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SOME APPLICATIONS OF CAUCHY'S MEAN VALUE THEOREM

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Abstract. In this note we prove some applications of Cauchy's mean value theorem.

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1. Introduction

The main motivation of this note is the paper of A.Mingarelli et. al. in [4] where they proposed the following problem: given $\phi : [0,1] \to \mathbb{R}$ a continuous function, determine continuous functions $f, g : [0,1] \to \mathbb{R}$ for which there exists $c \in (0,1)$ such that

$$\int_0^1 f(x)dx \int_0^c \phi(x)g(x)dx = \int_0^1 g(x)dx \int_0^c \phi(x)f(x)dx. \tag{1.1}$$

In this note we discuss a similar problem to that of Mingarelli. Namely; given continuous functions $\phi, u, v : [0,1] :\to \mathbb{R}$ determine $\eta \in (0,1)$ such that

$$\int_0^1 u(x)dx \int_\eta^1 \phi(x)v(x)dx = \int_0^1 v(x)dx \int_\eta^1 \phi(x)u(x)dx \tag{1.2}$$

• If $\int_0^1 u(x)dx = \int_0^1 v(x)dx = 0$ then (1.2) has infinitely many solutions namely the whole interval $(0,1)$.

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- If $u(x) = 1$, $\phi(x) = 1 - x$, and $v(x) = (1-x)^2$, it may have exactly one solution $\eta \approx 0.18$.
- In the case, $u(x) = 1$, $\phi(x) = x$ and $v(x) = x^2$, it may have no solution $\eta \in (0, 1)$.

The main tool employed to solve our problem (1.2) is a variant of Cauchy's mean value theorem (see Theorem 1.3).

Now we start by stating some results that we will use in this work.

Theorem 1.1 (Cauchy's Mean Value Theorem)**.** *Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions on an interval $[a, b]$, differentiable on (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$. Then, there exists $\eta \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\eta)}{g'(\eta)}. \tag{1.3}$$

In 2000, E.Wachnicki proved the following variant of Cauchy's mean value theorem.

Theorem 1.2 (Wachnicki's Theorem)**.** *Let $f, g : [a, b] \to \mathbb{R}$ be differentiable functions on $[a, b]$. Suppose that $g'(x) \neq 0$ for all $x \in [a, b]$ and*

$$\frac{f'(a)}{g'(a)} = \frac{f'(b)}{g'(b)}. \tag{1.4}$$

Then, there exists $\eta \in (a, b)$ such that

$$\frac{f(\eta) - f(a)}{g(\eta) - g(a)} = \frac{f'(\eta)}{g'(\eta)}. \tag{1.5}$$

Proof. The details of proof can be see in [6, Theorem 1.3]. But for reader's convenience we put here. Consider the function $\mathcal{W} : [a, b] \to \mathbb{R}$ given by

$$\mathcal{W}(x) = \begin{cases} \frac{f(x)-f(a)}{g(x)-g(a)}, & \text{if } x \in (a, b] \\ \frac{f'(a)}{g'(a)}, & \text{if } x = a. \end{cases} \tag{1.6}$$

The function $\mathcal{W}$ is continuous in $[a, b]$, differentiable in $(a, b]$ and

$$\begin{aligned} \mathcal{W}'(x) &= \frac{\big[g(x) - g(a)\big] f'(x) - \big[f(x) - f(a)\big] g'(x)}{\big[g(x) - g(a)\big]^2} \\ &= \frac{f'(x) - \mathcal{W}(x) g'(x)}{\big[g(x) - g(a)\big]}, \quad x \in (a, b]. \end{aligned}$$

By Weierstrass theorem $\mathcal{W}$ attains its bounds on $[a, b]$.

If $\mathcal{W}$ does not attain its bounds simultaneously in a and b, then there exists $\eta \in (a,b)$ where $\mathcal{W}$ attains its bounds. By Fermat's theorem, we have $\mathcal{W}'(\eta)=0$, i.e.,

$$f'(\eta)-\mathcal{W}(\eta)g'(\eta)=0 \Leftrightarrow \mathcal{W}(\eta)=\frac{f'(\eta)}{g'(\eta)}$$
$$\Leftrightarrow \frac{f(\eta)-f(a)}{g(\eta)-g(a)}=\frac{f'(\eta)}{g'(\eta)}.$$

If $\mathcal{W}$ attains its bounds in a and b, then we have the following possibilities:

$$\mathcal{W}(a)\leqslant \mathcal{W}(x)\leqslant \mathcal{W}(b), \quad \forall x\in[a,b] \tag{1.7}$$

or

$$\mathcal{W}(b)\leqslant \mathcal{W}(x)\leqslant \mathcal{W}(a), \quad \forall x\in[a,b]. \tag{1.8}$$

Without loss of generality we can assume that (1.7) holds. Also we can assume that $g'(x)>0$ for all $x\in[a,b]$, i.e., g is strictly increasing $[a,b]$. Thus, $g(a)<g(x)<g(b)$, $\forall x\in[a,b]$.

Using the second inequality in (1.7) we get

$$f(x)\leqslant f(a)+\mathcal{W}(b)[g(x)-g(a)], \ \forall x\in[a,b]. \tag{1.9}$$

Then, for all $x\in[a,b]$ we have

$$f(b)-f(x)\geqslant f(b)-f(a)-\mathcal{W}(b)[g(x)-g(a)].$$

Thus,

$$\frac{f(b)-f(x)}{g(b)-g(x)}\geqslant \frac{f(b)-f(a)-\mathcal{W}(b)[g(x)-g(a)]}{g(b)-g(x)}=\frac{f(b)-f(a)}{g(b)-g(a)}=\mathcal{W}(b).$$

Passing to the limit as $x\to b^-$, we obtain

$$\frac{f'(b)}{g'(b)}=\lim_{x\to b^-}\frac{f(b)-f(x)}{g(b)-g(x)}\geqslant \mathcal{W}(b). \tag{1.10}$$

Using (1.4) in (1.10) we get

$$\mathcal{W}(a)=\frac{f'(a)}{g'(a)}=\frac{f'(b)}{g'(b)}=\lim_{x\to b^-}\frac{f(b)-f(x)}{g(b)-g(x)}\geqslant \mathcal{W}(b). \tag{1.11}$$

Finally using (1.11) in (1.7) we conclude that $\mathcal{W}$ is a constant and therefore $\mathcal{W}'=0$. □

The ideas contained in the proof of Wachnicki's theorem are of C. Lupu in [3, Lemma 2.1, p. 2].

In [1] we proved the following variant of Wachnicki's Theorem.

Theorem 1.3 ([1, Theorem 2.4]). *Let* $f, g : [a,b] \to \mathbb{R}$ *be differentiable on* $[a,b]$*. Suppose that* $g'(x) \neq 0$ *for all* $x \in [a,b]$ *and*

$$\frac{f'(a)}{g'(a)} = \frac{f'(b)}{g'(b)}. \tag{1.12}$$

Then, there exists $\eta \in (a,b)$ *such that*

$$\frac{f(b) - f(\eta)}{g(b) - g(\eta)} = \frac{f'(\eta)}{g'(\eta)}. \tag{1.13}$$

Proof. The proof follows the same reasoning of Wachnicki's Theorem using the function $\mathcal{G} : [a,b] \to \mathbb{R}$ given by

$$\mathcal{G}(x) = \begin{cases} \frac{f(b)-f(x)}{g(b)-g(x)}, & \text{if } x \in [a,b) \\ \frac{f'(b)}{g'(b)}, & \text{if } x = b. \end{cases}$$

□

2. Applications of Cauchy's Mean Value Theorem

In this section we prove the main results of this note. Let $L^2([a,b])$ the space of square Lebesgue integrable functions and let $T : L^2([a,b]) \to L^2([a,b])$ be the bounded linear operator given by

$$(Tf)(t) = \int_t^b f(x)dx.$$

Let $\phi \in C^1([a,b])$ with $\phi'(x) \neq 0$ for all $x \in (a,b)$. Consider $T : C([a,b]) \to C([a,b])$ given by

$$(T\Phi)(t) = \int_t^b \Phi(x)dx, \quad \Phi \in C([a,b]),$$

and similarly define $T_\phi : C([a,b]) \to C([a,b])$ given by

$$(T_\phi\Phi)(t) = \int_t^b \phi(x)\Phi(x)dx, \quad \Phi \in C([a,b]).$$

Let $\mathcal{H} := \left\{\phi \in C^1([a,b]) : \phi'(x) \neq 0, \forall x \in [a,b], \phi(b) = 0\right\}$ and $\mathcal{C}_{\text{null}}([a,b]) := \left\{f \in C([a,b]) : \int_a^b f(x)dx = 0\right\}$.

Using the Wachnicki's theorem, C.Lupu in [3] proved the following result.

Theorem 2.1. *([3, Theorem 2.2]) Let $u \in \mathcal{C}_{\text{null}}([a,b])$ and $v \in C^1([a,b])$ with $v'(x) \neq 0$ for all $x \in [a,b]$. Then there exists $\eta \in (a,b)$ such that*

$$T_v u(\eta) = v(a) \cdot Tu(\eta), \tag{2.1}$$

where $T_v u(t) = \int_a^t u(x)v(x)dx$.

Now, we are ready to prove the main results of this note

Theorem 2.2. *Let $u \in \mathcal{C}_{\text{null}}([a,b])$ and $v \in C^1([a,b])$ with $v'(x) \neq 0$ for all $x \in [a,b]$. Then there exists $\eta \in (a,b)$ such that*

$$T_v u(\eta) = v(b) \cdot Tu(\eta). \tag{2.2}$$

Proof. To prove (2.2) we need to prove the existence of $\eta \in (a,b)$ such that

$$\int_\eta^b u(x)v(x)dx = v(b)\int_\eta^b u(x)dx$$

Consider the functions $f, g : [a,b] \to \mathbb{R}$ given by

$$\begin{cases} f(t) = \int_t^b u(x)v(x)dx - v(t)\int_t^b u(x)dx \\ g(t) = v(t). \end{cases}$$

Now, it is easy to see that $f'(t) = -v'(t)\int_t^b u(x)dx$. Also $\dfrac{f'(a)}{g'(a)} = \dfrac{f'(b)}{g'(b)}$. Then, by Theorem 1.3 there exists $\eta \in (a,b)$ such that

$$\frac{f(b) - f(\eta)}{g(b) - g(\eta)} = \frac{f'(\eta)}{g'(\eta)},$$

which is equivalent to

$$\frac{-\int_\eta^b u(x)v(x)dx + v(\eta)\int_\eta^b u(x)dx}{v(b) - v(\eta)} = \frac{-v'(\eta)\int_\eta^b u(x)dx}{v'(\eta)},$$

and thus we have

$$-\int_\eta^b u(x)v(x)dx + v(\eta)\int_\eta^b u(x)dx = -v(b)\int_\eta^b u(x)dx + v(\eta)\int_\eta^b u(x)dx,$$

which proves (2.2). $\square$

Remark 2.1. *In the same setting as Theorem 2.2 the function f satisfy the $f'(a) = f'(b)$. Then, by Myers' theorem* ([5, Theorem 1′]) *there exists $\eta \in (a, b)$ such that*

$$f(b) - f(\eta) = f'(\eta)(b - \eta),$$

which is equivalent to

$$-\int_{\eta}^{b} u(x)v(x)dx + v(\eta)\int_{\eta}^{b} u(x)dx = -v'(\eta)(b-\eta)\int_{\eta}^{b} u(x)dx$$

$$\int_{\eta}^{b} u(x)v(x)dx = \Big[v(\eta) + v'(\eta)(b-\eta)\Big]\int_{\eta}^{b} u(x)dx,$$

and this can be written as

$$T_v u(\eta) = \big[v(\eta) + (b-\eta)v'(\eta)\big]\cdot Tu(\eta). \tag{2.3}$$

The following result follows immediately from Theorem 2.2

Corollary 2.1. *If $u \in \mathcal{C}_{\text{null}}([a, b])$ and $v \in H$, then there exists $\eta \in (a, b)$ such that*

$$\int_{\eta}^{b} u(x)v(x)dx = 0. \tag{2.4}$$

Remark 2.2. *The Corollary 2.1 tell us that the functions in $T_u(\mathcal{C}_{\text{null}}([a, b]))$ have at least one zero in* $(0, 1)$.

Theorem 2.3. *If $u, v : [0, 1] \to \mathbb{R}$ are continuous functions, then there exists $\eta \in (0, 1)$ such that*

$$T_\phi u(\eta)\int_0^1 v(x)dx - T_\phi v(\eta)\int_0^1 u(x)dx = \phi(1)\Big[Tu(\eta)\int_0^1 v(x)dx - Tv(\eta)\int_0^1 u(x)dx\Big]. \tag{2.5}$$

Proof. Consider the functions $f, g : [0, 1] \to \mathbb{R}$ given by

$$\begin{cases} f(t) = \Big[\phi(t)Tu(t) - T_\phi u(t)\Big]\int_0^1 v(x)dx - \Big[\phi(t)Tv(t) - T_\phi v(t)\Big]\int_0^1 u(x)dx \\ g(t) = \phi(t). \end{cases}$$

It is easy to see that $f'(t) = \phi'(t)\Big[Tu(t)\int_0^1 v(x)dx - Tv(t)\int_0^1 u(x)dx\Big]$. Also $\dfrac{f'(0)}{g'(0)} = \dfrac{f'(1)}{g'(1)}$. Then, by Theorem 1.3 there exists $\eta \in (0, 1)$ such that

$$\frac{f(1) - f(\eta)}{g(1) - g(\eta)} = \frac{f'(\eta)}{g'(\eta)},$$

which is equivalent to

$$\frac{-\Big[\phi(\eta)Tu(\eta)-T_\phi u(\eta)\Big]\int\limits_0^1 v(x)dx+\Big[\phi(\eta)Tv(\eta)-T_\phi v(\eta)\Big]\int\limits_0^1 u(x)dx}{\phi(1)-\phi(\eta)}$$

$$=\frac{\phi'(\eta)\Big[Tu(\eta)\int\limits_0^1 v(x)dx-Tv(\eta)\int\limits_0^1 u(x)dx\Big]}{\phi'(\eta)}.$$

Thus

$$-\Big[\phi(\eta)Tu(\eta)-T_\phi u(\eta)\Big]\int\limits_0^1 v(x)dx+\Big[\phi(\eta)Tv(\eta)-T_\phi v(\eta)\Big]\int\limits_0^1 u(x)dx$$

$$=\Big[\phi(1)-\phi(\eta)\Big]\Big[Tu(\eta)\int\limits_0^1 v(x)dx-Tv(\eta)\int\limits_0^1 u(x)dx\Big].$$

From the last equality we get (2.5). □

Corollary 2.2. *If $\phi(1)=0$, then there exists $\eta\in(0,1)$ such that*

$$\int\limits_0^1 u(x)dx\cdot T_\phi v(\eta)=\int\limits_0^1 v(x)dx\cdot T_\phi u(\eta).$$

This corollary follows immediately from Theorem 2.3.

Corollary 2.3. *If $u,v:[0,1]\to\mathbb{R}$ are continuous functions, then there exists $\eta\in(0,1)$ such that*

$$\int\limits_0^1 u(x)dx\int\limits_\eta^1(1-x)v(x)dx=\int\limits_0^1 v(x)dx\int\limits_\eta^1(1-x)u(x)dx. \tag{2.6}$$

Proof. The proof follows by applying the Corollary 2.2 with $\phi(x)=1-x$. □

Remark 2.3. *The equation (2.6) in the Corollary 2.3 is a slight variation of a problem proposed by C.Lupu and T.Lupu ([2]).*

Remark 2.4. *In the proof of Therorem 2.2 we considered the auxiliary function $f:[0,1]\to\mathbb{R}$ given by*

$$f(t)=\int\limits_t^1 u(x)v(x)dx-v(t)\int\limits_t^1 u(x)dx.$$

Its derivative is given by $f'(t)=-v'(t)\int\limits_t^1 u(x)dx$. If $u\in\mathcal{C}_{\text{null}}([0,1])$, then $f'(0)=0=f'(1)$. By applying Myers' theorem ([5, Theorem 1′]) there exists $\eta\in(0,1)$ such that $f(1)-f(\eta)=(1-\eta)f'(\eta)$, *which is equivalent to*

$$\int_{\eta}^{1} u(x)v(x)dx = \Big[v(\eta) + (1-\eta)v'(\eta)\Big] \int_{\eta}^{1} u(x)dx.$$

This result is completely different from Therorem 2.2.

On the other hand, if $u \in C([0,1])$ is such that

$$\int_{0}^{1} u(x)v(x)dx = v(0) \int_{0}^{1} u(x)dx,$$

then $f(0) = 0 = f(1)$. By Rolle's theorem there exists $\eta \in (0,1)$ such that $f'(\eta) = 0$, i.e.,

$$\int_{\eta}^{1} u(x)dx = 0.$$

The following remarks are consequences of Theorem 2.2 on the interval $[0,1]$.

Remark 2.5. *If we replace the functions u, v by their p-powers u^p, v^p, $1 < p < \infty$, in Theorem 2.3 we have*

$$\int_{\eta}^{1} \phi(x)u^p(x)dx \int_{0}^{1} v^p(x)dx - \int_{\eta}^{1} \phi(x)v^p(x)dx \int_{0}^{1} u^p(x)dx$$
$$= \phi(1) \left[\int_{\eta}^{1} u^p(x)dx \int_{0}^{1} v^p(x)dx - \int_{\eta}^{1} v^p(x)dx \int_{0}^{1} u^p(x)dx \right].$$

This equality says that

$$\|u\|^p_{L^p_\phi(\eta,1)} \|v\|^p_{L^p(0,1)} - \|v\|^p_{L^p_\phi(\eta,1)} \|u\|^p_{L^p(0,1)}$$
$$= \phi(1) \Big[\|u\|^p_{L^p(\eta,1)} \|v\|^p_{L^p(0,1)} - \|v\|^p_{L^p(\eta,1)} \|u\|^p_{L^p(0,1)} \Big]$$

where the quantities are the norm in their respective spaces of (weighted) p-integrable functions. Moreover, if $\phi(1) = 0$ we get a slight modification of Example 3 in [4], i.e.,

$$\frac{\|u\|_{L^p_\phi(\eta,1)}}{\|v\|_{L^p_\phi(\eta,1)}} = \frac{\|u\|_{L^p(0,1)}}{\|v\|_{L^p(0,1)}}.$$

In particular, if u, v have equal $L^p(0,1)$-norms, then, for given $\phi \in \mathcal{H}$ there exists $\eta \in (0,1)$ such that their norms in the weighted Lebesgue space $L^p_\phi(\eta,1)$ are also equal.

Remark 2.6. *For $i \neq j$ consider $u(x) = P_i(x)P_j(x)$ where P_i, P_j are orthogonal functions on $[0,1]$, i.e.,*

$$\int_0^1 P_i(x)P_j(x)dx = 0.$$

By Theorem 2.2 there exists $\eta_{ij} \in (0,1)$ such that

$$\int_{\eta_{ij}}^1 \phi(x)P_i(x)P_j(x)dx = \phi(1)\int_{\eta_{ij}}^1 P_i(x)P_j(x)dx.$$

If $\phi(1) = 0$, then we obtain $\int_{\eta_{ij}}^1 \phi(x)P_i(x)P_j(x)dx = 0$ which is similar to the Example 4 in [4].

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