- interior ideal of M if B is a subsemigroup of M and $BMBM \subseteq B$

Definition 3.3. [23] Let M be a semigroup. L is said to be quasi- interior ideal of M if it is both a left quasi- interior ideal and a right quasi- interior ideal of M .

Every left quasi- interior ideal of a semigroup M need not be bi-ideal,quasi-ideal, interior ideal quasi-interior ideal. and bi-quasi ideals of semigroup M .

Example 3.4. Let N be a set of all natural numbers. N is additive abelian semigroup with respect to usual addition of integers. A subset $I = 2N$ of N is a left quasi- interior ideal of N but not bi-ideal,quasi-ideal, interior ideal, quasi-interior ideal and bi-quasi ideal of semigroup N .

Theorem 3.5. Every left quasi- interior ideal of a semigroup M is a bi-ideal of semigroup M .

Proof. Let B be a left quasi- interior ideal of the semigroup M . Then $MBMB \subseteq B$.

Therefore $BMB \subseteq MBMB \subseteq B$.

Hence every left quasi- interior ideal of the semigroup M is a bi-ideal of semigroup M . \square

Theorem 3.6. *Every interior ideal of a semigroup M is a left quasi- interior ideal of M .*

Proof. Let I be an interior ideal of the semigroup M .

Then $MIMI \subseteq MIM \subseteq I$.

Hence I is a left quasi- interior ideal of the semigroup M . \square

In the following theorems, we mention some important properties and we omit the proofs since proofs are straight forward.

Theorem 3.7. *Let M be a semigroup. Every left ideal is a left quasi- interior ideal of M .*

Theorem 3.8. *Let M be a semigroup. Every right ideal is a right quasi- interior ideal of M .*

Theorem 3.9. *Let M be a semigroup. Every quasi ideal is a quasi- interior ideal of M .*

Theorem 3.10. *Let M be a semigroup. Every ideal is a left quasi- interior ideal of M .*

Theorem 3.11. *Let M be a semigroup. The intersection of a right ideal and a left ideal of M is a left quasi- interior ideal of M .*

Theorem 3.12. *Let M be a semigroup. If L is a left ideal and R is a right ideal of a semigroup M then $B = RL$ is a quasi- interior ideal of M .*

Theorem 3.13. *Let M be a semigroup. If B is a left quasi- interior ideal and T is a subsemigroup of M then $B \cap T$ is a quasi- interior ideal of semigroup M .*

Theorem 3.14. *Let M be a semigroup and B be a subsemigroup of M . If $MMMMB \subseteq B$ then B is a left quasi- interior ideal of M .*

Theorem 3.15. *Let M be a semigroup and B be a subsemigroup of M . If $MMMMB \cap BMMMM \subseteq B$ then B is a left quasi- interior ideal of M .*

Theorem 3.16. *Let M be a semigroup and B be a subsemigroup of M . B is a left quasi- interior ideal of M if and only if there exist a left ideal L and a right ideal R of M such that $LR \subseteq B \subseteq R \cap L$.*

Proof. Suppose B is a quasi- interior ideal of the semigroup M .

Then $MBMB \subseteq B$. Let $R = BM$ and $L = MB$.

Then L and R are left ideal and right ideal of M respectively.

Therefore $LR \subseteq B \subseteq R \cap L$.

Conversely suppose that there exist L and R a left and right ideal of M respectively such that $LR \subseteq B \subseteq R \cap L$. Then

$$MBMB \subseteq M(R \cap L)M(R \cap L)$$

$$\subseteq RMRML$$

$$\subseteq RL \subseteq B.$$

Hence B is a left quasi- interior ideal of the semigroup M . □

Corollary 3.17. *Let M be a semigroup and B be a subsemigroup of M . B is a right quasi- interior ideal of M if and only if there exist a left ideal L and a right ideal R of M such that $RL \subseteq B \subseteq R \cap L$.*

Corollary 3.18. *Let M be a semigroup and B be a subsemigroup of M . B is a quasi- interior ideal of M if and only if there exist a left ideal L and a right ideal R of M such that $RL \cap LR \subseteq B \subseteq R \cap L$.*

Theorem 3.19. *The intersection of a left quasi- interior ideal B of a semigroup M and a right ideal A of M is always a left quasi- interior ideal of M .*

Proof. Suppose $C = B \cap A$.

$$MCMC \subseteq MBMB \subseteq B$$

$$MCMC \subseteq MAMA \subseteq A, \text{ since } A \text{ is a left ideal of } M$$

$$\text{Therefore } MCMC \subseteq B \cap A = C.$$

Hence the intersection of a left quasi- interior ideal B of the semigroup M and a subsemigroup A of M is always a left quasi- interior ideal of M . □

Corollary 3.20. *The intersection of a quasi- interior ideal B of a semigroup M and an ideal A of M is always a left quasi- interior ideal of M .*

Theorem 3.21. *Let A and C be left quasi- interior ideals of a semigroup M and $B = AC$. If $CC = C$ then B is a left quasi- interior ideal of M .*

Proof. Let A and C be left quasi- interior ideals of the semigroup M and $B = AC$. $BB = ACAC = ACCAC \subseteq AMCMC \subseteq AC = B$. Obviously $B = AC$ is a subsemigroup of M

$$\begin{aligned} MBMB &= MACMAC \\ &\subseteq MAMAC \subseteq AC = B. \end{aligned}$$

Hence B is a left quasi- interior ideal of M . □

Corollary 3.22. *Let A and C be quasi- interior ideals of a semigroup M and $B = CA$. If $CC = C$ then B is a quasi- interior ideal of M .*

Theorem 3.23. *Let A and C be subsemigroups of a semigroup M and $B = AC$. If A is the left ideal then B is a left quasi- interior ideal of M .*

Proof. Let A and C be subsemigroups of the semigroup M and $B = AC$ Suppose A is the left ideal of M . $BB = ACAC \subseteq AC = B$.

$$\begin{aligned} MBMB &= MACMAC \\ &\subseteq AC = B. \end{aligned}$$

Hence B is a left quasi- interior ideal of M . □

Corollary 3.24. *Let A and C be subsemigroups of a semigroup M and $B = AC$. If C is a right ideal then B is a left quasi- interior ideal of M .*

Theorem 3.25. *Let M be a semigroup and T be a non-empty subset of M . Then every subsemigroup of T containing $MTMT$ is a left quasi- interior ideal of semigroup M .*

Proof. Let B be a subsemigroup of T containing $MTMT$. Then

$$\begin{aligned} MBMB &\subseteq MTMT \\ &\subseteq B. \end{aligned}$$

Therefore $MBMB \subseteq B$.

Hence B is a left quasi- interior ideal of M . □

Theorem 3.26. *B is a left quasi- interior ideal of a semigroup M if and only if B is a left ideal of some ideal R of semigroup M .*

Proof. Suppose B is a left ideal of some ideal R of the semigroup M . Then $RB \subseteq B$, $RM \subseteq R$. Hence $MBMB \subseteq MRMB \subseteq RMB \subseteq RB \subseteq B$. Therefore B is a left quasi- interior ideal of a semigroup M . Conversely suppose that B is a left quasi- interior ideal of a semigroup M . Then $MBMB \subseteq B$. Therefore B is a left ideal of ideal MBM of the semigroup M . \square

Corollary 3.27. B is a right quasi- interior ideal of a semigroup M if and only if B is a right ideal of some ideal of semigroup M .

Theorem 3.28. If B is a left quasi- interior ideal of a semigroup M , T is a sub semigroup of M and $T \subseteq B$ then BT is a left quasi- interior ideal of M .

Proof. $BTBT \subseteq BT$.

Hence BT is a subsemigroup of M .

$$\text{We have } MBTM \subseteq MBM$$

$$\text{and } TMBT \subseteq BMB$$

$$\Rightarrow MBTMBT \subseteq MBMBT \subseteq BT.$$

Hence BT is a left quasi- interior ideal of the semigroup M . \square

Theorem 3.29. Let B be bi-ideal of a semigroup M and I be interior ideal of M . Then $B \cap I$ is a left quasi- interior ideal of M .

Proof. Obviously $B \cap I$ is subsemigroup of M . Suppose B is a bi-ideal of M and I is an interior ideal of M . Then

$$M(B \cap I)M(B \cap I) \subseteq MBMB \subseteq B,$$

$$M(B \cap I)M(B \cap I) \subseteq MIMI \subseteq I$$

Therefore $M(B \cap I)M(B \cap I)M \subseteq B \cap I$.

Hence $B \cap I$ is a left quasi- interior ideal of M . \square

Theorem 3.30. Let M be a semigroup and T be a subsemigroup of M . Then every subsemigroup of T containing $MTMT$ is a left quasi- interior ideal of M .

Proof. Let C be a subsemigroup of T containing $MTMT$. Then

$$MCMC \subseteq MTMT$$

$$\subseteq C.$$

Hence C is a left quasi- interior ideal of the semigroup M . □

Theorem 3.31. *Let M be a semigroup. If $M = Ma$, for all $a \in M$. Then every left quasi- interior ideal of M is a quasi ideal of M .*

Proof. Let B be a left quasi- interior ideal of the semigroup M and $a \in B$. Then

$$\begin{aligned} MBMB &\subseteq B \\ \Rightarrow Ma &\subseteq MB, \\ \Rightarrow M &\subseteq MB \subseteq M \\ \Rightarrow MB &= M \\ \Rightarrow BM &= BMB \subseteq MBMB \subseteq B \\ \Rightarrow MB \cap BM &\subseteq MM \cap BM \subseteq B. \end{aligned}$$

Therefore B is a quasi ideal of M . Hence the theorem. □

Theorem 3.32. *The intersection of $\{B_\lambda \mid \lambda \in A\}$ left quasi- interior ideals of a semigroup M is a quasi- interior ideal of M .*

Proof. Let $B = \bigcap_{\lambda \in A} B_\lambda$. Then B is a subsemigroup of M .

Since B_λ is a left quasi- interior ideal of M , we have

$$\begin{aligned} MB_\lambda MB_\lambda &\subseteq B_\lambda, \text{ for all } \lambda \in A \\ \Rightarrow M \cap B_\lambda M \cap B_\lambda &\subseteq \cap B_\lambda \\ \Rightarrow MBMB &\subseteq B. \end{aligned}$$

Hence B is a left quasi- interior ideal of M . □

Theorem 3.33. *Let B be a left quasi- interior ideal of a semigroup M , $e \in B$, $eB \subseteq B$ and e be an idempotent. Then eB is a left quasi- interior ideal of M .*

Proof. Let B be a left quasi- interior ideal of the semigroup M . Suppose $x \in B \cap eM$. Then $x \in B$ and $x = ey, y \in M$.

$$\begin{aligned} x &= ey \\ &= eey \\ &= e(ey) \\ &= ex \in eB. \end{aligned}$$

Therefore $B \cap eM \subseteq eB$

$$\begin{aligned} eB &\subseteq B \text{ and } eB \subseteq eM \\ \Rightarrow eB &\subseteq B \cap eM \\ \Rightarrow eB &= B \cap eM. \end{aligned}$$

Hence eB is a left quasi- interior ideal of M . □

Corollary 3.34. *Let M be a semigroup M and e be an idempotent. Then eM and Me are quasi- interior ideals of M .*

Theorem 3.35. *If B be a left quasi- interior ideal of a semigroup M , then B is a left bi- quasi ideal of M .*

Proof. Suppose B is a bi-quasi interior ideal of the semigroup M . Then $BMB \subseteq MBMB$.

Therefore $MB \cap BMB \subseteq BMB \subseteq MBMB \subseteq B$

Hence B is a left bi- quasir ideal of M . □

Corollary 3.36. *If B be a right quasi- interior ideal of a semigroup M , then B is a right bi- quasi ideal of M .*

Corollary 3.37. *If B be a quasi-i- interior ideal of a semigroup M , then B is a bi- quasi ideal of M .*

Theorem 3.38. *If B be a left quasi- interior ideal of a semigroup M , then B is a bi- interior ideal of a semigroup M .*

Proof. Suppose B is a left quasi- interior ideal of the semigroup M . Then $BMB \subseteq MBMB$

Therefore $MBM \cap BMB \subseteq BMB \subseteq MBMB \subseteq B$

Hence B is a bi- interior ideal of M . □

Theorem 3.39. Let B be subsemigroup of a semigroup M . If B can be represented as $B = LL$, where L is a left ideal of M then B is a left quasi- interior ideal of M .

Proof. Suppose $B = LL$, where L is a left ideal of M .

$$MBMB = MLLMLL$$

$$\subseteq LL = B.$$

Hence B is a left Lquasi- interior ideal of the semigroup M . □

Definition 3.40. A semigroup M is said to be quasi- interior simple semigroup if M has no quasi- interior ideals other than M itself.

Theorem 3.41. Let M be a simple semigroup. Every left quasi- interior ideal is a left ideal of M .

Proof. Let M be a simple semigroup and B be a left quasi- interior ideal of M .

Then $MBMB \subseteq B$ and MBM is an ideal of M .

Since M is a simple semigroup, we have $MBM = M$. Hence

$$MBMB \subseteq B$$

$$\Rightarrow MB \subseteq B.$$

Hence the theorem. □

Theorem 3.42. Let M be a semigroup. M is a quasi- interior simple semigroup if and only if $\langle a \rangle = M$, for all $a \in M$ and where $\langle a \rangle$ is the smallest left quasi- interior ideal generated by a .

Proof. Let M be a semigroup. Suppose M is a quasi- interior simple semigroup, $a \in M$ and $B = Ma$.

Then B is a left ideal of M .

Therefore, by Theorem[3.5], B is a left quasi- interior ideal of M .

Therefore $B = M$. Hence $Ma = M$, for all $a \in M$.

$$Ma \subseteq \langle a \rangle \subseteq M$$

$$\Rightarrow M \subseteq \langle a \rangle \subseteq M.$$

Therefore $M = \langle a \rangle$.

Suppose $\langle a \rangle$ is the smallest left quasi- interior ideal of M generated by a and $\langle a \rangle = M$ and A is the left quasi- interior ideal and $a \in A$. Then

$$\begin{aligned} \langle a \rangle &\subseteq A \subseteq M \\ \Rightarrow M &\subseteq A \subseteq M. \end{aligned}$$

Therefore $A = M$. Hence M is a quasi- interior simple semigroup. \square

Theorem 3.43. *Let M be a semigroup. Then M is a left quasi- interior simple semigroup if and only if $MaMa = M$, for all $a \in M$.*

Proof. Suppose M is left quasi- interior simple semigroup and $a \in M$.

Therefore $MaMa$ is a left quasi- interior ideal of M .

Hence $MaMa = M$, for all $a \in M$.

Conversely suppose that $MaMa = M$, for all $a \in M$.

Let B be a left quasi- interior ideal of the semigroup M and $a \in B$.

$$\begin{aligned} M &= MaMa \\ &\subseteq BMBMB \subseteq B \end{aligned}$$

Therefore $M = B$.

Hence M is a quasi- interior simple semigroup. \square

Definition 3.44. A semigroup M is a left (right) simple semigroup if M has no proper left (right) ideal of M

Definition 3.45. A semigroup M is said to be simple semigroup if M has no proper ideals.

Theorem 3.46. *If semigroup M is a left simple semigroup then every left quasi- interior ideal of M is a right ideal of M .*

Proof. Let B be a left quasi- interior of the left simple semigroup M . Then MB is a left ideal of M and $MB \subseteq M$. Therefore $MB = M$. Then

$$\begin{aligned} MBM &= MM \subseteq M \\ \Rightarrow BM &= BMB \\ \Rightarrow BM &= BMB \subseteq MBMB \subseteq B \\ \Rightarrow BM &\subseteq B. \end{aligned}$$

Hence every left quasi- interior ideal is a right ideal of M .

□

Corollary 3.47. *If semigroup M is a right simple semigroup then every left quasi- interior ideal of M is a left ideal of M .*

Corollary 3.48. *Every left quasi- interior ideal of left and right simple semigroup M is an ideal of M .*

Theorem 3.49. *Let M be a semigroup and B be a left quasi- interior ideal of M . Then B is a minimal left quasi- interior ideal of M if and only if B is a left quasi- interior simple subsemigroup of M .*

Proof. Let B be a minimal left quasi- interior ideal of the semigroup M and C be a left quasi- interior ideal of B . Then $BCBC \subseteq C$.

Therefore $CBCB$ is a left quasi- interior ideal of M .

Since C is a left quasi- interior ideal of B ,

$$\begin{aligned} CBCB &= B \\ \Rightarrow B &= CBCB \subseteq C \\ \Rightarrow B &= C. \end{aligned}$$

Conversely suppose that B is a left quasi- interior simple subsemigroup of M . Let C be a left quasi- interior ideal of M and $C \subseteq B$.

$$\begin{aligned} \Rightarrow BCBC &\subseteq MCMC \subseteq MBMB \subseteq B, \\ \Rightarrow B &= C, \text{ since } B \text{ is a quasi- interior simple semigroup.} \end{aligned}$$

Hence B is a minimal left quasi- interior ideal of M .

□

Theorem 3.50. *Let M be a semigroup and $B = LL$, where L is a minimal left ideal of M . Then B is a minimal left quasi- interior ideal of M .*

Proof. Obviously $B = LL$ is a left quasi- interior ideal of M . Let A be a left quasi- interior ideal of M such that $A \subseteq B$.

We have MA is a left ideal. Then

$$\begin{aligned} MA &\subseteq MB \\ &= MLL \\ &\subseteq L, \text{ since } L \text{ is a left ideal of } M. \end{aligned}$$

Therefore $MA = L$,

$$\begin{aligned} \text{Hence } B &= MAMA \\ &\subseteq A \end{aligned}$$

Therefore $A = B$. Hence B is a minimal left quasi- interior ideal of M .

□

Corollary 3.51. *Let M be a semigroup and $B = RR$, where R is a minimal right ideal of M . Then B is a minimal right quasi- interior ideal of M .*

Corollary 3.52. *Let M be a semigroup and $B = II$, where I is a minimal ideal of M . Then B is a minimal quasi- interior ideal of M .*

We characterize regular semigroup using left quasi- interior ideals of semigroup.

Theorem 3.53. *Let M be a regular semigroup. Then every left quasi- interior ideal of M is an ideal of M .*

Proof. Let B be a left quasi- interior ideal of M . Then

$$\begin{aligned} MBMB &\subseteq B \\ \Rightarrow B &\subseteq BMB, \text{ since } M \text{ is regular} \\ \Rightarrow MB &\subseteq MBMB \subseteq B. \end{aligned}$$

Similarly, we can show that $BM \subseteq MBMB \subseteq B$.

Hence the theorem.

□

Theorem 3.54. *M is a regular semigroup if and only if $AB = A \cap B$ for any right ideal A and left ideal B of a semigroup M .*

Theorem 3.55. *Let M be a regular semigroup. Then B is a quasi- interior ideal of M if and only if $MBMB = B$, and $BMBM = B$, for all quasi- interior ideals B of M .*

Proof. Suppose M is a regular semigroup, B is a left quasi-interior ideal of M and $x \in B$. Then $MBMB \subseteq B$ and there exists $y \in M$, such that $x = xyx \in BMB$ and $BMB \subseteq MBMB$. Therefore $x \in MBMB$.

Hence $MBMB = B$. Similarly we can show that $BMBM = B$. Conversely suppose that $MBMB = B$, and $BMBM = B$, for all quasi- interior ideals B of M .

Let $B = R \cap L$, where R is a right ideal and L is a left ideal of M .

Then B is a quasi-interior ideal of M .

Therefore $M(R \cap L)M(R \cap L) = R \cap L$

$$\begin{aligned} R \cap L &= (R \cap L)M(R \cap L)M \\ &\subseteq RMLM \\ &\subseteq RLM \\ R \cap L &= (R \cap L)M(R \cap L)M \subseteq RLMRLM \subseteq RL \\ &\subseteq R \cap L \text{ (since } RL \subseteq L \text{ and } RL \subseteq R). \end{aligned}$$

Therefore $R \cap L = RL$. Hence M is a regular semigroup. □

4. WEAK-INTERIOR IDEALS OF SEMIGROUPS

In this section, we introduce the notion of weak-interior ideal as a generalization of quasi-ideal and interior ideal of semigroup and study the properties of weak-interior ideal of semigroup. Throughout this paper M is a semigroup with unity element.

Definition 4.1. A non-empty subset B of a semigroup M is said to be left weak-interior ideal of M if B is a subsemigroup of M and $MBB \subseteq B$.

Definition 4.2. A non-empty subset B of a semigroup M is said to be right weak-interior ideal of M if B is a subsemigroup of M and $BBM \subseteq B$.

Definition 4.3. A non-empty subset B of a semigroup M is said to be weak-interior ideal of M if B is a subsemigroup of M and B is left and right weak-interior ideal of M .

Example 4.4. Let Q be the set of all rational numbers, $M = \left\{ \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \mid b, d \in Q \right\}$ be the set of matrices. The binary operation AB is defined as usual matrix multiplication of A, B , for

all $A, B \in M$. Then M is a semigroup. If $R = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid 0 \neq b \in Q \right\}$ then R is a left weak interior ideal of the semigroup M and R is neither a left ideal nor a right ideal, not a weak interior ideal and not an interior ideal of the semigroup M .

In the following theorems, we mention some important properties and we omit the proofs since they are straight forward.

Theorem 4.5. *Let M be a semigroup. Then the following are hold.*

- (1) *Every left ideal is a left weak-interior ideal of M .*
- (2) *Every right ideal is a right weak-interior ideal of M .*
- (3) *Every quasi ideal is a weak-interior ideal of M .*
- (4) *Every ideal is a weak-interior ideal of M .*
- (5) *If B is a weak-interior ideal and T is a subsemigroup of M then $B \cap T$ is a weak-interior ideal of ring M .*
- (6) *If L is a left ideal and R is a right ideal of a semigroup M then $B = LR$ is a weak-interior ideal of M .*
- (7) *M is a left weak-interior simple semigroup if and only if $\langle a \rangle = M$, for all $a \in M$ and where $\langle a \rangle$ is the smallest left weak-interior ideal generated by a .*

Theorem 4.6. *Let M be a semigroup. Then the following are hold.*

- (1) *If B be an interior ideal of a semigroup M , then B is a left(right) weak-interior ideal of M .*
- (2) *The intersection of a left weak-interior ideal B of a semigroup M and a right ideal A of M is always a left weak-interior ideal of M .*
- (3) *Let A and C be subsemigroups of a semigroup M and $B = AC$ and B is subsemigroup of M . If A is the left ideal of M then B is a weak-interior ideal of M .*
- (4) *Let M be a semigroup and T be a non-empty subset of M . If subsemigroup B of M containing MTT and $B \subseteq T$ then B is a left weak-interior ideal of semigroup M .*
- (5) *If B is a left weak-interior ideal of semigroup M , BT is an additively subsemigroup of M and $T \subseteq B$ then BT is a left weak-interior ideal of M .*
- (6) *Let M be a semigroup and T be a subsemigroup of M . Then every subsemigroup of T containing MTT is a left weak-interior ideal of M .*

- (7) The intersection of $\{B_\lambda \mid \lambda \in A\}$ left weak-interior ideals of a semigroup M is a left weak-interior ideal of M .
- (8) Let M be a semigroup and B be a subsemigroup of M . If $MMB \subseteq B$ and $BMM \subseteq B$ then B is a weak-interior ideal of M .

Theorem 4.7. Let M be a semigroup and B be a subsemigroup of M . B is a left weak-interior ideal of M if and only if there exists left ideal L such that $LL \subseteq B \subseteq L$.

Proof. Suppose B is a left weak-interior ideal of the semigroup M . Then $MBB \subseteq B$. Let $L = MB$. Then L is a left ideal of M . Therefore $LL \subseteq B \subseteq L$. Conversely suppose that there exists left ideal L of M such that $LL \subseteq B \subseteq L$. Then $MBB \subseteq MLL \subseteq B$. Hence B is a left weak-interior ideal of M . \square

Corollary 4.8. Let M be a semigroup and B be a subsemigroup of M . B is a weak-interior ideal of M if and only if there exist ideal R such that $RR \subseteq B \subseteq R$.

Theorem 4.9. Let A and C be left weak-interior ideals of a semigroup M , and B is subsemigroup of M . If $B = AC = CA$ then B is a left quasi-interior ideal of M .

Proof. Let A and C be left weak-interior ideals of the semigroup M and $B = AC$. Then $BB = ACAC = ACCCAC \subseteq ACMCMC \subseteq AC = B$. Therefore $B = AC$ is a subsemigroup of M . Then $MBB = MACAC \subseteq MAAC \subseteq AC = B$. Hence B is a left weak-interior ideal of M . \square

Theorem 4.10. Let M be a semigroup. If $M = Ma$, for all $a \in M$. Then every left weak-interior ideal of M is a quasi ideal of M .

Proof. Let B be a left weak-interior ideal of the semigroup M and $a \in B$. Then $\Rightarrow Ma \subseteq MB, \Rightarrow M \subseteq MB \subseteq M, \Rightarrow MB = M, \Rightarrow MB = MBB \subseteq B, \Rightarrow MB \cap BM \subseteq B \cap BM \subseteq B$. Therefore B is a quasi ideal of M . Hence the theorem. \square

Theorem 4.11. B is a left weak-interior ideal of a semigroup M if and only if B is a left ideal of some left ideal of a semigroup M .

Proof. Suppose B is a left ideal of left ideal R of the semigroup M . Then $RB \subseteq B, MR \subseteq R$ and $MBB \subseteq MRB \subseteq RB \subseteq B$. Therefore B is a left weak-interior ideal of a semigroup M . Conversely suppose that B is a left weak-interior ideal of a semigroup M . Then $MBB \subseteq B$. Therefore B is a left ideal of left ideal MB of the semigroup M . \square

Definition 4.12. A semigroup M is a left (right) simple semigroup if M has no proper left (right) ideals of M .

Definition 4.13. A semigroup M is said to be simple semigroup if M has no proper ideals of M .

Definition 4.14. A semigroup M is said to be left(right) weak-interior simple semigroup if M has no left(right) weak-interior ideal other than M itself.

Definition 4.15. A semigroup M is said to be weak-interior simple semigroup if M has no weak-interior ideal other than M itself.

Theorem 4.16. *If M is a semigroup and every non zero element has inverse, then M is a left weak-interior simple semigroup.*

Proof. Let B be a proper left weak-interior ideal of the semigroup M , $x \in M$ and $0 \neq a \in B$. There exists $b \in M$ such that $ab = 1$. Therefore $abx = x$. $x \in BM$ and $M \subseteq BM$. We have $BM \subseteq M$. Hence $M = BM$. Similarly we can prove $MB = M$. $M = MB = MBB \subseteq B$, $M \subseteq B$. Therefore $M = B$. Hence the theorem. \square

Theorem 4.17. *Let M be a left simple semigroup. Every left weak-interior ideal of M is an ideal of M .*

Proof. Let M be a left simple semigroup and B be a left weak-interior ideal of M . Then $MBB \subseteq B$ and MB is a left ideal of M . Since M is a left simple semigroup, we have $MB = M$. Therefore $MB \subseteq MBB \subseteq B$. Then $BM = BMB = BMBB \subseteq BB \subseteq B$. Hence every left weak-interior ideal is an ideal of M . \square

Theorem 4.18. *Let M be a semigroup. Then M is a left weak-interior simple semigroup if and only if $Maa = M$, for all $a \in M$.*

Proof. Suppose M is the left-quasi interior simple semigroup and $a \in M$. Then Maa is a weak-interior ideal of M . Hence $Maa = M$, for all $a \in M$.

Conversely suppose that $Maa = M$, for all $a \in M$. Let B be a left weak-interior ideal of the semigroup M and $a \in B$. $M = Maa \subseteq MBB \subseteq B$. Therefore $M = B$. Hence M is a left weak-interior simple semigroup. \square

Theorem 4.19. *Let M be a semigroup and B be a left weak-interior ideal of M . Then B is a minimal left weak-interior ideal of M if and only if B is a left weak-interior simple subsemigroup of M .*

Proof. Let B be a minimal left weak-interior ideal of the semigroup M and C be a left weak-interior ideal of B . Then $BCC \subseteq C$.

and BCC is a left weak-interior ideal of M .

Since B is a minimal weak-interior ideal of M , $BCC = B, \Rightarrow B = BCC \subseteq C, \Rightarrow B = C$.

Conversely suppose that B is the left weak-interior simple subsemigroup of M . Let C be a left weak-interior ideal of M and $C \subseteq B. \Rightarrow BCC \subseteq MCC \subseteq MBB \subseteq B$. Therefore C is a left weak-interior of $B. \Rightarrow B = C$, since B is a left weak-interior simple subsemigroup of M . Hence B is a minimal left weak-interior ideal of M . \square

Theorem 4.20. *Let M be a semigroup, any left weak-interior ideal of M commutes with M and $B = LL$, where L is a minimal left ideal of M . Then B is a minimal left weak-interior ideal of M .*

Proof. Obviously $B = LL$ is a left weak-interior ideal of M . Let A be a left weak-interior ideal of M such that $A \subseteq B$.

We have MA is a left ideal of M . Then $MA \subseteq MB = MLL \subseteq L$, since L is a left ideal of M .

Therefore $MA = L$, since L is a minimal left ideal of M . Hence $B = MAMA \subseteq MMAA \subseteq MAA \subseteq A$. Therefore $A = B$. Hence B is a minimal left weak-interior ideal of M . \square

Theorem 4.21. *Let M be semigroup Then B is a weak-interior ideal of an idempotent regular semigroup M if and only if $BBM = B$ and $MBB = B$ for all weak-interior ideals B of M .*

Proof. Suppose M is the regular semigroup, B is the weak-interior ideal of M and $x \in B$. Then $MBB \subseteq B, y \in M$, such that $x = xx = xxyxx \in MBB$. Therefore $x \in MBB$. Hence $MBB = B$. Similarly we can prove $BBM = B$. Conversely suppose that $BBM = B$ and $MBB = B$ for all weak-interior ideals B of M . Let $B = R \cap L$ where R is a right ideal and L is a left ideal of M . Then B is a weak-interior ideal of M .

Therefore $(R \cap L)(R \cap L)M = R \cap L, R \cap L = (R \cap L)(R \cap L)M \subseteq RRM, R \cap L = M(R \cap L)(R \cap L) \subseteq MLL, R \cap L = (R \cap L)(R \cap L)M \subseteq RRMMLL \subseteq RL$, since $RL \subseteq L$ and $RL \subseteq R, RL \subseteq R \cap L$. Therefore $R \cap L = RL$. Hence M is a regular semigroup. \square

5. CONCLUSION

As a further generalization of ideals, we introduced the notion of quasi-interior ideals and weak -interior ideals as a generalization of quasi ideal, bi-ideal and interior ideal of semigroups. We study the properties of quasi-interior ideals and weak-interior ideals of semigroups and characterized the quasi-interior (weak-interior) simple semigroup, regular semigroup

using quasi-interior ideals and weak-interior ideal ideals of semigroups. In continuity of this paper, we study prime, maximal and minimal quasi- interior (weak-interior) ideals of semigroups.

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