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SOLVING A LINEAR SYSTEM WITH NON-SQUARE COEFFICIENT MATRIX USING DETERMINANTS

RUBEN D. CORDERO JR.[1], LOUIE RESTI S. RELLON[1,*], JOE VINCENT B. DELUAO[2,*]

[1]Department of Mathematics, University of Mindanao, Philippines

[2]Department of Mathematics, University of Southeastern Philippines, Philippines

Corresponding authors: louie_rellon@umindanao.edu.ph; joedeluao@gmail.com

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Abstract. One method of solving a linear system $AX = B$ with square coefficient matrix A when the solution exists is by using determinants and this is known as Cramer's Rule. This paper presents the Extended Cramer's formula of solving a linear system $AX = B$ when the coefficient matrix A is an $m \times n$ matrix with $n = m + 1$ as an alternate process of elimination method.

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1. Introduction

Consider solving a linear system $AX = B$ when the coefficient matrix is $m \times n$ where $n = m + 1$. It has been known that solving a linear system with equal number of equations and variables can be done by elimination or by substitution or by Gauss-Jordan reduction or by Cramer's rule with the use of determinants or by using the inverse of the coefficient matrix. On the other hand, a general formula of Cramer's rule was presented by Leiva [5] (2015) given in Theorems 2.11 and 2.12, which provides a unique solution of the system when the $m \times n$ coefficient matrix has linearly independent rows by extending the idea of Burgstahler [1] (1983). However, this method seems different as compared to elimination method as we have known that there are many solutions of the system when there are more unknowns than the number of equations. In our paper [2], we presented the solution of a linear system with

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an $m \times n$ coefficient matrix A when $n = m + 1$ which is given by $X = A^{\dagger}B$ where $A^{\dagger}$ is the quasi-inverse of A. In this paper, we discuss the Extended Cramer's Rule by employing the quasi-inverse matrix and it is found out that this formula is equivalent to solving a linear system with $m \times n$ coefficient matrix when $n = m + 1$ when applying elimination method. For the basic terminologies not stated here, the reader is advised to refer to the book of Kolman and Hill [3] or the book of Larson [4].

2. Basic Concepts

Definition 2.1. A *linear system* is a collection of m linear equations with n unknowns $x_1, x_, \ldots, x_n$,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Definition 2.2. An $m \times n$ *matrix* $A = [a_{ij}]$ of *order* mn is a rectangular array of numbers having m *rows* and n *columns*

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \ldots & a_{mn} \end{bmatrix}.$$

The row vectors of A are $(a_{11}\ a_{12} \ldots\ a_{1n}), (a_{21}\ a_{22} \ldots\ a_{2n}), \ldots, (a_{m1}\ a_{m2} \ldots\ a_{mn})$ while the column vectors are $\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2}, \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$. If $m = n$, then we say that A is a *square matrix* of *order* n. If $m \neq n$, we say that A is *non-square* matrix.

Definition 2.3. If A is an $m \times n$ matrix, then the matrix $B = [a_{pk}]$ obtained by deleting a row/s or column/s or both is called a *submatrix* of A.

Definition 2.4. Let $S = \{1, 2, \ldots, n\}$ arranged in ascending order. A rearrangement $j_1j_2 \ldots j_n$ of the elements of S is called a *permutation* of S and is denoted by S_n. The number of permutations of S is given by $n!$. A permutation $j_1j_2 \ldots j_n$ is said to have an *inversion* if larger integer j_r precedes a smaller one j_s. A permutation is called *even* or *odd* according to whether the total number of inversions is even or odd. It should be noted that there are $n!/2$ even as well as odd permutations.

Definition 2.5. Let A be an $n \times n$ matrix. The *determinant* of A, written $|A|$ is given by $|A| = \sum(\pm)a_{1j_1}a_{2j_2}\dots a_{nj_n}$ where the summation ranges over all permutations $j_1j_2\dots j_n$ of the set $S = \{1, 2, \dots, n\}$. The sign $+$ or $-$ depends on whether the permutation $j_1j_2\dots j_n$ is even or odd.

Definition 2.6. Let A be an $n \times n$ matrix. Let M_{ij} be the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and jth column of A. The determinant $|M_{ij}|$ is called the *minor* of a_{ij}. The cofactor A_{ij} of a_{ij} is defined as $A_{ij} = (-1)^{i+j}|M_{ij}|$.

Definition 2.7. Let A be a square matrix of order n. The *adjoint* of A, denoted by $adjA$ is an $n \times n$ matrix whose i, jth element is the cofactor A_{ji} of a_{ji}. That is,

$$adjA = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \dots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}.$$

Theorem 2.8. (Cramer's Rule) If a given system of n linear equations in n variables has a coefficient matrix A with a nonzero determinant $|A|$, then the solution of the system is given by

$$x_1 = \frac{|A_1|}{|A|}, x_2 = \frac{|A_2|}{|A|}, \dots, x_n = \frac{|A_n|}{|A|}$$

where the i^{th} column of A_i is the column of constants in the system of equations.

Definition 2.9. [8] Let A be an $m \times n$ matrix. If A^TA is invertible then the *left pseudo-inverse* A_L^+ of A is defined as

$$A_L^+ = (A^TA)^{-1}A^T \text{ and } A_L^+A = I \text{ or } E.$$

If AA^T is invertible then the *right pseudo-inverse* A_R^+ is defined as

$$A_R^+ = A^T(AA^T)^{-1} \text{ and } AA_R^+ = I \text{ or } E.$$

Remark 2.10. We use the notation A^+ to mean A_R^+ or A_L^+.

Theorem 2.11. [5] For all $B \in R^m$, the system given in definition 2.1 is solvable if and only if $\det(AA^T) \neq 0$, where A is the $m \times n$ coefficient matrix and $B = [b_1, b_2, \dots, b_m]^T$. Moreover, one solution of the system is given by $X = A^T(AA^T)^{-1}B$. The Cramer's formula for the solution is given by $x_i = \sum_{j=1}^{m} a_{ji}\frac{det(AA^T)_j}{det(AA^T)}$, $i = 1, 2, \ldots, n$, where $(AA^T)_j$ is the matrix obtained by replacing the entries in the j^{th} column of AA^T by the entries in B.

Theorem 2.12. [5] The solution of the system given in Theorem 2.11 can be written as

$$x_i = \frac{\begin{vmatrix} \|l_1\|^2+a_{1i}b_1 \ \langle l_1,l_2\rangle+a_{2i}b_1 \ldots \ \langle l_1,l_m\rangle+a_{mi}b_1 \\ \langle l_2,l_1\rangle+a_{1i}b_2 \ |l_2\|^2+a_{2i}b_2 \ldots \ \langle l_2,l_m\rangle+a_{mi}b_2 \\ \vdots \\ \langle l_m,l_1\rangle+a_{1i}b_m \ \langle l_m,l_2\rangle+a_{2i}b_m \ldots \ \|l_m\|^2+a_{mi}b_m \end{vmatrix}}{det(AA^T)} - 1,$$

where $l_1, l_2, \ldots, l_m$ are column vectors of A, which is obtained by extending the idea of Burgstahler [1].

Definition 2.13. [2] Let A be an $m \times n$ matrix. The one-sided inverse $A^\dagger$, not necessarily equal to A^+, such that $AA^\dagger = I$ or $A^\dagger A = I$ where $A^\dagger$ can provide non-unique solutions to the linear system $AX = B$ is called a *quasi-inverse* of A. The solution is given by $X = A^\dagger B$. We describe matrix A as quasi-invertible.

Theorem 2.14. [2] Let A be an $m \times n$ quasi-invertible matrix where $n = m + 1$ and let A_j be a square submatrix of A whose order is m and is obtained by deleting the j^{th} column of A. Then the quasi-inverse of A is an $n \times m$ matrix $A^\dagger = [b_{ji}]$ such that

$$b_{ji} = \begin{cases} \frac{A_{ij}+|A_j|r}{|A_n|}, & \text{if } j \ is \ odd \\ \frac{A_{ij}-|A_j|r}{|A_n|}, & \text{if } j \ is \ even \end{cases}$$

when the number of rows is even and

$$b_{ji} = \begin{cases} \frac{A_{ij}-|A_j|r}{|A_n|}, & \text{if } j \ is \ odd \\ \frac{A_{ij}+|A_j|r}{|A_n|}, & \text{if } j \ is \ even \end{cases}$$

when the number of rows is odd

where in both cases, $1 \leq j \leq m$ and $b_{ni} = r$ for $i = 1, 2, \ldots, m$, A_{ij} is the cofactor of a_{ij} of the invertible submatrix A_n obtained from A by deleting the n^{th} column. That is,

$$A^\dagger = \begin{bmatrix} \frac{A_{11}\pm|A_1|r}{|A_n|} & \frac{A_{21}\pm|A_1|r}{|A_n|} & \ldots & \frac{A_{m1}\pm|A_1|r}{|A_n|} \\ \frac{A_{12}\pm|A_2|r}{|A_n|} & \frac{A_{22}\pm|A_2|r}{|A_n|} & \ldots & \frac{A_{m2}\pm|A_2|r}{|A_n|} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{A_{1m}\pm|A_m|r}{|A_n|} & \frac{A_{2m}\pm|A_m|r}{|A_n|} & \ldots & \frac{A_{mm}\pm|A_m|r}{|A_n|} \\ r & r & \ldots & r \end{bmatrix}.$$

Theorem 2.15. [2] Let A be an $m \times n$ matrix with $n = m + 1$ and let $AX = B$ be a linear system where $X = [x_1\ x_2\ \ldots\ x_n]^T$ and $B = [c_1\ c_2\ \ldots\ c_m]^T$. If the system has a solution, then a solution is given by

$$X = A^{\dagger}B$$

where $A^{\dagger}$ is a quasi-inverse of A.

3. Extended Cramer's Rule

Theorem 3.1. Let

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = c_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = c_m$$

be a linear system with an $m \times n$ coefficient matrix A where $m = n - 1$. If the given system has a solution, then the Extended Cramer's Rule of a solution is given by
$x_1 = \frac{|A_{nx_1}| \pm |A_1| r \sum_{i=1}^{m} c_i}{|A_n|}$, $x_2 = \frac{|A_{nx_2}| \pm |A_2| r \sum_{i=1}^{m} c_i}{|A_n|}$, $x_3 = \frac{|A_{nx_3}| \pm |A_3| r \sum_{i=1}^{m} c_i}{|A_n|}, \ldots,$
$x_{n-1} = \frac{|A_{nx_{n-1}}| \pm |A_{n-1}| r \sum_{i=1}^{m} c_i}{|A_n|}$, $x_n = r\sum_{i=1}^{m} c_i$ where $r \in R$, A_j, $1 \leq j \leq n$ is the submatrix of A obtained by deleting the j^{th} column, $X = [x_1\ x_2\ \ldots\ x_n]^T$, $B = [c_1\ c_2\ \ldots\ c_m]^T$, A_{nx_i} is the matrix obtained from A_n by replacing the x_i column with the constants c_j, $1 \leq i \leq n-1$ and the choice of sign depends on the conditions given in Theorem 2.14.

Proof 3.2. Let $AX = B$ be the linear system given above. Suppose that $AX = B$ has a solution. Then by Theorem 2.15, a non-unique solution is given by $X = A^{\dagger}B$ where $A^{\dagger}$ is the quasi-inverse of A. By Theorem 2.14,

$$A^{\dagger} = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1m} \\ b_{21} & b_{22} & \ldots & b_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \ldots & b_{mm} \\ r & r & \ldots & r \end{bmatrix}$$

such that $b_{ji} = \begin{cases} \frac{A_{ij} + |A_j| r}{|A_n|}, & \text{if } j \textit{ is odd} \\ \frac{A_{ij} - |A_j| r}{|A_n|}, & \text{if } j \textit{ is even} \end{cases}$
when the number of rows is even and

$$b_{ji} = \begin{cases} \frac{A_{ij} - |A_j| r}{|A_n|}, & \text{if } j \; is \; odd \\ \frac{A_{ij} + |A_j| r}{|A_n|}, & \text{if } j \; is \; even \end{cases}$$

when the number of rows is odd, where in both cases, $1 \leq j \leq n$ and $b_{ni} = r$ for $i = 1, 2, \ldots, m$, A_{ij} is the cofactor of a_{ij} of the invertible submatrix A_n obtained from A by deleting the n^{th} column. That is,

$$A^{\dagger} = \begin{bmatrix} \frac{A_{11} \pm |A_1| r}{|A_n|} & \frac{A_{21} \pm |A_1| r}{|A_n|} & \ldots & \frac{A_{m1} \pm |A_1| r}{|A_n|} \\ \frac{A_{12} \pm |A_2| r}{|A_n|} & \frac{A_{22} \pm |A_2| r}{|A_n|} & \ldots & \frac{A_{m2} \pm |A_2| r}{|A_n|} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{A_{1m} \pm |A_m| r}{|A_n|} & \frac{A_{2m} \pm |A_m| r}{|A_n|} & \ldots & \frac{A_{mm} \pm |A_m| r}{|A_n|} \\ r & r & \ldots & r \end{bmatrix}.$$

Now,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{A_{11} \pm |A_1| r}{|A_n|} & \frac{A_{21} \pm |A_1| r}{|A_n|} & \ldots & \frac{A_{m1} \pm |A_1| r}{|A_n|} \\ \frac{A_{12} \pm |A_2| r}{|A_n|} & \frac{A_{22} \pm |A_2| r}{|A_n|} & \ldots & \frac{A_{m2} \pm |A_2| r}{|A_n|} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{A_{1m} \pm |A_m| r}{|A_n|} & \frac{A_{2m} \pm |A_m| r}{|A_n|} & \ldots & \frac{A_{mm} \pm |A_m| r}{|A_n|} \\ r & r & \ldots & r \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(A_{11} \pm |A_1| r)c_1 + (A_{21} \pm |A_1| r)c_2 + \cdots + (A_{m1} \pm |A_1| r)c_m}{|A_n|} \\ \frac{(A_{12} \pm |A_2| r)c_1 + (A_{22} \pm |A_2| r)c_2 + \cdots + (A_{m2} \pm |A_2| r)c_m}{|A_n|} \\ \vdots \\ \frac{(A_{1m} \pm |A_m| r)c_1 + (A_{2m} \pm |A_2| r)c_2 + \cdots + (A_{mm} \pm |A_2| r)c_m}{|A_n|} \\ rc_1 + rc_2 + \cdots + rc_m \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(A_{11} \pm |A_1| r)c_1 + (A_{21} \pm |A_1| r)c_2 + \cdots + (A_{m1} \pm |A_1| r)c_m}{|A_n|} \\ \frac{(A_{12} \pm |A_2| r)c_1 + (A_{22} \pm |A_2| r)c_2 + \cdots + (A_{m2} \pm |A_2| r)c_m}{|A_n|} \\ \vdots \\ \frac{(A_{1m} \pm |A_m| r)c_1 + (A_{2m} \pm |A_2| r)c_2 + \cdots + (A_{mm} \pm |A_2| r)c_m}{|A_n|} \\ r \sum_{i=1}^{m} c_i \end{bmatrix}$$

$$= \begin{bmatrix} \frac{A_{11}c_1 \pm |A_1| rc_1 + A_{21}c_2 \pm |A_1| rc_2 + \cdots + A_{m1}c_m \pm |A_1| rc_m}{|A_n|} \\ \frac{A_{12}c_1 \pm |A_2| rc_1 + A_{22}c_2 \pm |A_2| rc_2 + \cdots + A_{m2}c_m \pm |A_2| rc_m}{|A_n|} \\ \vdots \\ \frac{A_{1m}c_1 \pm |A_m| rc_1 + A_{2m}c_2 \pm |A_2| rc_2 + \cdots + A_{mm}c_m \pm |A_2| rc_m}{|A_n|} \\ r \sum_{i=1}^{m} c_i \end{bmatrix}$$

$$= \begin{bmatrix} \frac{A_{11}c_1+A_{21}c_2+\cdots+A_{m1}c_m\pm|A_1|rc_1\pm|A_1|rc_2\pm\cdots\pm|A_1|rc_m}{|A_n|} \\ \frac{A_{12}c_1+A_{22}c_2+\cdots+A_{m2}c_m\pm|A_2|rc_1\pm|A_2|rc_2\pm\cdots\pm|A_2|rc_m}{|A_n|} \\ \vdots \\ \frac{A_{1m}c_1+A_{2m}c_2+\cdots+A_{mm}c_m\pm|A_m|rc_1\pm|A_2|rc_2\pm\cdots\pm|A_2|rc_m}{|A_n|} \\ r\sum_{i=1}^m c_i \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(A_{11}c_1+A_{21}c_2+\cdots+A_{m1}c_m)\pm|A_1|r(c_1+c_2+\cdots+c_m)}{|A_n|} \\ \frac{(A_{12}c_1+A_{22}c_2+\cdots+A_{m2}c_m)\pm|A_2|r(c_1+c_2+\cdots+c_m)}{|A_n|} \\ \vdots \\ \frac{(A_{1m}c_1+A_{2m}c_2+\cdots+A_{mm}c_m)\pm|A_m|r(c_1+c_2+\cdots+c_m)}{|A_n|} \\ r\sum_{i=1}^m c_i \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(A_{11}c_1+A_{21}c_2+\cdots+A_{m1}c_m)\pm|A_1|r\sum_{i=1}^m c_i}{|A_n|} \\ \frac{(A_{12}c_1+A_{22}c_2+\cdots+A_{m2}c_m)\pm|A_2|r\sum_{i=1}^m c_i}{|A_n|} \\ \vdots \\ \frac{(A_{1m}c_1+A_{2m}c_2+\cdots+A_{mm}c_m)\pm|A_m|r\sum_{i=1}^m c_i}{|A_n|} \\ r\sum_{i=1}^m c_i \end{bmatrix}.$$

From the last matrix, we obtain $x_n = r\sum_{i=1}^m c_i$.

Consider the submatrix

$$\begin{bmatrix} \frac{(A_{11}c_1+A_{21}c_2+\cdots+A_{m1}c_m)\pm|A_1|r\sum_{i=1}^m c_i}{|A_n|} \\ \frac{(A_{12}c_1+A_{22}c_2+\cdots+A_{m2}c_m)\pm|A_2|r\sum_{i=1}^m c_i}{|A_n|} \\ \vdots \\ \frac{(A_{1m}c_1+A_{2m}c_2+\cdots+A_{mm}c_m)\pm|A_m|r\sum_{i=1}^m c_i}{|A_n|} \end{bmatrix}$$

which is equal to

$$\frac{1}{|A_n|}\begin{bmatrix} (A_{11}c_1+A_{21}c_2+\cdots+A_{m1}c_m)\pm|A_1|r\sum_{i=1}^m c_i \\ (A_{12}c_1+A_{22}c_2+\cdots+A_{m2}c_m)\pm|A_2|r\sum_{i=1}^m c_i \\ \vdots \\ (A_{1m}c_1+A_{2m}c_2+\cdots+A_{mm}c_m)\pm|A_m|r\sum_{i=1}^m c_i \end{bmatrix}$$

$$= \frac{1}{|A_n|}\begin{bmatrix} A_{11}c_1+A_{21}c_2+\cdots+A_{m1}c_m \\ A_{12}c_1+A_{22}c_2+\cdots+A_{m2}c_m \\ \vdots \\ A_{1m}c_1+A_{2m}c_2+\cdots+A_{mm}c_m \end{bmatrix} \pm \frac{1}{|A_n|}\begin{bmatrix} |A_1|r\sum_{i=1}^m c_i \\ |A_2|r\sum_{i=1}^m c_i \\ \vdots \\ |A_m|r\sum_{i=1}^m c_i \end{bmatrix}.$$

But note that

$$|A_{nx_1}| = A_{11}c_1 + A_{21}c_2 + \cdots + A_{m1}c_m,$$
$$|A_{nx_2}| = A_{12}c_1 + A_{22}c_2 + \cdots + A_{m2}c_m,$$
$$\vdots$$
$$|A_{nx_m}| = A_{1m}c_1 + A_{2m}c_2 + \cdots + A_{mm}c_m.$$

Thus, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} = \frac{1}{|A_n|}\begin{bmatrix} |A_{nx_1}| \\ |A_{nx_2}| \\ \vdots \\ |A_{nx_m}| \end{bmatrix} \pm \frac{1}{|A_n|}\begin{bmatrix} |A_1|r\sum_{i=1}^m c_i \\ |A_2|r\sum_{i=1}^m c_i \\ \vdots \\ |A_m|r\sum_{i=1}^m c_i \end{bmatrix}$$
$$= \frac{1}{|A_n|}\begin{bmatrix} |A_{nx_1}| \pm |A_1|r\sum_{i=1}^m c_i \\ |A_{nx_2}| \pm |A_2|r\sum_{i=1}^m c_i \\ \vdots \\ |A_{nx_m}| \pm |A_m|r\sum_{i=1}^m c_i \end{bmatrix} = \begin{bmatrix} \frac{|A_{nx_1}|\pm|A_1|r\sum_{i=1}^m c_i}{|A_n|} \\ \frac{|A_{nx_2}|\pm|A_2|r\sum_{i=1}^m c_i}{|A_n|} \\ \vdots \\ \frac{|A_{nx_m}|\pm|A_m|r\sum_{i=1}^m c_i}{|A_n|} \end{bmatrix} = \begin{bmatrix} \frac{|A_{nx_1}|\pm|A_1|r\sum_{i=1}^m c_i}{|A_n|} \\ \frac{|A_{nx_2}|\pm|A_2|r\sum_{i=1}^m c_i}{|A_n|} \\ \vdots \\ \frac{|A_{nx_{n-1}}|\pm|A_{n-1}|r\sum_{i=1}^m c_i}{|A_n|} \end{bmatrix},$$

since $m = n - 1$.

Therefore, we obtain $x_1 = \frac{|A_{nx_1}|\pm|A_1|r\sum_{i=1}^m c_i}{|A_n|}$, $x_2 = \frac{|A_{nx_2}|\pm|A_2|r\sum_{i=1}^m c_i}{|A_n|}$, $x_3 = \frac{|A_{nx_3}|\pm|A_3|r\sum_{i=1}^m c_i}{|A_n|}, \ldots,$ $x_{n-1} = \frac{|A_{nx_{n-1}}|\pm|A_{n-1}|r\sum_{i=1}^m c_i}{|A_n|}$, $x_n = r\sum_{i=1}^m c_i$, $r \in R$. □

Remark 3.3. In the solution of a linear system in Theorem 3.1, if we discard $\sum_{i=1}^m c_i$, then $x_i = \frac{|A_{nx_i}|\pm|A_j|r}{|A_n|}$, $1 \leq i \leq n-1$, $x_n = r$ is also a solution of the system.

Theorem 3.4. Let $AX = B$ be a linear system with $m \times n$ coefficient matrix A where $m = n-1$. The following statements are equivalent:

$i.)$ $x_i = \frac{|A_{nx_i}|\pm|A_j|r\sum_{i=1}^m c_i}{|A_n|}$ $(1 \leq i \leq m)$, $x_n = r\sum_{i=1}^m c_i$, $r \in R$, is a solution of the Extended Cramer's Rule.

$ii.)$ $x_i = \frac{|A_{nx_i}|\pm|A_j|t}{|A_n|}$ $(1 \leq i \leq m)$, $x_n = t$, where $t = r\sum_{i=1}^m c_i$ or $t = r \in R$ is a solution in the elimination method.

Proof 3.5. $(i \longrightarrow ii)$ Suppose that

$x_1 = \frac{|A_{nx_1}|\pm|A_1|r\sum_{i=1}^m c_i}{|A_n|}$

$x_2 = \frac{|A_{nx_2}|\pm|A_2|r\sum_{i=1}^m c_i}{|A_n|}$

$x_3 = \frac{|A_{nx_3}|\pm|A_3|r\sum_{i=1}^m c_i}{|A_n|}$

$\vdots$

$x_{n-1} = \frac{|A_{nx_{n-1}}| \pm |A_{n-1}| r \sum_{i=1}^{m} c_i}{|A_n|}$

$x_n = r \sum_{i=1}^{m} c_i$

is the solution of the linear system $AX = B$ obtained by Extended Cramer's Rule. Then by the proof of Theorem 3.1,

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{|A_{nx_1}| \pm |A_1| r \sum_{i=1}^{m} c_i}{|A_n|} \\ \frac{|A_{nx_2}| \pm |A_2| r \sum_{i=1}^{m} c_i}{|A_n|} \\ \vdots \\ \frac{|A_{nx_{n-1}}| \pm |A_{n-1}| r \sum_{i=1}^{m} c_i}{|A_n|} \\ r \sum_{i=1}^{m} c_m \end{bmatrix}.$$

Now,

$$\begin{bmatrix} \frac{|A_{nx_1}| \pm |A_1| r \sum_{i=1}^{m} c_i}{|A_n|} \\ \frac{|A_{nx_2}| \pm |A_2| r \sum_{i=1}^{m} c_i}{|A_n|} \\ \vdots \\ \frac{|A_{nx_{n-1}}| \pm |A_{n-1}| r \sum_{i=1}^{m} c_i}{|A_n|} \\ r \sum_{i=1}^{m} c_i \end{bmatrix} = \begin{bmatrix} \frac{|A_{nx_1}|}{|A_n|} \\ \frac{|A_{nx_2}|}{|A_n|} \\ \vdots \\ \frac{|A_{nx_{n-1}}|}{|A_n|} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{\pm |A_1| r \sum_{i=1}^{m} c_i}{|A_n|} \\ \frac{\pm |A_2| r \sum_{i=1}^{m} c_i}{|A_n|} \\ \vdots \\ \frac{\pm |A_{n-1}| r \sum_{i=1}^{m} c_i}{|A_n|} \\ r \sum_{i=1}^{m} c_i \end{bmatrix}$$

$$= \begin{bmatrix} \frac{|A_{nx_1}|}{|A_n|} \\ \frac{|A_{nx_2}|}{|A_n|} \\ \vdots \\ \frac{|A_{nx_{n-1}}|}{|A_n|} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{\pm |A_1|}{|A_n|} \\ \frac{\pm |A_2|}{|A_n|} \\ \vdots \\ \frac{\pm |A_{n-1}|}{|A_n|} \\ 1 \end{bmatrix} r \sum_{j=1}^{m} c_i.$$

Let $t = r \sum_{i=1}^{m} c_i$. Then we obtain

$$\begin{bmatrix} \frac{|A_{nx_1}|}{|A_n|} \\ \frac{|A_{nx_2}|}{|A_n|} \\ \vdots \\ \frac{|A_{nx_{n-1}}|}{|A_n|} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{\pm |A_1|}{|A_n|} \\ \frac{\pm |A_2|}{|A_n|} \\ \vdots \\ \frac{\pm |A_{n-1}|}{|A_n|} \\ 1 \end{bmatrix} t = \begin{bmatrix} \frac{|A_{nx_1}|}{|A_n|} \\ \frac{|A_{nx_2}|}{|A_n|} \\ \vdots \\ \frac{|A_{nx_{n-1}}|}{|A_n|} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{\pm |A_1| t}{|A_n|} \\ \frac{\pm |A_2| t}{|A_n|} \\ \vdots \\ \frac{\pm |A_{n-1}| t}{|A_n|} \\ t \end{bmatrix} = \begin{bmatrix} \frac{|A_{nx_1}| \pm |A_1| t}{|A_n|} \\ \frac{|A_{nx_2}| \pm |A_2| t}{|A_n|} \\ \vdots \\ \frac{|A_{nx_{n-1}}| \pm |A_{n-1}| t}{|A_n|} \\ t \end{bmatrix}.$$

The last matrix exhibits the solution of the elimination method.

Therefore, $x_i = \frac{|A_{nx_i}| \pm |A_j| t}{|A_n|}$ $(1 \leq i \leq m)$, $x_n = t$, where $t = r \sum_{i=1}^{m} c_i$, $r \in R$. Moreover, by Remark 3.3, we can delete $\sum_{i=1}^{m} c_i$ to have $t = r$. This provides a solution in the elimination method.

$(ii \longrightarrow i)$ Suppose that $x_i = \frac{|A_{nx_i}| \pm |A_j| t}{|A_n|}$, $x_n = t$, $t \in R$ $(1 \leq i \leq m)$ is a solution in the elimination method. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{|A_{nx_1}|\pm|A_1|t}{|A_n|} \\ \frac{|A_{nx_2}|\pm|A_2|t}{|A_n|} \\ \vdots \\ \frac{|A_{nx_m}|\pm|A_m|t}{|A_n|} \\ t \end{bmatrix}.$$

By the proof of Theorem 3.1,

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{(A_{11}c_1+A_{21}c_2+\cdots+A_{m1}c_m)\pm|A_1|r\sum_{i=1}^m c_i}{|A_n|} \\ \frac{(A_{12}c_1+A_{22}c_2+\cdots+A_{m2}c_m)\pm|A_2|r\sum_{i=1}^m c_i}{|A_n|} \\ \vdots \\ \frac{(A_{1m}c_1+A_{2m}c_2+\cdots+A_{mm}c_m)\pm|A_m|r\sum_{i=1}^m c_i}{|A_n|} \\ r\sum_{i=1}^m c_i \end{bmatrix} = \begin{bmatrix} \frac{|A_{nx_1}|\pm|A_1|r\sum_{i=1}^m c_i}{|A_n|} \\ \frac{|A_{nx_2}|\pm|A_2|r\sum_{i=1}^m c_i}{|A_n|} \\ \vdots \\ \frac{|A_{nx_m}|\pm|A_m|r\sum_{i=1}^m c_i}{|A_n|} \\ r\sum_{i=1}^m c_i \end{bmatrix}.$$

So, by the equality of matrices, $t = r\sum_{i=1}^m c_i$ in which we obtain the solution of the Extended Cramer's Rule. By Remark 3.3, $\begin{bmatrix} \frac{|A_{nx_1}|\pm|A_1|r}{|A_n|} \\ \frac{|A_{nx_2}|\pm|A_2|r}{|A_n|} \\ \vdots \\ \frac{|A_{nx_m}|\pm|A_m|r}{|A_n|} \\ r \end{bmatrix}$ is also a solution of the linear system. By taking $r = t$, we obtain the desired statement. □

Example 3.6. Solve the system

$$\begin{aligned} x_1 - x_3 + 2x_4 &= -3, \\ 3x_1 + 3x_2 - 3x_3 + 4x_4 &= 0, \\ 4x_1 + 2x_2 + 5x_3 + x_4 &= 39 \end{aligned}$$

using the Extended Cramer's Rule.

Solution. The coefficient matrix of the given system is $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 3 & -3 & 4 \\ 4 & 2 & 5 & 1 \end{bmatrix}$. Then we have $A_1 = \begin{bmatrix} 0 & -1 & 2 \\ 3 & -3 & 4 \\ 2 & 5 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 4 \\ 4 & 5 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 3 & 4 \\ 4 & 2 & 1 \end{bmatrix}$, and $A_4 = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 3 & -3 \\ 4 & 2 & 5 \end{bmatrix}$, the submatrices of A obtained by deleting the 1^{st}, 2^{nd}, and 3^{rd}, and the 4^{th} columns of A respectively. We also have $A_{4x_1} = \begin{bmatrix} -3 & 0 & -1 \\ 0 & 3 & -3 \\ 39 & 2 & 5 \end{bmatrix}$, $A_{4x_2} = \begin{bmatrix} 1 & -3 & -1 \\ 3 & 0 & -3 \\ 4 & 39 & 5 \end{bmatrix}$, $A_{4x_3} = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 3 & 0 \\ 4 & 2 & 39 \end{bmatrix}$. Computing the determinants of the submatrices, we obtain $|A_1| = 37$, $|A_2| = 18$, $|A_3| = -17$, and $|A_4| = 27$, $|A_{4x_1}| = 54$, $|A_{4x_2}| = 81$, and $|A_{4x_3}| = 135$. Since the number of rows of the coefficient matrix A is odd, the choice of sign on the values of $x_i$depends on the entries of the quasi-inverse matrix given in Theorem 2.14. That is,

$$b_{ji} = \begin{cases} \frac{A_{ij} - |A_j| r}{|A_n|}, & \text{if } j \textit{ is odd} \\ \frac{A_{ij} + |A_j| r}{|A_n|}, & \text{if } j \textit{ is even.} \end{cases}$$

Note that $\sum_{j=1}^{3} c_j = -3 + 0 + 39 = 36$ and from the above condition, we have

$x_1 = \frac{|A_{4x_1}| - |A_1| 36r}{|A_4|} = \frac{54-(37)(36r)}{27} = \frac{54-1332r}{27}$

$x_2 = \frac{|A_{4x_2}| + |A_2| 36r}{|A_4|} = \frac{81+(18)(36r)}{27} = \frac{81+648r}{27}$

$x_3 = \frac{|A_{4x_3}| - |A_3| 36r}{|A_4|} = \frac{135-(-17)(36r)}{27} = \frac{135+612r}{27}$

$x_4 = 36r$, $r \in R$

which provides a solution of the given linear system. If $r = 0$, then $x_1 = 2$, $x_2 = 3$, $x_3 = 5$, and $x_4 = 0$ satisfy the equation. On the other hand, if $r = 2$, then $x_1 = -\frac{290}{3}$, $x_2 = 51$, and $x_3 = \frac{151}{3}$, $x_4 = 72$ also satisfy the equation.

Example 3.7. Obtain a solution of the linear system in Example 3.6 in the elimination method without performing it.

Solution. From the above discussion, we obtain the solution of the system $x_1 = \frac{54-1332r}{27} = \frac{54-(37)(36r)}{27}$, $x_2 = \frac{81+648r}{27} = \frac{81+(18)(36r)}{27}$, $x_3 = \frac{135+612r}{27} = \frac{135+(17)(36r)}{27}$, $x_4 = 36r$. By Corollary 3.4, $t = 36r$ and the solution in the elimination method is, $x_1 = \frac{54-37t}{27}$, $x_2 = \frac{81+18t}{27}$, $x_3 = \frac{135+17t}{27}$, $x_4 = t$, $t \in R$. Observe that if $t = 1$, $x_1 = \frac{17}{27}$, $x_2 = \frac{11}{3}$, $x_3 = \frac{152}{27}$, $x_4 = 1$ provides a solution of the given system.

Remark 3.8. In Example 3.6, if we apply the methods in Theorem 2.11 given by Leiva, we obtain the unique solution of the system as $x_1 = \frac{7827}{2711}$, $x_2 = \frac{6963}{2711}$, $x_3 = \frac{12450}{2711}$, $x_4 = -\frac{1755}{2711}$.

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