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ON COMMUTATIVE TRIPLES ASSOCIATED TO FINITE GROUPS

ARMEL N'GAZA AKA, IBRAHIMA TOURE*

Department of Mathematics and Informatic, University Félix Houphouet-Boigny, Abidjan, Côte d'Ivoire

*Corresponding author: ibrahima.toure@univ-fhb.edu.ci

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ABSTRACT. Let G be a finite group and K a subgroup of G. Let τ denote a unitary equivalence class of irreducible representations of K. (G, K, τ) is a commutative triple if the algebra of τ-radial functions on G is commutative under convolution. In this paper, we give some characterizations of these triples.

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1. INTRODUCTION

Let G be a locally compact group and K a compact subgroup of G. The pair (G, K) is called a Gelfand pair if $L^1(G\backslash\backslash K)$, the algebra of complex-valued integrable and bi-K-invariant functions on G, is commutative for convolution. This notion has been studied on finite and infinite groups by a number of authors [3–7] and is by now well understood. Since the trivial representation of K appears and is fundamental for the notion of Gelfand pairs, some authors in the papers such as [3,7,9,10,14], have introduced a generalization of Gelfand pair for infinite groups called commutative triple. Let G be a locally compact group, K a compact subgroup of G and τ a unitary irreducible representation of K. The triple (G, K, τ) is commutative if the algebra $L^1(G, K, \tau, \tau)$ of τ- radial functions on G is commutative for convolution. On infinite groups they have extended a lot of properties of Gelfand pairs to commutative triples. Our purpose in this paper is to study commutative triples on finite groups and extend some characterizations of finite Gelfand pairs obtained in [3]. The permutation representation

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of G, defined on $L(G/K)$, the space of complex valued functions on G/K, fundamental in the study of finite Gelfand pair, is replaced by generalized permutation representation of G, defined on cross sections of homogeneous vector bundle over G/K associated with τ. Our work is organized as follows. In section 2, we introduce preliminaries and basic notations useful for a well understanding of the paper. In section 3, we prove that the triple is commutative if and only if the generalized permutation representation is multiplicity free which is equivalent to the commutativity of the algebra of intertwining operators of generalized permutation representation. Thanks to this result we obtain a necessary condition: if (G, K, τ) is a commutative triple then (G, K) is a Gelfand pair. This result has been obtained in [3] for Lie groups with G/K connected and in [14] for connected Lie groups. We have also given some sufficient conditions to obtain a commutative triple. Finally we obtain some conditions to construct commutative triples from Gelfand pairs. We end the section by some examples of commutative triples on symmetric groups.

2. PRELIMINARIES

Let G be a finite group, $K \leq G$ a subgroup and τ an irreducible representation of K. We denote by V_τ the realization space of τ. $L(G, End(V_\tau))$ designates the space of all $End(V_\tau)$-valued functions defined on G. A function $F : G \to End(V_\tau)$ is said τ-radial if it verifies the property:

$$F(k_1 g k_2) = \tau(k_2^{-1})F(g)\tau(k_1^{-1})$$

$\forall\, k_1, k_2 \in K$ and $g \in G$. Denote by $L(G, K, \tau, \tau)$ the space of τ-radial functions. This is a natural generalization of the classical notion of bi-K-invariant functions. We define a scalar product on $L(G, End(V_\tau))$ by

$$\langle F_1, F_2\rangle = \sum_{g\in G} Tr[F_1(g)F_2^*(g)]$$

where F^* is the adjoint operator of F. For $F_1, F_2 \in L(G, End(V_\tau))$, define the convolution by: $\forall\, g \in G$

$$F_1 * F_2(g) = \sum_{h\in G} F_1(h^{-1}g)F_2(h).$$

If $F_1, F_2 \in L(G, K, \tau, \tau)$, we show that $F_1 * F_2 \in L(G, K, \tau, \tau)$ that is $L(G, K, \tau, \tau)$ is a subalgebra of $L(G, End(V_\tau))$.

The map $F \mapsto F_\tau^\sharp$ is the projection of $L(G, End(V_\tau))$ onto $L(G, K, \tau, \tau)$, where $\forall\, k_1, k_2 \in K$ and

$g \in G$

$$F^{\sharp}_{\tau}(g) = \frac{1}{|K|^2} \sum_{k_1,k_2 \in K} \tau(k_2)F(k_1gk_2)\tau(k_1)$$

where $|.|$ designates the cardinality in this paper. Set $X = G/K$ the homogeneous space associated to the pair (G, K). For $x \in X$, we denote by δ_x the Dirac function at x, that is $\delta_x(y) = \begin{cases} 1, \ if\ x = y \\ 0, \ if\ x \neq y \end{cases}$
The set $\{\delta_x, x \in X\}$ is a natural basis for $L(X)$ the vector space of complex-valued functions on X. We denote by $L(X, V_\tau) = \{f : X \to V_\tau\}$ the vector space of all V_τ-valued functions defined on X. Let $\mathcal{B} = \{v_j : 1 \leq j \leq d_\tau\}$ be an orthonormal basis of V_τ, where d_τ is the dimension of V_τ. Then the functions $\delta_{x,v_j} : X \to V_\tau$ $(x \in X, 1 \leq j \leq d_\tau)$ given by

$$\delta_{x,v_j}(y) = \begin{cases} v_j, \ if\ x = y \\ 0, \ if\ x \neq y \end{cases}$$

for $y \in X$, form a basis for $L(X, V_\tau)$. In fact for $f \in L(X, V_\tau)$, we have $f = \sum_{j=1}^{d_\tau} f_j v_j$ where $f_j \in L(X)$. But $f_j = \sum_{x \in X} f_j(x)\delta_x$, so $f = \sum_{x \in X, 1 \leq j \leq d_\tau} f_j(x)\delta_x v_j$. Notice that $\delta_{x,v_j} = \delta_x v_j$. The space $L(X, V_\tau)$ is endowed with the scalar product defined by setting

$$\langle f_1, f_2 \rangle = \sum_{x \in X} \langle f_1(x), f_2(x) \rangle,$$

for $f_1, f_2 \in L(X, V_\tau)$. Note that the basis $\{\delta_{x,v_j}, x \in X, 1 \leq j \leq d_\tau\}$ is orthonormal with respect to $\langle ., . \rangle$. If $A : L(X, V_\tau) \to L(X, V_\tau)$ is a linear operator, setting $V(x, y, v_j) = [A\delta_{y,v_j}](x) \in V_\tau$ for $x, y \in X$ and $v_j \in \mathcal{B}$, we have that $[Af](x) = \sum_{1 \leq j \leq d_\tau, y \in X} \langle f(y), v_j \rangle V(x, y, v_j)$ for all $f \in L(X, V_\tau)$ and we say that the set of vectors $\{V(x, y, v_j) : x, y \in X, v_j \in \mathcal{B}\}$, represents the operator A. Let $E^\tau = X \times_\tau V_\tau$ be the homogeneous vector bundle over X associated with τ. We write an element of $E^\tau = X \times_\tau V_\tau$ by $[x, v]$ where $x \in X$ and $v \in V_\tau$. We designate by $\Gamma(E^\tau)$ the set of cross sections of E^τ. We write the action by $g \in G$ on $x \in X$ as $g.x$. This action is transitive. Let x_0 be the point stabilized by K. Suppose that the representation τ of K extends to a representation ρ of G on V_τ. Then $\Gamma(E^\tau)$ and $L(X, V_\tau)$ are G-isomorphic [12]. The morphism is defined as follows: to each $s \in \Gamma(E^\tau)$ given by $s(y) = [gH, u(g)]$ where $y = g.x_0$ we associate the function f defined by $f(y) = \rho(g)u(g)$. The representation ρ defines an action of G on $v \in V_\tau$ as follows $g.v = \rho(g)v$ and in particular $k.v = \tau(k)v$. The actions of G on X and V_τ respectively induce an action of G on $X \times V_\tau$, E^τ and $\Gamma(E^\tau)$ defined as follows:

$$g.(x, v) = (g.x, g.v)$$

$$g.[x, v] = [g.x, v]$$

and

$$(g.s)(x) = g.s(g^{-1}.x)$$

for $g \in G, x \in X, v \in V_\tau$ and $s \in \Gamma(E^\tau)$. We recall that the generalized permutation representation of G is the representation of G on $\Gamma(E^\tau)$ defined by $\forall g \in G, \forall s \in \Gamma(E^\tau)$ and $\forall x \in X$:

$$\lambda^\tau(g)(s)(x) = g.s(g^{-1}x)$$

A cross section of E^τ may be identified with a vector valued function $f : G \to V_\tau$ (see [15], section 5.3. or [13]) which is right K-covariant of type τ i.e.

$$f(gk) = \tau(k^{-1})f(g)$$

We denote by $L(G, K, \tau)$ the space of functions on G which are right K-covariant of type τ. The action of G on $L(G, K, \tau)$ defined by : $g.f(h) = U(g)f(h) = f(g^{-1}h)$ defines a representation U of G on $L(G, K, \tau)$ called induced representation of τ on G and denoted by $U = Ind_K^G(\tau)$. By the identification mentioned above between $\Gamma(E^\tau)$ and $L(G, K, \tau)$, the representations λ^τ and $Ind_K^G(\tau)$ are equivalent. λ^τ is unitary for the following inner product

$$\langle s, s'\rangle = \sum_{x \in G} \langle s(x), s'(x)\rangle_\tau$$

In this paper, we confound sometimes the representation and its realization space. We denote by
$Hom_G(\Gamma(E^\tau), \Gamma(E^\tau)) = \{T : \Gamma(E^\tau) \longrightarrow \Gamma(E^\tau), T(\lambda^\tau(g))s = \lambda^\tau(g)(T)s,$
$\forall\ g \in G\ ,\ s \in \Gamma(E^\tau)\}$ the space of intertwining operators of the generalized permutation representation.

3. Commutative Triples

Let G be a finite group, K a subgroup of G, and τ a unitary irreducible representation of K on V_τ.

Definition 3.1. The triple (G, K, τ) is a commutative triple if the convolution algebra $L(G, K, \tau, \tau)$ of τ-radial functions on G is commutative.

If τ is the one-dimensional trivial representation of K then $L(G, K, \tau, \tau)$ is the algebra of bi-K-invariant functions on G and we obtain the notion of finite Gelfand pair.

Remark 3.2. If (G, K, τ) is commutative then so is (G, K, τ') for any unitary irreducible representation τ' unitary equivalent to τ

In fact, if $A : V_\tau \to V'_\tau$ is an interwiner operator of τ and τ', the map from $L(G, K, \tau, \tau)$ to $L(G, K, \tau', \tau')$ defined by $F \mapsto AFA^{-1}$ where $AFA^{-1}(x) = AF(x)A^{-1}$ is an isomorphism of algebras.
The remark shows that it suffices to work with an element of the class. We denote by $\widehat{K}$ the set of unitary equivalence class of unitary irreducible representations of K.
The following is a characterization of finite commutative triples.

Theorem 3.3. *Let $K \leq G$ be finite groups and $\tau \in \widehat{K}$. $X = G/K$.*
The following are equivalent:

(i) *(G, K, τ) is a commutative triple i.e. $L(G, K, \tau, \tau)$ is commutative*
(ii) *$Hom_G(\Gamma(E^\tau), \Gamma(E^\tau))$ is commutative*
(iii) *the generalized permutation representation λ^τ of G is multiplicity-free.*

Proof. $(i) \Leftrightarrow (ii)$. It suffices to show that

$$L(G, K, \tau, \tau) \cong Hom_G(\Gamma(E^\tau), \Gamma(E^\tau))$$

.

An operator $T : \Gamma(E^\tau) \to \Gamma(E^\tau)$ is expressed by a set of vectors $\{V(x, y, v_j) : x, y \in X, v_j \in \mathcal{B}\}$ in the sense that :
$[Ts](x) = \sum_{1\leq j\leq d_\tau, y\in X} \langle s(y), v_j\rangle V(x, y, v_j)$, where $\{v_j\}_{1\leq j\leq d_\tau}$ is an orthonormal basis of V_τ. If T is an intertwiner, it satisfies the condition $T[\lambda^\tau(g)]s = \lambda^\tau(g)[T]s$ which yields

$$g.V(g^{-1}x, y, v_j) = V(x, g.y, g.v_j) \tag{3.1}$$

Considering the action of G on G/K, let x_0 be the point stabilized by K. The function $v \mapsto V(x, y, v)$ is linear. Indeed for $v, u \in V_\tau$ and $\lambda \in \mathbb{C}, y \in X$: $\delta_{y,\lambda v+u} = \lambda\delta_{y,v} + \delta_{y,u}$. Now we consider the function $\psi : X \times V_\tau \to V_\tau$ defined by $\psi(x, v_j) = V(x_0, x, v_j)$. ψ is K-equivariant thanks to equation (3.1). Let us consider the function $\varphi : X \to End(V_\tau)$, defined by $\varphi(x)v_j = V(x_0, x, v_j)$.
We associate to φ a function $\widetilde{\varphi}$ defined on G by $\widetilde{\varphi}(g) = g^{-1}.\varphi(g^{-1}.x_0)$, that is

$$\widetilde{\varphi}(g)[v_j] = g^{-1}.\varphi(g^{-1}.x_0)[v_j] = \varphi(g^{-1}.x_0)[g^{-1}.v_j] \quad \forall\ v_j \in V_\tau$$

$\widetilde{\varphi}$ is τ-radial. Indeed for $k_1, k_2 \in K,\ g \in G,\ v_j \in V_\tau$.

$$\begin{aligned}
\widetilde{\varphi}(k_1 g k_2)[v_j] &= (k_1 g k_2)^{-1}.\varphi[(k_1 g k_2)^{-1}.x_0][v_j] \\
&= k_2^{-1} g^{-1} k_1^{-1}.\varphi[k_2^{-1} g^{-1} k_1^{-1}.x_0][v_j] \\
&= k_2^{-1} g^{-1}.\varphi(k_2^{-1} g^{-1}.x_0)[k_1^{-1}.v_j] \\
&= k_2^{-1} g^{-1}.\varphi(k_2^{-1} g^{-1}.x_0)[\tau(k_1^{-1})v_j] \\
&= \varphi(k_2^{-1} g^{-1}.x_0)[k_2^{-1} g^{-1}\tau(k_1^{-1})v_j] \\
&= V(x_0, k_2^{-1} g^{-1}.x_0, k_2^{-1} g^{-1}\tau(k_1^{-1})v_j) \\
&= k_2^{-1}.V(x_0, g^{-1}.x_0, g^{-1}\tau(k_1^{-1})v_j) \\
&= \tau(k_2^{-1})V(x_0, g^{-1}.x_0, g^{-1}\tau(k_1^{-1})v_j) \\
&= \tau(k_2^{-1})\varphi(g^{-1}.x_0)[g^{-1}\tau(k_1^{-1})v_j] \\
&= \tau(k_2^{-1})g^{-1}\varphi(g^{-1}.x_0)[\tau(k_1^{-1})v_j] \\
&= \tau(k_2^{-1})\widetilde{\varphi}(g)[\tau(k_1^{-1})v_j] \\
&= \tau(k_2^{-1})\widetilde{\varphi}(g)\tau(k_1^{-1})[v_j]
\end{aligned}$$

Thus we can define the map: $\Delta : T \mapsto \widetilde{\varphi}$ from $Hom_G(\Gamma(E^\tau), \Gamma(E^\tau))$ to $L(G, K, \tau, \tau)$.

This map is naturally injective. Let us show that it is a morphism of algebras. In fact, let $T \equiv \{V(x, y, v_j)\}$, $S \equiv \{R(x, y, v_j)\}$ and $TS \equiv \{W(x, y, v_j)\}$. Let put $\Delta(T) = \widetilde{\varphi}$, $\Delta(S) = \widetilde{\phi}$ and $\Delta(TS) = \widetilde{\Theta}$. Let $g \in G$ and $v_i \in V_\tau$, one has

$$\begin{aligned}
\widetilde{\Theta}(g)[v_i] &= g^{-1}.\Theta(g^{-1}x_0)[v_i] \\
&= W(x_0, g^{-1}.x_0, g^{-1}.v_i) \\
&= [TS](\delta_{g^{-1}.x_0, g^{-1}.v_i})(x_0) \\
&= T[S(\delta_{g^{-1}.x_0, g^{-1}.v_i})](x_0) \\
&= \sum_{1 \le j \le d_\tau, y \in X} \langle S(\delta_{g^{-1}.x_0, g^{-1}.v_i})(y), v_j\rangle V(x_0, y, v_j) \\
&= \sum_{1 \le j \le d_\tau, y \in X} \langle R(y, g^{-1}.x_0, g^{-1}.v_i), v_j\rangle V(x_0, y, v_j) \\
&= \sum_{1 \le j \le d_\tau, t \in G} \langle R(t.x_0, g^{-1}.x_0, g^{-1}.v_i), v_j\rangle V(x_0, t.x_0, v_j) \\
&= \sum_{1 \le j \le d_\tau, t \in G} \langle t.R(x_0, t^{-1}g^{-1}.x_0, t^{-1}g^{-1}.v_i), v_j\rangle V(x_0, t.x_0, v_j)
\end{aligned}$$

$$\begin{aligned}\widetilde{\Theta}(g)[v_i] &= \sum_{1\le j\le d_\tau, t\in G} \langle R(x_0, t^{-1}g^{-1}.x_0, t^{-1}g^{-1}.v_i), v_j\rangle V(x_0, t.x_0, t.v_j)\\ &= \sum_{1\le j\le d_\tau, t\in G} \langle \phi(t^{-1}g^{-1}.x_0)[t^{-1}g^{-1}.v_i], v_j\rangle \varphi(t.x_0)[t.v_j]\\ &= \sum_{1\le j\le d_\tau, t\in G} \langle \widetilde{\phi}(tg).[v_i], v_j\rangle \widetilde{\varphi}(t^{-1}).[v_j]\\ &= \sum_{t\in G}(\sum_{1\le j\le d_\tau} \langle \widetilde{\phi}(t)[v_i], v_j\rangle \widetilde{\varphi}(t^{-1}g).[v_j])\\ &= \sum_{t\in G} \widetilde{\varphi}(t^{-1}g)[\widetilde{\phi}(t)[v_i]]\end{aligned}$$

So $\widetilde{\Theta}(g)[v_i] = \widetilde{\varphi} * \widetilde{\phi}(g)[v_i]$ that is $\Delta(TS) = \Delta(T) * \Delta(S)$.

$(iii) \Rightarrow (ii)$ Suppose now that λ^τ is multiplicity-free, that is $\Gamma(E^\tau) = \bigoplus_{i=0}^N S_i$ with S_i inequivalent irreducible representations $\forall i \in \{0, 1, ..., N\}$. Let $T \in Hom_G(\Gamma(E^\tau), \Gamma(E^\tau))$ and denote by $T_i = T_{|S_i}$ the restriction of T to S_i. Since $KerT_i$ is a G-invariant subspace of S_i and S_i is irreducible then $KerT_i = S_i$ either $KerT_i = \{0\}$. Hence if T_i is not trivial, then S_i and $ImT_i = \{T_i v, v \in S_i\}$ are isomorphic. Since $ImT_i = \{T_i v, v \in S_i\}$ is a G-invariant subspace and $\Gamma(E^\tau)$ is multiplicity-free then $S_i = ImT_i$. By Schur lemma there exists $\lambda_i \in \mathbb{C}$ such that $T_i s = \lambda_i s \ \forall\ s \in S_i$. Hence for any $T \in Hom_G(\Gamma(E^\tau), \Gamma(E^\tau))$, there exists $(\lambda_0, \lambda_1, ..., \lambda_d) \in \mathbb{C}^{N+1}$ such that $Ts = \sum_{k=0}^N \lambda_k s_k$. If $Q \in Hom_G(\Gamma(E^\tau), \Gamma(E^\tau))$ and $(\beta_0, \beta_1, ..., \beta_d) \in \mathbb{C}^{N+1}$ such that $Qs = \sum_{k=0}^N \beta_k s_k$ then:

$$TQs = T[\sum_{k=0}^N \beta_k s_k] = \sum_{k=0}^N \beta_k T(s_k) = \sum_{k=0}^N \lambda_k \beta_k s_k = \sum_{k=0}^N \lambda_k Q(s_k) = QTs.$$

So $Hom_G(\Gamma(E^\tau), \Gamma(E^\tau))$ is commutative.

$(ii) \Rightarrow (iii)$ Suppose that $\Gamma(E^\tau)$ is not multiplicity-free. Thus there exists two G-modules irreducible S_i and S_j which are equivalent. Let $L : S_i \longrightarrow S_j$ be an intertwiner operator.

$$\Gamma(E^\tau) = S_i \oplus S_j \oplus V,$$

where V is an orthogonal complement of $S_i \oplus S_j$. We define two linear operators T_i and T_j on $\Gamma(E^\tau)$ by setting $T_i : s = s_i + s_j + v \longmapsto Ls_i$ and $T_j : s = s_i + s_j + v \longmapsto L^{-1}s_j$. We have

$T_i, T_j \in Hom_G(\Gamma(E^\tau), \Gamma(E^\tau))$. In fact, $\forall\ s \in \Gamma(E^\tau), \forall\ x \in X,\ \forall\ g \in G$,

$$\begin{aligned}\lambda^\tau(g)[T_i]s(x) &= g.[T_i s](g^{-1}.x) = g.[T_i(s_i + s_j + v)](g^{-1}.x)\\ &= g.[Ls_i](g^{-1}.x) = L[g.s_i](g^{-1}.x) = L[\lambda^\tau(g)s_i](x)\\ &= T_i[\lambda^\tau(g)s](x)\end{aligned}$$

and

$$\begin{aligned}\lambda^\tau(g)[T_j]s(x) &= g.[Qs](g^{-1}x) = g.[T_j(s_i + s_j + v)](g^{-1}.x)\\ &= g.[L^{-1}s_j](g^{-1}.x)\\ &= L^{-1}[g.s_j](g^{-1}.x) = L^{-1}[\lambda^\tau(g)s_j](x)\\ &= T_j[\lambda^\tau(g)]s(x)\end{aligned}$$

Now for $s_j \in S_j$, we have $[T_iT_j](s_j) = T_i(L^{-1}s_j) = LL^{-1}s_j = s_j$ and $[QT](s_j) = Q(L(0)) = Q(0) = 0$. So $Hom_G(\Gamma(E^\tau), \Gamma(E^\tau))$ is not commutative. □

From the previous proof, we deduce the following results.

Corollary 3.4. *Let (G, K, τ) be a commutative triple and $\Gamma(E^\tau) = \bigoplus_{i=0}^{N} S_i$ be the decomposition of $\Gamma(E^\tau)$ into irreducible inequivalent subrepresentations. Then*

(i) *If $T \in Hom_G(\Gamma(E^\tau), \Gamma(E^\tau))$, then any S_i is an eigenspace of T.*

(ii) *If $T \in Hom_G(\Gamma(E^\tau), \Gamma(E^\tau))$ and λ_i is the eigenvalue of the restriction $T_{|S_i}$, then the map*

$$T \longmapsto (\lambda_0, \lambda_1, ..., \lambda_N)$$

is an isomorphism between $Hom_G(\Gamma(E^\tau), \Gamma(E^\tau))$ and $\mathbb{C}^{N+1}$.

(iii) $N + 1 = dim(L(G, K, \tau, \tau)) = dim(Hom_G(\Gamma(E^\tau), \Gamma(E^\tau)))$.

An other consequence of the previous theorem is this necessary condition.

Corollary 3.5. *If (G, K, τ) is a commutative triple then (G, K) is a Gelfand pair.*

Proof. As mentioned in the preliminaries, $\Gamma(E^\tau)$ is isomorphic to $L(X, V_\tau)$. Since $L(X, V_\tau)$ and $L(X) \otimes V_\tau$ are isomorphic then $\Gamma(E^\tau)$ is isomorphic to $L(X) \otimes V_\tau$. The generalized permutation representation λ^τ is equivalent to the G-representation on $L(X) \otimes V_\tau$ defined by $\gamma^\tau(g)(f \otimes v) = \lambda(g)f \otimes g.v$, where λ is the permutation representation defined by $\lambda(g)f(x) = f(g^{-1}.x)$. In fact, if we consider the isomorphism $S : L(X) \otimes V_\tau \to \Gamma(E^\tau)$ defined by: $S(f \otimes v)(x) = [x, \rho(g_0^{-1})f(x)v]$, where $x = g_0.x_0$, we have :

$$\begin{aligned}
\lambda^\tau(g)(S(f \otimes v))(x) &= g.S(f \otimes v)(g^{-1}.x) \\
&= g.[g^{-1}.x, \rho(g_0^{-1}g)f(g^{-1}.x)v] \\
&= [x, \rho(g_0^{-1}g)f(g^{-1}.x)v] \\
&= [x, \rho(g_0^{-1})\lambda(g)f(x)\rho(g)v] \\
&= S(\lambda(g)f \otimes g.v)(x) = S(\gamma^\tau(g)(f \otimes v)(x)
\end{aligned}$$

So, it follows that the intertwiner operators algebras $Hom_G(\Gamma(E^\tau), \Gamma(E^\tau))$ and $Hom_G(L(X) \otimes V_\tau, L(X) \otimes V_\tau)$ are isomorphic. Now let us consider $T \in Hom_G(L(X), L(X))$ where $Hom_G(L(X), L(X))$ is the algebra of intertwiner operators of the permutation representation λ. Denoting by Id_{V_τ} the identity operator on V_τ, we show that $T \otimes Id_{V_\tau}$ is an element of $Hom_G(L(X) \otimes V_\tau, L(X) \otimes V_\tau)$. In fact,

$$\begin{aligned}
\gamma^\tau(g)(T \otimes Id_{V_\tau})(f \otimes v)(x) &= \gamma^\tau(g)(Tf \otimes v)(x) \\
&= \lambda(g)Tf(x) \otimes g.v \\
&= T\lambda(g)f \otimes g.v \\
&= (T \otimes Id_{V_\tau}))(\lambda(g)f \otimes g.v)(x) \\
&= (T \otimes Id_{V_\tau}))\gamma^\tau(g)(f \otimes v)(x).
\end{aligned}$$

Since (G, K, τ) is a commutative triple then according to Theorem 3.2, $Hom_G(\Gamma(E^\tau), \Gamma(E^\tau))$ is commutative and so is $Hom_G(L(X) \otimes V_\tau, L(X) \otimes V_\tau)$. Thus for $T, T' \in Hom_G(L(X), L(X))$ we have $(T \otimes Id_{V_\tau})o(T' \otimes Id_{V_\tau}) = (T' \otimes Id_{V_\tau})o(T \otimes Id_{V_\tau})$. So $TT' = T'T$ that is $Hom_G(L(X), L(X))$ is commutative. Thanks to Theorem 4.4.2 in ([3] page 125), (G, K) is a Gelfand pair. □

We denote by χ^τ the generalized permutation character that is the character associated with the generalized permutation representation λ^τ. We know that if $\mathcal{B} = \{v_i : 1 \le i \le d_\tau\}$ is an orthonormal basis of V_τ then the set $\{\delta_{x,v_i} : x \in X, v_i \in \mathcal{B}\}$ is an orthonormal basis of $\Gamma(E^\tau)$. For $g \in G$, the Frobenius fixed point character formula [12] gives

$$\chi^\tau(g) = \sum_{x \in X} \sum_{v_i \in \mathcal{B}} \langle \lambda^\tau(g)\delta_{x,v_i}, \delta_{x,v_i} \rangle = |\{(x, v_i) \in X \times \mathcal{B}, g.(x, v_i) = (x, v_i)\}|.$$

and it is straightforward to verify Burnside Lemma [3] that is

$$\frac{1}{|G|}\sum_{g\in G}\chi^{\tau}(g) = \text{number of G orbits in } E^{\tau}$$

Since G acts on G/K transitively, then it is also straightforward to prove that there exists a bijection between the set of orbits of G on $E^{\tau}\times E^{\tau}$ and those of K on E^{τ} and obtain the Wielandt lemma [3] which is cornerstone for the proof of the following result.

Theorem 3.6. *Let G be a finite group, K be a subgroup of G, $\tau\in\widehat{K}$ and denote by $X = G/K$ the corresponding homogeneous space. Suppose we have a decomposition $\Gamma(E^{\tau}) = \bigoplus_{k=0}^{N} S_k$ into pairwise inequivalent G- subrepresentations with $N+1$ =the number of K-orbits on X. Then the S_k are irreducible $\forall\ k\in\{0,1,...,N\}$ and (G,K,τ) is a commutative triple.*

Proof. We consider if necessary another decomposition of $\Gamma(E^{\tau})$ into irreducibles that is $\Gamma(E^{\tau}) = \bigoplus_{k=0}^{d} m_k S_k$ where m_k is the multiplicity of S_k in $\Gamma(E^{\tau})$ and the S_k are irreducible G-subrepresentations. It is clear that $N\le d$ and we have, using Wielandt's lemma, that:
$N+1\le d+1\le\sum_{k=0}^{d} m_k\le\sum_{k=0}^{d} m_k^2 = N+1$. It comes that $d=N$ and $\sum_{k=0}^{d} m_k = N+1$. So $\sum_{k=0}^{d} m_k = d+1$. Since $m_k\ge 1$ for all $k\in\{1,2,...,d\}$, we have $m_k = 1$ for all k. □

Theorem 3.7. *Suppose that K and K' are two subgroups conjugate in G and τ (resp. τ') an irreducible representation of K (resp. of K') such that $V_{\tau} = V_{\tau'}$. Then (G,K,τ) is a commutative triple if and only if (G,K',τ') is a commutative triple.*

Proof. Since K and K' are two subgroups conjugate in G, there exists a G-spaces isomorphism ϕ from $X = G/K$ onto $X' = G/K'$ (See [3]; page 81). Let us consider the map $\beta:\Gamma(E^{\tau})\longrightarrow\Gamma(E^{\tau'})$ defined by:

$$\beta(s)(x) = s\circ\phi^{-1}(x)\ \forall\ s\in\Gamma(E^{\tau}),\ x\in X'.$$

β is an isomorphism and intertwines $\lambda^{\tau'}$ and λ^{τ}.

In fact, first it is clear that β is linear. Then, $\forall\ s \in \Gamma(E^{\tau}), \forall\ x \in X' and\ \forall\ g \in G$,

$$\begin{aligned}
\lambda^{\tau'}(g) \circ \beta(s)(x) &= g.\beta(s)(g^{-1}x) \\
&= g.[(s \circ \phi^{-1})(g^{-1}x)] \\
&= g.s[\phi^{-1}(g^{-1}x)] \\
&= g.s(g^{-1}\phi^{-1}(x)) \\
&= \lambda^{\tau}(g)s(\phi^{-1}(x)) \\
&= [\lambda^{\tau}(g)s](\phi^{-1}(x)) \\
&= \beta \circ \lambda^{\tau}(g)s(x)
\end{aligned}$$

and we have $\lambda^{\tau'}(g) \circ \beta = \beta \circ \lambda^{\tau}(g)$. Finally, $\forall\ s_1, s_2 \in S(E_{\tau}), \forall\ x \in X'$ $\beta(s_1)(x) = \beta(s_2)(x) \Rightarrow s_1 \circ \phi^{-1}(x) = s_2 \circ \phi^{-1}(x)$ hence $s_1 = s_2$ since ϕ is bijective. β induces an isomorphism $\Phi : Hom_G(\Gamma(E^{\tau}), \Gamma(E^{\tau})) \to Hom_G(\Gamma(E^{\tau'}), \Gamma(E^{\tau'}))$ given by: for $T \in Hom_G(\Gamma(E^{\tau}), \Gamma(E^{\tau}))$,

$$\Phi(T) = \beta \circ T \circ \beta^{-1}.$$

We have $\forall\ T \in Hom_G(\Gamma(E^{\tau}), \Gamma(E^{\tau}))$

$$\begin{aligned}
\lambda^{\tau'}(g) \circ \Phi(T) &= \lambda^{\tau'}(g) \circ \beta \circ T \circ \beta^{-1} \\
&= \beta \circ \lambda^{\tau}(g) \circ T \circ \beta^{-1} \\
&= \beta \circ T \circ \lambda^{\tau}(g) \circ \beta^{-1} \\
&= \beta \circ T \circ \beta^{-1} \circ \lambda^{\tau'}(g) \\
&= \Phi(T) \circ \lambda^{\tau'}(g)
\end{aligned}$$

Φ is clearly linear. Now let $T_1, T_2 \in Hom_G(\Gamma(E^{\tau}), \Gamma(E^{\tau}))$

$\Phi(T_1) = \Phi(T_2) \Rightarrow \beta \circ T_1 \circ \beta^{-1} = \beta \circ T_2 \circ \beta^{-1} \Rightarrow T_1 = T_1$

then Φ is an isomorphism. So using Theorem 3.2. above, we have done. □

Remark 3.8. The equivalence between (i) and (iii) in Theorem 3.2. implies, thanks to Frobenius reciprocity Theorem, that (G, K, τ) is commutative if and only if τ appears at most one times in any unitary irreducible representation of G.

We denote by $Res_B^A(U)$ the restriction to B of the representation U of A.

Theorem 3.9. *Let K, L be two subgroups of G such that $K \leq L$. Let τ and δ be irreducible representations respectively of K and L such that $m(\tau, Res_K^L(\delta)) = 1$. If (G, K, τ) is a commutative triple then so is (G, L, δ).*

Proof. Let U be a unitary irreducible representation of G. $Res_L^G(U)$ is written

$$Res_L^G(U) = \sum_{\rho \in \widehat{L}} m(\rho, Res_L^G(U))\rho$$

and

$$Res_K^L(Res_L^G(U)) = \sum_{\rho \in \widehat{L}} m(\rho, Res_L^G(U)) Res_K^L(\rho).$$

By the transitivity of the restriction, we have $Res_K^L(Res_L^G(U)) = Res_K^G(U)$ and

$$\begin{aligned} m(\tau, Res_K^G(U)) &= \sum_{\rho \in \widehat{L}} m(\rho, Res_L^G(U)) m(\tau, Res_K^L(\rho)) \\ &= m(\delta, Res_L^G(U)) m(\tau, Res_K^L(\delta)) + \\ &\sum_{\rho \in \widehat{L}, \rho \not\simeq \delta} m(\rho, Res_L^G(U)) m(\tau, Res_K^L(\rho)) \end{aligned}$$

But (G, K, τ) is a commutative triple, so thanks to above remark we have $m(\tau, Res_K^G(U)) \leq 1$. It follows from the last equality that
$m(\delta, Res_L^G(U)) m(\tau, Res_K^L(\delta)) \leq 1$. Now by hypothesis $m(\tau, Res_K^L(\delta)) = 1$ so we deduce that $m(\delta, Res_L^G(U)) \leq 1$. Consequently, thanks to above remark, (G, L, δ) is a commutative triple. □

Corollary 3.10. *Let K, L be two subgroups of G such that $K \leq L$. If (G, K) is a Gelfand pair then for all $\delta \in \widehat{L}$ containing the one dimensional trivial representation 1_K of K, (G, L, δ) is a commutative triple.*

Proof. : Since (G, K) is a Gelfand pair then (L, K) is a Gelfand pair [1]. So according to a classical property of Gelfand pair [5], we have $m(1_K, Res_K^L(\delta)) \leq 1$. But δ contains 1_K so $m(1_K, Res_K^L(\delta)) = 1$ and using the previous Theorem we obtain that (G, L, δ) is a commutative triple. □

We now give some examples to achieve the work. Let n be a positive integer. A composition of n is an ordered sequence of non negative integers $a = (a_1, a_2, ..., a_h)$ such that $a_1 + a_2 + ... + a_h = n$.

A partition λ of n is a composition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ of n such that $\lambda_1 \geq \lambda_2 \geq ...\lambda_k \geq 1$. We recall that there exists a canonical one to one correspondence between the set of all partitions of n and the set of all irreducible representations of the symmetric group S_n [11].
If λ is a partition of n, the irreducible representation canonically associated to λ is denoted by S^λ.

(1) Let μ be a partition of $n-1$. We know by the Branching rules [8,11] that :

$$Res^{S_n}_{S_{n-1}}(S^\lambda) = \bigoplus_{\lambda^-} S^{\lambda^-},$$

where λ^- is a inner corner of λ [11]. So we have :

$$m(S^\mu, Res^{S_n}_{S_{n-1}}(S^\lambda)) = \begin{cases} 1 & \text{, if } \mu \text{ is a inner corner of } \lambda \\ 0 & \text{, if otherwise} \end{cases}$$

So, it follows that (S_n, S_{n-1}, S^μ) is a commutative triple.

(2) Let a be a non negative integer such that $1 \leq a \leq n-1$. Let $\mu = (\mu_1, \mu_2, ..., \mu_h)$ be a partition of $n-a$. We say $\lambda \geq \mu$ if $h \leq k$ and $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \geq \lambda_h \geq \mu_h$.
According Littlewood-Richardson rule (see [8,11]) (it is a particular case) we have:

$$m(S^\mu \otimes S^{(a)}, Res^{S_n}_{S_{n-a}\times S_a}(S^\lambda)) = \begin{cases} 1 & \text{, if } \lambda \geq \mu \\ 0 & \text{, otherwise} \end{cases}.$$

So for any partition $\mu = (\mu_1, \mu_2, ..., \mu_h)$ of $n-a$, the triple
$(S_n, S_{n-a} \times S_a, S^\mu \otimes S^{(a)})$ is a commutative triple.

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