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STEPANOV-LIKE PSEUDO ALMOST AUTOMORPHIC DYNAMICS OF QVRNNS WITH MIXED DELAYS ON TIME SCALES VIA A DIRECT METHOD

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ABSTRACT. In this paper, we are concerned with a class of quaternion-valued recurrent neutral neural networks (QVRNNs) with time-varying delays and infinite distributed delays on time scales. With the help of the Stepanov like pseudo almost automorphy on time scales, Banach's fixed point theorem, the theory of time scale calculations, and constructing a suitable Lyapunov functional, a set of sufficient criteria that guarantee the existence, uniqueness and the globally S^p-exponential stability of Stepanov-like pseudo almost automorphic solution on time scales of this class of neural networks are established via a direct method. In other words, we do not decompose the considered QVRNNs into real-valued systems or complex-valued systems. Finally, two numerical examples and simulations (for $\mathbb{T} = \mathbb{R}$ and $\mathbb{Z}$) are performed to verify our theoretical results.

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1. INTRODUCTION

The artificial neural network has been used in recent decades to simulate the structure and function of the biological neural network. As a consequence, the neural networks (NNs) proposed by Chua and Yang ([2]) have attracted growing attention because of their broad range of applications in, for example, function approximation, pattern recognition, associative memory, computing technology, nonlinear programming and combinatorial optimization ([22, 24, 27]). Recently, recurrent neural networks (RNNs) have become a good tool for the approximating dynamical systems and are capable

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of learning the characteristics and modeling sequential data. They can be trained to reproduce any target dynamics, up to a certain degree of accuracy. As a result they have become of increasing interest to the various scientific research communities ([22, 29, 35]). As is well known, the time delays that inevitably appear in both biological and artificial neural networks caused by the finite switching speed of neurons and amplifiers will affect the stability of neural networks. Moreover, the effect of time-delay on the dynamic behavior of systems has become a primary problem of delay, based on engineering technology and natural sciences ([13, 15]). Consequently, over the last decades, neural networks with different types of delay (discrete, distributed,...) and their dynamic characteristics have been widely studied ([3, 20, 23]).

On the one hand, the theory of time scales, which originated in the work of Setefan Hilger in his Ph.D. thesis in 1988 ([17]), has as its main goal the unification continuous analysis and discrete analysis. Therefore, the study of dynamic equations on time scales allows us to avoid producing separate results for differential equations on the one hand, and for difference equations on the other hand ([5, 11, 19, 26]). Hence, when we choose $\mathbb{T} = \mathbb{R}$ (resp. $\mathbb{T} = \mathbb{Z}$), we get the result corresponding to the differential equations (resp. difference equations). Besides, since there are other time scales, not just the set of real numbers and the set of integers numbers, we have a much more general result. Furthermore, discrete recurrent neural networks are more practical for computation and numerical simulation than continuous recurrent neural networks. Then, it is useful to study the dynamics of neural networks on time scales([10, 32]).

On the other hand, almost automorphy is a very important and powerful dynamic behavior of neural networks that have been extensively investigated by a number of researchers ([8, 9, 34, 36, 37]). The notion of Stepanov-like pseudo almost automorphy (S^pPAA) on time scales, which is a natural generalization of the concepts of pseudo almost periodic, pseudo almost automorphic, Stepanov-like almost automorphic and Stepanov-like pseudo almost periodic and much more general and plays a very important role in better understanding the almost periodicity, was introduced in the literature by M. Es-saiydy an M. Zitane ([12]).

Moreover, in 1840, the Irish mathematician Hamilton ([18]) invented the quaternion algebra, it includes real numbers and complex numbers. Quaternion algebra has been widely used in a variety of fields, including theoretical physics, modern mathematics, geometry, NNs, image processing and robotics ([16, 21, 25]). Another practical application of applying quaternion is that it can exploit and treat 3-or 4-dimensional vectors in a single entity, which facilitates computation in 3-or 4-dimensional situations, so that the efficient processing of information can be carried out through quaternionic variable operations. Nevertheless, because the multiplication of quaternion numbers does not satisfy the commutative law, the research of quaternion-valued NNs (QVNNs) has had great difficulties. In addition, most of the existing results are obtained by decomposing QVNNs into real-VNNs ([7, 33]) or a complex-VNNs ([14, 28]). As a result, current results on QVNNs dynamics through direct methods are still very rare.

To the best of our knowledge, no such work has been done on the S^pPAA solution and S^p-globally exponential stability of real-valued RNNs, complex-valued RNNs or quaternion-valued RNNs on time scales. It is therefore a difficult and important problem in theories and applications. Motivated by the above analysis and discussion, the main purpose of this paper is to study the existence and S^p-global exponential stability of S^pPAA solution of quaternion-valued recurrent neural networks (QVRNNs) with mixed time-varying delays on time scales by direct methods. Further, our methods proposed in this paper can be used to study the problem of Stepanov-like almost periodic solution and Stepanov-like almost automorphic solution for other types of discrete- or continuous- QVNNs such as quaternion-valued Hopfield NNs, Cohen-Grossberg NNs and quaternion-valued BAM.

This paper is organized as follows: In section 2, the QVRNNs with mixed time-varying delays on time scales are presented and we introduce some necessary definitions and lemmas which are needed in later sections. In section 3, we establish some sufficient conditions for the existence and S^p-global exponential stability of S^pPAA solution for QVRNNs with mixed time-varying delays on time scales. In section 4, two numerical examples and simulations are given to demonstrate the feasibility of our theoretical results.

2. Model description and Preliminaries

In this section, we shall first recall some fundamental definitions, lemmas which are used in what follows. Throughout this paper we fix $p \geq 1$ and $(X, \|.\|)$ is a Banach space. We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ the set of positive integers, the set of integers, the set of real, the set of complex numbers and the algebra of quaternion respectively. The skew field of the quaternion is determined by $\mathbb{H} := \{x; x = x^R + x^I i + x^J j + x^K k\}$, where x^R, x^I, x^J and x^K are real numbers and the elements i, j, and k obey the Hamilton's multiplication rules: $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$, $i^2 = j^2 = k^2 = -1$. Moreover, the quaternion conjugate is defined as $\bar{x} = x^R - x^I i - x^J j - x^K k$, and the norm of x is defined by $|x|_{\mathbb{H}} = \sqrt{x\bar{x}}$.

In this paper, we consider the following quaternion-valued recurrent neural networks (QVRNNs) with mixed time-varying delays on time scales:

$$
\begin{aligned}
x_l^{\Delta}(t) &= -a_l(t)x_l(t)+\sum_{m=1}^{n} b_{lm}(t)f_m(x_m(t))+\sum_{m=1}^{n} c_{lm}(t)g_m(x_m(t-\xi_m(t)))\\
&\quad+\sum_{m=1}^{n} d_{lm}(t)\int_{-\infty}^{t} N_{lm}(t-z)h_m(x_m(z))\Delta z+I_l(t), \quad t\in\mathbb{T}.
\end{aligned}
\tag{2.1}
$$

Where $l\in\{1,2,\dots,n\}$, n corresponds to the number of units in neural networks, $\mathbb{T}$ is an almost periodic time scale; $\mathbb{H}$ is a Quaternion algebra; $x_l(t)\in\mathbb{H}$ corresponds to the state of the lth unit at time t, $a_l(t)=\mathrm{diag}\{a_i(t)\}_{i=1}^{i=n}$ represents the rate with which the ith neuron will reset its potential to the resting state in isolation when they are disconnected from the network and the external inputs at time t, f_m, g_m and $h_m:\mathbb{H}\to\mathbb{H}$ are output transfer functions, $b_{lm}(.)$, $c_{lm}(.)$, $d_{lm}(.)$ present the connection weights, the discretely delayed connection weights, and the distributively delayed connection weights, of the mth neuron on the l neuron, respectively. $\xi_m(.)$ corresponds to transmission delays at time t and satisfy $t-\xi_m(t)\in\mathbb{T}$ for $t\in\mathbb{T}$, N_{lm} is the delay kernel function, $I_l(.)$ denote the state bias of the lth neuron. The initial condition of system (2.1) is of the form

$$x_l(s)=\psi_l(s), \quad s\in(-\infty,0]_{\mathbb{T}},$$

where ψ_l is rd-continuous and $\psi_l\in L^p_{loc}((-\infty,0]_{\mathbb{T}},\mathbb{H})$ $l=1,\dots,n$.

Remark 2.1. 1) *If $\mathbb{T}=\mathbb{Z}$, Then sys.(2.1) can be transformed into the form below :*

$$
\begin{aligned}
x_l(k+1)-x_l(k) &= -a_l(k)x_l(k)+\sum_{m=1}^{n} b_{lm}(k)f_m(x_m(k))\\
&\quad+\sum_{m=1}^{n} c_{lm}(k)g_m(x_m(k-\xi_m(k)))\\
&\quad+\sum_{m=1}^{n} d_{lm}(k)\int_{-\infty}^{k} N_{lm}(k-z)h_m(x_m(k))dz+I_l(k), \quad k\in\mathbb{Z}.
\end{aligned}
$$

2) *If $\mathbb{T}=\mathbb{R}$, Then sys.(2.1) can be transformed into the form below :*

$$
\begin{aligned}
x_l'(t) &= -a_l(t)x_l(t)+\sum_{m=1}^{n} b_{lm}(t)f_m(x_m(t))+\sum_{m=1}^{n} c_{lm}(t)g_m(x_m(t-\xi_m(t)))\\
&\quad+\sum_{m=1}^{n} d_{lm}(t)\int_{-\infty}^{t} N_{lm}(t-z)h_m(x_m(z))dz+I_l(t), \quad t\in\mathbb{R}.
\end{aligned}
$$

2.1. **Time scales.**

Definition 2.1 ([6]). An arbitrary nonempty closed subset $\mathbb{T}$ of the set of real numbers $\mathbb{R}$ is called a time scale. The forward and backward jump operators $\sigma,\psi:\mathbb{T}\longrightarrow\mathbb{T}$ and the graininess $\mu:\mathbb{T}\longrightarrow\mathbb{R}^+$

are defined, respectively, by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, $\psi(t) = \sup\{s \in \mathbb{T} : s < t\}$, $\mu(t) = \sigma(t) - t$. A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\psi(t) = t$, left-scattered if $\psi(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called right-dense continuous or rd-continuous provided that it is continuous at all right-dense points in $\mathbb{T}$ and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called continuous if and only if it is both left-dense continuous and right-dense continuous. A function $p : \mathbb{T} \longrightarrow \mathbb{R}$ is called regressive provided if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T} \setminus \max(\mathbb{T})$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \longrightarrow \mathbb{R}$ will be denoted by $\mathfrak{R} = \mathfrak{R}(\mathbb{T}) = \mathfrak{R}(\mathbb{T};\mathbb{R})$. We define the set $\mathfrak{R}^+$ of all positively regressive elements by $\mathfrak{R}^+ = \mathfrak{R}^+(\mathbb{T}) = \mathfrak{R}^+(\mathbb{T};\mathbb{R}) = \{p \in \mathfrak{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$. Let $a, b \in \mathbb{T}$, with $a \leq b$, $[a,b]$, $[a,b)$, $(a,b]$, (a,b) being the usual intervals on the real line. The intervals $[a,a)$, $(a,a]$, (a,a) are understood as the empty set, and we use the following symbols :

$$[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T} \quad [a,b)_{\mathbb{T}} = [a,b) \cap \mathbb{T}$$

$$(a,b]_{\mathbb{T}} = (a,b] \cap \mathbb{T} \quad (a,b)_{\mathbb{T}} = (a,b) \cap \mathbb{T}.$$

Definition 2.2 ([6]). A time scale $\mathbb{T}$ is called invariant under translations if

$$\Pi = \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}; \forall t \in \mathbb{T}\} \neq \{0\}.$$

Definition 2.3 ([6]). For $f : \mathbb{T} \to X$ and $s \in \mathbb{T} \setminus \{\max \mathbb{T}\}$, $f^{\Delta}(t) \in X$ is the delta derivative of f at s if for $\varepsilon > 0$, there is a neighborhood V of s such that for $t \in V$,

$$\| f(\sigma(s)) - f(t) - f^{\Delta}(s)(\sigma(s) - t) \| < \varepsilon \mid \sigma(s) - t \mid .$$

Moreover, f is delta differentiable on $\mathbb{T}$ provided that $f^{\Delta}(s)$ exists for $s \in \mathbb{T}$.

Lemma 2.1 ([6]). Considering that f, g be delta differentiable functions on $\mathbb{T}$, then:

1) $(\lambda_1 f + \lambda_2 g)^{\Delta} = \lambda_1 f^{\Delta} + \lambda_2 g^{\Delta}$, for any constants λ_1, λ_2;

2)

$$\begin{aligned}(fg)^{\Delta}(t) &= (f)^{\Delta}(t)g(t) + f(\sigma(t))(g)^{\Delta}(t) \\ &= f(t)(g)^{\Delta}(t) + (f)^{\Delta}(t)g(\sigma(t));\end{aligned}$$

3) If f and f^{Δ} are continuous, then

$$\left(\int_a^t f(t,s)\Delta s\right)^{\Delta} = f(\sigma(t),t) + \int_a^t f(t,s)\Delta s.$$

Definition 2.4 ([6]). If $p \in \mathfrak{R}$, then we define the exponential function by :

$$\hat{e}_p(t,s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right\},$$

for $s,t \in \mathbb{T}$, with the cylinder transformation

$$\xi_m(z) = \begin{cases} \frac{\log(1+hz)}{h}, & if\, h \neq 0, \\ z, & if\, h = 0. \end{cases}$$

Definition 2.5 ([6]). If $p,q \in \mathfrak{R}$, then we define a circle plus addition by

$$(p \oplus q)(t) := p(t) + q(t) + p(t)q(t)\mu(t),$$

for all $t \in \mathbb{T} \setminus \max(\mathbb{T})$. For $p \in \mathfrak{R}$, define a circle minus p by

$$\ominus p = -\frac{p}{1+\mu p}.$$

Lemma 2.2 ([6]). Let $p,q \in \mathfrak{R}$, and $t,s,r \in \mathbb{T}$. Then,

1) $\hat{e}_0(t,s) = 1$ and $\hat{e}_p(t,t) = 1$;
2) $\hat{e}_p(\sigma(t),s) = (1+p(t)\mu(t))\hat{e}_p(t,s)$;
3) $\hat{e}_p(t,s) = \frac{1}{\hat{e}_p(s,t)} = \hat{e}_{\ominus p}(s,t)$;
4) $\hat{e}_p(t,r)\hat{e}_p(r,s) = \hat{e}_p(t,s)$;
5) $(\hat{e}_p(t,s))^\Delta = p(t)\hat{e}_p(t,s)$;
6) If $a,b,c \in \mathbb{T}$. Then,

$$\int_a^b \hat{e}_p(c,\sigma(t))p(t)\Delta t = \hat{e}_p(c,a) - \hat{e}_p(c,b).$$

7) For $t_0 \in \mathbb{T}$, $\hat{e}_{\ominus\lambda}(t_0,.)$ is increasing on $(-\infty,t_0]_{\mathbb{T}}$.

Lemma 2.3 ([6]). Assume $p \in \mathfrak{R}$, and $t_0 \in \mathbb{T}$. If $1+\mu(t)p(t) > 0$ for all $t \in \mathbb{T}$, then, $\hat{e}_p(t,t_0) > 0$ for all $t \in \mathbb{T}$.

Definition 2.6 ([1]). Let $F_1 = \{[t,s[_{\mathbb{T}} : t,s \in \mathbb{T}$ with $t \leq s\}$. Define a countably additive measure m_1 on F_1 by assigning to every $[t,s[_{\mathbb{T}} \in F_1$ its length, i.e;

$$m_1([t,s[_{\mathbb{T}}) = s - t.$$

Using m_1, we can generate the outer measure m_1^* on $P(\mathbb{T})$ (the power set of $\mathbb{T}$): for $E \in P(\mathbb{T})$,

$$m_1^*(E) = \begin{cases} \inf_{\mathfrak{B}}\{\sum_{i\in I_B}(s_m - t_m)\} \in \mathbb{R}^+, & \beta \notin E, \\ +\infty, & \beta \in E, \end{cases}$$

where $\beta = \sup \mathbb{T}$, and,

$$\mathfrak{B} = \{\{[t_m, s_m[_{\mathbb{T}} \in F_1\}_{i \in I_B} : I_B \subset \mathbb{N},\ E \subset \cup_{i \in I_B} [t_m, s_m[_{\mathbb{T}}\}.$$

A set $A \subset \mathbb{T}$ is called $\Delta-$measurable if for $E \subset \mathbb{T}$,

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (\mathbb{T} \setminus A)).$$

Let

$$\mathscr{M}^*(m_1^*) = \{A,\, A \,\text{is}\, \Delta - \text{measurable}\ \ \text{subset in}\, \mathbb{T}\}.$$

Restricting m_1^* to $\mathscr{M}^*(m_1^*)$, we get the Lebesgue $\Delta-$measure, which is denoted by μ_Δ.

Definition 2.7 ([1]). $f : \mathbb{T} \to X$ is a $\Delta-$measurable function if there exists a simple function sequence $\{f_k : k \in \mathbb{N}\}$ such that, $f_k(s) \to f(s)$ a.e. in $\mathbb{T}$.

Definition 2.8 ([1]). $f : \mathbb{T} \to X$ is a $\Delta-$integrable function if there exists a simple function sequence $\{f_k : k \in \mathbb{N}\}$ such that $f_k(s) \to f(s)$ a.e. in $\mathbb{T}$ and,

$$\lim_{k \to \infty} \int_{\mathbb{T}} \| f_k(s) - f(s) \| \Delta s = 0.$$

Then, the integral of f is defined as

$$\int_{\mathbb{T}} f(s) \Delta s = \lim_{k \to \infty} \int_{\mathbb{T}} f_k(s) \Delta s.$$

Definition 2.9 ([1]). For $p \geq 1$, $f : \mathbb{T} \to X$ is called locally L^p $\Delta-$integrable if f is $\Delta-$measurable and for any compact $\Delta-$measurable set $E \subset \mathbb{T}$, the $\Delta-$integral

$$\int_E \| f(s) \|^p \Delta s < \infty.$$

The set of all L^p $\Delta-$integrable functions is denoted by $L^p_{loc}(\mathbb{T}; X)$.

Theorem 2.1 ([4]). If $a, b \in \mathbb{T}$, with $a \leq b$, then,

1) $\mu_\Delta([a, b)) = b - a$,
2) $\mu_\Delta((a, b)) = b - \sigma(a)$.

Theorem 2.2 ([4]). If $a, b \in \mathbb{T} \setminus \{\max \mathbb{T}\}$, with $a \leq b$, then,

1) $\mu_\Delta((a, b]) = \sigma(b) - \sigma(a)$,
2) $\mu_\Delta([a, b]) = \sigma(b) - a$.

2.2. **Stepanov-like pseudo almost automorphic functions on $\mathbb{T}$.** This subsection is devoted to definitions, the important properties of Stepanov-like pseudo almost automorphic functions on time scales introduced by M. Es-saiydy and M. Zitane ([12]).

Definition 2.10 ([12]). We say that $f:\mathbb{T}\to\mathbb{H}$ is almost automorphic if from every sequence $\{s_n\}_{n=1}^{\infty}\subset\Pi$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that :

$$g(t)=\lim_{n\to\infty}f(t+\tau_n)$$

is well defined for each $t\in\mathbb{T}$ and

$$\lim_{n\to\infty}g(t-\tau_n)=f(t)$$

for each $t\in\mathbb{T}$. Denote by $AA(\mathbb{T},\mathbb{H})$ the set of all such functions.

Definition 2.11 ([12]). Let $t_0\in\mathbb{T}$ and $r\in\Pi$. A function $f\in BC(\mathbb{T},\mathbb{H})$, is said to be ergodic if

$$\lim_{r\to+\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\parallel f(s)\parallel_{\mathbb{H}}\Delta s=0,$$

The space of all such functions is denoted by $PAA_0(\mathbb{T},\mathbb{H})$.

Definition 2.12 ([12]). A functions $f\in BC(\mathbb{T},\mathbb{H})$ is called pseudo almost automorphic if $f=g+h$ where $g\in AA(\mathbb{T},\mathbb{H})$ and $h\in PAA_0(\mathbb{T},\mathbb{H})$.

We set,

$$K=\begin{cases}\inf\{\mid\tau\mid;\,\tau\in\mathbb{T},\,\tau\neq 0\}, & if\,\mathbb{T}\neq\mathbb{R},\\ 1, & if\,\mathbb{T}=\mathbb{R}.\end{cases}$$

Let $f\in L^p_{loc}(\mathbb{T},\mathbb{H})$, for $1\leq p<\infty$. Define :

- $\parallel.\parallel_{S^p}: L^p_{loc}(\mathbb{T},\mathbb{H})\to\mathbb{R}^+$ as : $\parallel f\parallel_{S^p}=\sup_{t\in\mathbb{T}}\left(\frac{1}{K}\int_t^{t+K}\mid f(s)\mid^p_{\mathbb{H}}\Delta s\right)^{\frac{1}{p}}$.
- $C_{rd}(\mathbb{T};\mathbb{H})=\{f:\mathbb{T}\to\mathbb{H}: f \text{ is rd-continuous}\}$.
- $BC_{rd}(\mathbb{T};\mathbb{H})=\{f:T\to\mathbb{H}: f \text{ is bounded and rd-continuous}\}$.
- $L^p_{loc}(\mathbb{T};\mathbb{H})=\{f:\mathbb{T}\to\mathbb{H}: f \text{ is locally } L^p\ \Delta-\text{integrable}\}$.
- $BS^p(\mathbb{T};\mathbb{H})=\left\{f\in L^p_{loc}(\mathbb{T};\mathbb{H}):\parallel f\parallel_{S^p}<\infty\right\}$.

Definition 2.13 ([12]). Let $f\in BS^p(\mathbb{T},\mathbb{H})$ and $F\in BS^p(\mathbb{T}\times\mathbb{H},\mathbb{H})$.

1) We say that $f:\mathbb{T}\to\mathbb{H}$ is Stepanov-like almost automorphic if for every sequence $\{s_n\}_{n=1}^{\infty}\subset\Pi$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$\parallel g(t)-f(t+\tau_n)\parallel_{S^p}\to 0,\quad as\quad n\to\infty,$$

is well defined for each $t \in \mathbb{T}$ and

$$\| g(t-\tau_n) - f(t) \|_{S^p} \to 0, \quad as \quad n \to \infty,$$

for each $t \in \mathbb{T}$. Denote by $S^pAA\,(\mathbb{T},\mathbb{H})$ the set of all such functions.

2) A function $F : \mathbb{T}\times\mathbb{H} \to \mathbb{H}$, $(t,x) \to F(t,x)$ is said to be Stepanov-like almost automorphic if $t \to F(t,x)$ is Stepanov almost automorphic in $t \in \mathbb{T}$ uniformly for each $x \in \mathbb{H}$. Denote by $S^pAA\,(\mathbb{T}\times\mathbb{H},\mathbb{H})$ the collection of such functions.

Definition 2.14 ([12]). A function $f \in BS^p(\mathbb{T},\mathbb{H})$ is said to be Stepanov-like ergodic on $\mathbb{T}$ if :

$$\lim_{r\to+\infty} \frac{1}{2r}\int_{t_0-r}^{t_0+r} \left(\frac{1}{K}\int_t^{t+K} | f(s) |_{\mathbb{H}}^p \Delta s\right)^{\frac{1}{p}} \Delta t = 0.$$

The space of all such functions will be denoted by $S^pPAA_0(\mathbb{T},\mathbb{H})$.

Definition 2.15 ([12]). A function $f \in BS^p(\mathbb{T},\mathbb{H})$ is said to be Stepanov-like pseudo almost automorphic on $\mathbb{T}$ or briefly S^p pseudo almost periodic if f is written in the following form :

$$f = g + \phi,$$

where $g \in S^pAP(\mathbb{T},\mathbb{H})$ and $\phi \in S^pPAA_0(\mathbb{T},\mathbb{H})$. The space of all such functions will be denoted by $S^pPAA(\mathbb{T},\mathbb{H})$.

Definition 2.16 ([12]). A function $f : \mathbb{T}\times\mathbb{H} \to \mathbb{H}$ such that $f(.,u) \in BS^p(\mathbb{T},\mathbb{H})$ for each $u \in \mathbb{H}$ is said to be Stepanov-like pseudo almost automorphic if f is written in the following form:

$$f = g + \phi,$$

where $g \in S^pAP(\mathbb{T}\times\mathbb{H},\mathbb{H})$ and $\phi \in S^pPAA_0(\mathbb{T}\times\mathbb{H},\mathbb{H})$. The space of all such functions will be denoted by $S^pPAA(\mathbb{T}\times\mathbb{H},\mathbb{H})$.

Next, we recall the Bochner-like transform for general time scales.

If $\mathbb{T} \neq \mathbb{R}$, we fix a left scattered point $\omega \in \mathbb{T}$, there is a unique $n_t \in \mathbb{Z}$ such that $t - n_tK \in [\omega, \omega + k)_{\mathbb{T}}$. Let

$$N_t = \begin{cases} t, & \mathbb{T} = \mathbb{R}, \\ n_t & \mathbb{T} \neq \mathbb{R}. \end{cases}$$

Definition 2.17 ([10]). Let $f \in BS^p(\mathbb{T},\mathbb{H})$. The Bochner-like transform of f is the function f^c : $\mathbb{T}\times\mathbb{T} \to \mathbb{H}$ defined for all $t,s \in \mathbb{T}$ by

$$f^c(t,s) = f(N_tK + s).$$

And we have

$$\| f \|_{S^p} = \| f^c \|_\infty .$$

Definition 2.18 ([12]). A function $f \in BS^p(\mathbb{T},\mathbb{H})$ is said to be Stepanov-like pseudo almost automorphic if its Bochner-like transform f^c is pseudo almost automorphic in the sense that there exist two functions g,ϕ such that $f^c = g^c + \phi^c$, where $g^c \in AA(\mathbb{T},BS^p(\mathbb{T},\mathbb{H}))$ and $\phi^c \in PAA_0(\mathbb{T},BS^p(\mathbb{T},\mathbb{H}))$.

Lemma 2.4 ([12]).

1) If $h,g \in S^pPAA(\mathbb{T},\mathbb{H})$, then $h+g, hg \in S^pPAA(\mathbb{T},\mathbb{H})$.
2) If $h \in S^pPAA(\mathbb{T},\mathbb{H})$ and $g \in S^pAP(\mathbb{T},\mathbb{H})$, then $hg \in S^pPAA(\mathbb{T},\mathbb{H})$.

Proposition 2.1 ([12]). $(S^pPAA(\mathbb{T},\mathbb{H}), \| \,.\, \|_{S^p})$ is a Banach space.

3. Main Results

3.1. **Existence of S^p-pseudo almost automorphic solution of (2.1) on $\mathbb{T}$.** In this section, we will study the existence and S^p-global exponential stability of Stepanov-like pseudo almost automorphic solution of system (2.1) on time scales.

We will list a few hypotheses which will be used for the rest of this paper.

(S_1) : For all $1 \le l \le n$, the function $\xi_m(.) \in S^pAA(\mathbb{T},\mathbb{H}) \cap C^1_{rd}(\mathbb{T},\mathbb{H})$ such that

$$0 \le \xi_m(.) \le \overline{\xi}, \quad 0 \le \xi^* - \xi_m^\Delta(.) < 1 - \xi_m^\Delta(.).$$

(S_2) : For all $1 \le l,m \le n$, the functions $a_{lm}(.)$, $b_{lm}(.)$, $c_{lm}(.)$, $d_{lm}(.) \in S^pPAA(\mathbb{T},\mathbb{H})$.

(S_3) : There exist positive constants L_l^f, L_l^g, L_{h_l} such that for any u, $v \in \mathbb{H}$, the activity functions f_l, g_l, $h_l \in C_{rd}(\mathbb{H},\mathbb{H})$ satisfy

$$| f_l(u) - f_l(v) |_{\mathbb{H}} \le L_l^f \, | u - v |_{\mathbb{H}},$$

$$| g_l(u) - g_l(v) |_{\mathbb{H}} \le L_l^g \, | u - v |_{\mathbb{H}},$$

$$| h_l(u) - h_l(v) |_{\mathbb{H}} \le L_l^h \, | u - v |_{\mathbb{H}} .$$

Furthermore, we suppose that $f_l(0) = g_l(0) = h_l(0) = 0$.

(S_4) : For all $1 \le l,m \le n$, the delay kernel $N_{lm} : [0,+\infty)_{\mathbb{T}} \to [0,+\infty)$ is rd-continuous, $\Delta-$integrable and there exists $\lambda > 0$ such that $\int_0^{+\infty} N_{lm}(z)\Delta z = N^*$.

(S_5) : $\varpi < 1$.

As a convenience, we have introduced these notations which simplify the writing of the equations:

$$b_{lm}^* = \sup_{t\in\mathbb{T}} | b_{lm}(t) |_{\mathbb{H}}, \;\; c_{lm}^* = \sup_{t\in\mathbb{T}} | c_{lm}(t) |_{\mathbb{H}}, \;\; d_{lm}^* = \sup_{t\in\mathbb{T}} | d_{lm}(t) |_{\mathbb{H}}, \;\; \bar{\mu} = \sup_{t\in\mathbb{T}} \mu(t)$$

$$\overline{a}_l = \inf_{t\in\mathbb{T}} a_l(t) \quad a_l^* = \sup_{t\in\mathbb{T}} a_l(t) > 0 \quad \check{a}_l = \inf_{t\in\mathbb{T}} \overline{a}_l + \inf_{t\in\mathbb{T}} a_l \quad l = 1, \dots, n,$$

$$\varpi = \max_{1\le l\le n}\left\{ r_l^{\frac{1}{q}}(q).r_l^{\frac{1}{p}}(p)\times\left[\sum_{m=1}^n b_{lm}^* L_m^f + \sum_{m=1}^n \frac{c_{lm}^* L_m^g}{(1-\xi_m^*)^{\frac{1}{p}}} + \sum_{m=1}^n d_{lm}^* L_m^h N^*\right]\right\},$$

$$\text{with } r_l(q) = \frac{2+\overline{a}_l\bar{\mu}q}{\overline{a}_l q}, \quad \rho_I = \max_{l=1,\dots,n} r_l^{\frac{1}{q}}(q) r_l^{\frac{1}{p}}(p) \parallel I \parallel_{S^p}, \quad r^* = \frac{\varpi\rho_I}{1-\varpi}.$$

Lemma 3.1. *Suppose that condition (S_1) holds. If $g \in S^pPAA(\mathbb{T},\mathbb{H})$ and $\xi \in S^pAA(\mathbb{T},\mathbb{H})$, then $g(.-\xi(.)) \in S^pPAA(\mathbb{T},\mathbb{H})$.*

Proof. Since $f \in S^pPAA(\mathbb{T},\mathbb{H})$, then,

$$g(t-\xi(t)) = g_1(t-\xi(t)) + g_2(t-\xi(t)) := G_1(t) + G_2(t), \quad \forall t \in \mathbb{T}$$

where $g_1 \in S^pAA(\mathbb{T},\mathbb{H})$ and $g_2 \in S^pPAA_0(\mathbb{T},\mathbb{H})$. It's obvious that $G_1(t) = g_1(t-\xi(t)) \in S^pAA(\mathbb{T},\mathbb{H})$. It remains to show that $G_2(.) = g_2(.-\xi(.)) \in S^pPAA_0(\mathbb{T},\mathbb{H})$. Since $g_2 \in S^pPAA_0(\mathbb{T},\mathbb{H})$, we have for all $t \in \mathbb{T}$,

$$F(t) = \left(\frac{1}{K}\int_t^{t+K} \mid g_2(s) \mid_{\mathbb{H}}^p \Delta s\right)^{\frac{1}{p}} \in S^pPAA_0(\mathbb{T},\mathbb{H}).$$

Thus,

$$\begin{aligned}
&\lim_{r\to+\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_t^{t+K} \mid g_2(s-\xi(s)) \mid_{\mathbb{H}}^p \Delta s\right)^{\frac{1}{p}}\Delta t\\
&= \frac{1}{(1-\xi^{\Delta}(s))^{\frac{1}{p}}}\lim_{r\to+\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_{t-\xi(t)}^{t+K-\xi(t+k)} \mid g_2(z) \mid_{\mathbb{H}}^p \Delta z\right)^{\frac{1}{p}}\Delta t,\\
&\le \frac{1}{(1-\xi^*)^{\frac{1}{p}}}\lim_{r\to+\infty}\frac{1}{2r}\left(\frac{1}{K}\int_{t-\bar{\xi}}^{t+K-\bar{\xi}} \mid g_2(z) \mid_{\mathbb{H}}^p \Delta z\right)^{\frac{1}{p}}\Delta t,\\
&\le \frac{1}{(1-\xi^*)^{\frac{1}{p}}}\lim_{r\to+\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r} \mid F(t-\bar{\xi}) \mid_{\mathbb{H}} \Delta t,\\
&\le \frac{1}{(1-\xi^*)^{\frac{1}{p}}}\lim_{r\to+\infty}\frac{1}{2r}\int_{t_0-r-\bar{\xi}}^{t_0+r-\bar{\xi}} \mid F(t) \mid_{\mathbb{H}} \Delta t,\\
&\le \frac{1}{(1-\xi^*)^{\frac{1}{p}}}\lim_{r\to+\infty}\frac{1}{2r}\int_{t_0-r-\bar{\xi}}^{t_0+r+\bar{\xi}} \mid F(t) \mid_{\mathbb{H}} \Delta t,\\
&= \frac{1}{(1-\xi^*)^{\frac{1}{p}}}\lim_{r\to+\infty}\frac{r+\bar{\xi}}{r}\frac{1}{2(r+\bar{\xi})}\int_{t_0-r-\bar{\xi}}^{t_0+r+\bar{\xi}} \mid F(t) \mid_{\mathbb{H}} \Delta t,\\
&= 0.
\end{aligned}$$

So, $G_2 \in S^pPAA_0(\mathbb{T},\mathbb{H})$. $\square$

Lemma 3.2. If a function $g \in C_{rd}(\mathbb{H},\mathbb{H})$ satisfies condition (S_4) and $x \in S^pPAA(\mathbb{T},\mathbb{H})$, then $g \circ x \in S^pPAA(\mathbb{T},\mathbb{H})$.

Proof. We have $x \in S^pPAA(\mathbb{T},\mathbb{H})$, then $x = x_1 + x_2$ where $x_1 \in S^pAA(\mathbb{T},\mathbb{H})$ and $x_2 \in S^pPAA_0(\mathbb{T},\mathbb{H})$. Hence,

$$g \circ x(.) = g \circ x_1(.) + g \circ x(.) - g \circ x_1(.) = G_1(.) + G_2(.),$$

where, $G_1(.) = g \circ x_1(.)$ and $G_2(.) = g \circ x(.) - g \circ x_1(.)$. Since $x_1 \in S^pAA(\mathbb{T},\mathbb{H})$, then for every sequence $\{s_n\}_{n=1}^{\infty} \subset \Pi$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$\sup_{t\in\mathbb{T}} \left(\frac{1}{K} \int_t^{t+K} | x_1(s+\tau_n) - \varphi(s) |_{\mathbb{H}}^p \, \Delta s \right)^{\frac{1}{p}} \to 0, \quad \text{as} \quad n \to \infty.$$

We set $\varphi_1(.) := g \circ \varphi$, therefore,

$$\begin{aligned}
&\sup_{t\in\mathbb{T}} \left(\frac{1}{K} \int_t^{t+K} | g \circ x_1(s+\tau_n) - \varphi_1(s) |_{\mathbb{H}}^p \, \Delta s \right)^{\frac{1}{p}} \\
&\qquad = \sup_{t\in\mathbb{T}} \left(\frac{1}{K} \int_t^{t+K} | g \circ x_1(s+\tau_n) - g \circ \varphi(s) |_{\mathbb{H}}^p \, \Delta s \right)^{\frac{1}{p}}, \\
&\qquad \le L^g \sup_{t\in\mathbb{T}} \left(\frac{1}{K} \int_t^{t+K} | x_1(s+\tau_n) - \varphi(s) |_{\mathbb{H}}^p \, \Delta s \right)^{\frac{1}{p}}.
\end{aligned}$$

Then,

$$\sup_{t\in\mathbb{T}} \left(\frac{1}{K} \int_t^{t+K} | g \circ x_1(s+\tau_n) - \varphi_1(s) |_{\mathbb{H}}^p \, \Delta s \right)^{\frac{1}{p}} \to 0, \quad \text{as} \quad n \to \infty.$$

Because of that $x_1(.) \in S^pAA(\mathbb{T},\mathbb{H})$. Now,

$$\begin{aligned}
&\lim_{r\to+\infty} \frac{1}{2r} \int_{t_0-r}^{t_0+r} \left(\frac{1}{K} \int_t^{t+K} | G_2(s) |_{\mathbb{H}}^p \, \Delta s \right)^{\frac{1}{p}} \Delta t \\
&\quad = \lim_{r\to+\infty} \frac{1}{2r} \int_{t_0-r}^{t_0+r} \left(\frac{1}{K} \int_t^{t+K} | g \circ x(s) - g \circ x_1(s) |_{\mathbb{H}}^p \, \Delta s \right)^{\frac{1}{p}} \Delta t, \\
&\quad \le \lim_{r\to+\infty} \frac{L^g}{2r} \int_{t_0-r}^{t_0+r} \left(\frac{1}{K} \int_t^{t+K} | x(s) - x_1(s) |_{\mathbb{H}}^p \, \Delta s \right)^{\frac{1}{p}} \Delta t, \\
&\quad \le \lim_{r\to+\infty} \frac{L^g}{2r} \int_{t_0-r}^{t_0+r} \left(\frac{1}{K} \int_t^{t+K} | x_2(s) |_{\mathbb{H}}^p \, \Delta s \right)^{\frac{1}{p}} \Delta t, \\
&\quad = 0.
\end{aligned}$$

As a consequence , $G_2(.) \in S^pPAA_0(\mathbb{T},\mathbb{H})$. In conclusion, $g \circ x \in S^pPAA(\mathbb{T},\mathbb{H})$. ☐

Lemma 3.3. *Assume that (S_3) and (S_4) hold and $x(.) \in S^pPAA(\mathbb{T},\mathbb{H})$, then for $l,m = 1,2,...,n$, the function $t \to \int_{-\infty}^{t} N_{lm}(t-s)h_m(x_m(s))\Delta s$ belongs to $S^pPAA(\mathbb{T},\mathbb{H})$, for all $t,s \in \mathbb{T}$.*

Proof. We have $x_m \in S^pPAA(\mathbb{T},\mathbb{H})$. Then,

$$\begin{aligned}|\phi_{lm}(t)|_{\mathbb{H}} &\leq \int_{-\infty}^{t} \Big|N_{lm}(t-s)h_m(x_m(s))\Delta s\Big|_{\mathbb{H}} \\ &\leq L_m^h \parallel x_m \parallel_{S^p} \int_0^{+\infty} \big|N_{lm}(s)\Delta s \\ &\leq N^* L_m^h \parallel x_m \parallel_{S^p} .\end{aligned}$$

which proves that the integral $\int_{-\infty}^{t} N_{lm}(t-s)h_m(x_m(s))\Delta s$ is absolutely convergent and $\phi_{lm} \in BS^p(\mathbb{T},\mathbb{H})$. Hence, it is easy to prove the rd-continuity of function ϕ_{lm}. Now, from lemma (3.2) we have, $h_m^c(x_m^c(.)) \in PAA(\mathbb{T},BS^p(\mathbb{T},\mathbb{H}))$. Then, for all $1 \leq m \leq n$, we get

$$h_m^c(x_m^c(.)) = y_m^c(.) + z_m^c(.),$$

Where $y_m^c(.) \in AA(\mathbb{T},BS^p(\mathbb{T},\mathbb{H}))$ and $z_m^c(.) \in PAA_0(\mathbb{T},BS^p(\mathbb{T},\mathbb{H}))$. Thus, for all $1 \leq m,l \leq n$, we can write ϕ_{lm} in the following form

$$\begin{aligned}\phi_{lm}^c(t) &= \int_{-\infty}^{t} N_{lm}(t-s)y_m^c(s)\Delta s + \int_{-\infty}^{t} N_{lm}(t-s)z_m^c(s)\Delta s \\ &= Y_{lm}^c(t) + Z_{lm}^c(t).\end{aligned}$$

On the one hand, let us prove that $Y_{lm}^c(.) \in AA(\mathbb{T},BS^p(\mathbb{T},\mathbb{H}))$. Since $y_m^c(.) \in AA(\mathbb{T},BS^p(\mathbb{T},\mathbb{H}))$, so for every sequence $\{s_n\}_{n=1}^{\infty} \subset \Pi$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} y_m^c(t+\tau_n) = \varphi_m(t) \quad \text{and} \quad y_m^c(t) = \lim_{n\to\infty} \varphi_m(t-\tau_n).$$

We pose $W_{lm} := \int_{-\infty}^{t} N_{lm}(t-s)\varphi_m(s)\Delta s$. Then we have

$$\begin{aligned}\parallel Y_{lm}(t+\tau_n) - W_{lm}(t) \parallel_{S^p} = \parallel Y_{lm}^c(t+\tau_n) - W_{lm}(t) \parallel_\infty & \\ = \parallel \int_{-\infty}^{t+\tau_n} N_{lm}(t+\tau_n-s)y_m^c(s)\Delta s & \\ - \int_{-\infty}^{t} N_{lm}(t-s)\varphi_m(s)\Delta s \parallel_\infty & \\ \leq \int_{-\infty}^{t} N_{lm}(t-s) \parallel y_m^c(s+\tau_n) - \varphi_m(s) \parallel_\infty \Delta s. &\end{aligned}$$

By the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty} Y_m^c(t+\tau_n) = W_{lm}(t) \quad \forall t \in \mathbb{T}.$$

Similarly, we can obtain

$$Y_m^c(t) = \lim_{n\to\infty} W_{lm}(t-\tau_n) \quad \forall t \in \mathbb{T}.$$

As a result, $Y^c_{lm}(.) \in AA(\mathbb{T}, BS^p(\mathbb{T},\mathbb{H}))$.
On the other hand, it remains to show that $Z^c_{lm}(.) \in PAA_0(\mathbb{T}, BS^p(\mathbb{T},\mathbb{H}))$. Indeed,

$$\begin{aligned}
\lim_{r\to\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r} \| Z_{lm}(s) \|_{S^p}\,\Delta t &= \lim_{r\to\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r} \| Z^c_{lm}(s) \|_{\infty}\,\Delta t \\
&\le \lim_{r\to\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\int_0^{+\infty} N_{lm}(s) \| z^c_m(t-s) \|_{\infty}\,\Delta s\Delta t, \\
&\le \int_0^{+\infty} N_{lm}(s) \lim_{r\to\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r} \| z^c_m(t-s) \|_{\infty}\,\Delta t\Delta s, \\
&= 0.
\end{aligned}$$

Thus, $Y^c_{lm}(.) \in PAA_0(\mathbb{T}, BS^p(\mathbb{T},\mathbb{H}))$, then, $\phi^c_{lm}(.) \in PAA(\mathbb{T}, BS^p(\mathbb{T},\mathbb{H}))$. Finally, the function ϕ_{lm} belongs to $S^pPAA(\mathbb{T},\mathbb{H})$. $\square$

Theorem 3.1. *Let $\psi = (\psi_1, ..., \psi_n) \in S^pPAA(\mathbb{T},\mathbb{H})$. Under assumptions (S_1)-(S_4), the nonlinear operator defined by,*

$$(\Pi_\psi)_l(t) = \int_{-\infty}^{t} \hat{e}_{\ominus a_l}(t,\sigma(s))J_l(s)\Delta s, \quad l = 1, ..., n.$$

Where

$$\begin{aligned}
J_l(t) = &\sum_{m=1}^{n} b_{lm}(t) f_m(\psi_m(t)) + \sum_{m=1}^{n} c_{lm}(t) g_m(\psi_m(t-\xi_m(t))) \\
&+ \sum_{m=1}^{n} d_{lm}(t)\int_{-\infty}^{t} N_{lm}(t-s) h_m(\psi_m(s))\Delta s + I_l(t)
\end{aligned}$$

maps $S^pPAA(\mathbb{T},\mathbb{H})$ into itself.

Proof. It's easy to see that $(\Pi_\psi)_l$ is well defined and continuous. Again from lemma (3.1) and Lemma (3.3), we have $J_l(.)$ belongs to $S^pPAA(\mathbb{T},\mathbb{H})$. Now, let $J_l = Y_l + Z_l$ with $Y_l \in S^pAA(\mathbb{T},\mathbb{H})$, $Z_l(.) \in S^pPAA_0(\mathbb{T},\mathbb{H})$. For the sake of convenience, we break the proof in two steps.
First : We will prove that $(F_\psi)_l(t) = \int_{-\infty}^{t} \hat{e}_{\ominus a_l}(t,\sigma(s))Y_l(s)\Delta s \in S^pAA(\mathbb{T},\mathbb{H})$. Since $Y_l(.) \in S^PAA(\mathbb{T},\mathbb{H})$, then for every sequence $\{s_n\}_{n=1}^{\infty} \subset \Pi$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$\sup_{t_1\in\mathbb{T}}\left(\frac{1}{K}\int_{t_1}^{t_1+K} | Y_l(t+\tau_n) - \varphi_l(t) |^p_{\mathbb{H}}\,\Delta t\right)^{\frac{1}{p}} \to 0, \quad \text{as} \quad n\to\infty.$$

We set $(\mathscr{U}_\psi)_l(t)=\int_{-\infty}^{t}\hat{e}_{\ominus a_l}(t,\sigma(s))\varphi_l(s)\Delta s$. Thus,

$$\begin{aligned}
&|(F_\psi)_l(t+\tau_n)-(\mathscr{U}_\psi)_l(t)\,|_{\mathbb{H}}\\
&=\left|\int_{-\infty}^{t}\hat{e}_{\ominus a_l}(t,\sigma(s))Y_l(s+\tau_n)\Delta s-\int_{-\infty}^{t}\hat{e}_{\ominus a_l}(t,\sigma(s))\varphi_l(s)\Delta s\right|_{\mathbb{H}},\\
&\leq\int_{-\infty}^{t}\hat{e}_{\ominus\bar{a}_l}(t,\sigma(s))\big|Y_l(s+\tau_n)-\varphi_l(s)\big|_{\mathbb{H}}\Delta s,\\
&\leq\int_{-\infty}^{0}\hat{e}_{\ominus\bar{a}_l}(0,\sigma(s))\big|Y_l(t+s+\tau_n)-\varphi_l(t+s)\big|_{\mathbb{H}}\Delta s,\\
&\leq r_l^{\frac{1}{q}}(q)\left(\int_{-\infty}^{0}\hat{e}_{\ominus(\frac{\bar{a}_lp}{2})}(0,\sigma(s))\big|Y_l(t+s+\tau_n)-\varphi_l(t+s)\big|_{\mathbb{H}}^{p}\Delta s\right)^{\frac{1}{p}}.
\end{aligned}$$

Fubini's theorem implies that

$$\begin{aligned}
&\sup_{t_1\in\mathbb{T}}\left(\frac{1}{K}\int_{t_1}^{t_1+K}\left|(F_\psi)_l(t+\tau_n)-(\mathscr{U}_\psi)_l(t)\right|_{\mathbb{H}}^{p}\Delta t\right)^{\frac{1}{p}}\\
&\leq\sup_{t_1\in\mathbb{T}}\left(\frac{1}{K}\int_{t_1}^{t_1+K}r_l^{\frac{p}{q}}.\int_{-\infty}^{0}\hat{e}_{\ominus(\frac{\bar{a}_lp}{2})}(0,\sigma(s))\left|Y_l(s+t+\tau_n)-\varphi_l(s+t)\right|_{\mathbb{H}}^{p}\Delta s\Delta t\right)^{\frac{1}{p}},\\
&\leq r_l^{\frac{1}{q}}(q).\left(\int_{-\infty}^{0}\hat{e}_{\ominus(\frac{\bar{a}_lp}{2})}(0,\sigma(s))\sup_{\bar{t}\in\mathbb{T}}\frac{1}{K}\int_{\bar{t}}^{\bar{t}+K}\left|Y_l(t+\tau_n)-\varphi_l(t)\right|_{\mathbb{H}}^{p}\Delta t\Delta s\right)^{\frac{1}{p}},\\
&\leq r_l^{\frac{1}{q}}(q).r_l^{\frac{1}{p}}(p).\left(\sup_{\bar{t}\in\mathbb{T}}\frac{1}{K}\int_{\bar{t}}^{\bar{t}+K}\left|Y_l(t+\tau_n)-\varphi_l(t)\right|_{\mathbb{H}}^{p}\Delta t\Delta s\right)^{\frac{1}{p}},\\
&<\max_{1\leq l\leq n}r_l^{\frac{1}{q}}(q)r_l^{\frac{1}{p}}(p)\left(\sup_{\bar{t}\in\mathbb{T}}\frac{1}{K}\int_{\bar{t}}^{\bar{t}+K}\left|Y_l(t+\tau_n)-\varphi_l(t)\right|_{\mathbb{H}}^{p}\Delta t\Delta s\right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore,

$$\sup_{t_1\in\mathbb{T}}\left(\frac{1}{K}\int_{t_1}^{t_1+K}\left|(F_\psi)_l(t+\tau_n)-(\mathscr{U}_\psi)_l(t)\right|_{\mathbb{H}}^{p}\Delta t\right)^{\frac{1}{p}}\to 0,\quad\text{as}\quad n\to\infty.$$

Which means that $(F_\psi)_l(.)\in S^pAA(\mathbb{T},\mathbb{H})$.

Second : Let us prove that $(G_\psi)_l(t)=\int_{-\infty}^{t}\hat{e}_{\ominus a_l}(t,\sigma(s))Z_l(s)\Delta s\in S^pPAA_0(\mathbb{T},\mathbb{H})$. It follows from Hölder's inequality and Fubini's theorem that

$$\begin{aligned}
&\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_{t}^{t+K}|\,(G_\psi)_l(s)\,|_{\mathbb{H}}^{p}\,\Delta s\right)^{\frac{1}{p}}\Delta t\leq\\
&\qquad(2r)^{\frac{1}{q}}\left[\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_{t}^{t+K}|\,(G_\psi)_l(s)\,|_{\mathbb{H}}^{p}\,\Delta s\right)\Delta t\right]^{\frac{1}{p}}.
\end{aligned}$$

Since,

$$\begin{aligned}
&\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_t^{t+K} |(G_\psi)_l(s)|_{\mathbb{H}}^p \Delta s\right)^{\frac{1}{p}}\Delta t\\
&\leq (2r)^{\frac{1}{q}}\left[\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_t^{t+K} r_l^{\frac{p}{q}}(q)\int_{-\infty}^{0}\hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(z))\,|Z_m(s+z)|_{\mathbb{H}}^p \Delta z\Delta s\right)\Delta t\right]^{\frac{1}{p}},\\
&\leq (2r)^{\frac{1}{q}} r_l^{\frac{1}{q}}(q)\left[\int_{-\infty}^{0}\hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(z))\int_{t_0-r}^{t_0+r}\frac{1}{K}\int_t^{t+K}|Z_l(s+z)|_{\mathbb{H}}^p \Delta s\Delta t\Delta z\right]^{\frac{1}{p}}.
\end{aligned}$$

According to Lebesgue dominated theorem and $Z_l(.)\in S^pPAA_0(\mathbb{T},\mathbb{H})$, we get

$$\begin{aligned}
&\lim_{r\to\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_t^{t+K}|(G_\psi)_l(s)|_{\mathbb{H}}^p\Delta s\right)^{\frac{1}{p}}\Delta t \leq \frac{1}{2r}(2r)^{\frac{1}{q}}(2r)^{\frac{1}{p}} r_l^{\frac{1}{q}}(q)\\
&\times\left(\int_{-\infty}^{0}\hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(z))\lim_{r\to\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\frac{1}{K}\int_t^{t+K}|Z_l(s+z)|_{\mathbb{H}}^p\Delta s\Delta t\Delta z\right)^{\frac{1}{p}},\\
&\leq r_l^{\frac{1}{q}}(q)\left[\int_{-\infty}^{0}\hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(z))\lim_{r\to\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\frac{1}{K}\int_t^{t+K}|Z_l(s+z)|_{\mathbb{H}}^p\Delta s\Delta t\Delta z\right]^{\frac{1}{p}},\\
&\leq r_l^{\frac{1}{q}}(q)\left[\int_{-\infty}^{0}\hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(z))\lim_{r\to\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_t^{t+K}|Z_l(s+z)|_{\mathbb{H}}^q\Delta s\right)^{\frac{p}{q}}\Delta t\Delta z\right]^{\frac{1}{p}},\\
&\leq r_l^{\frac{1}{q}}(q)\left[\int_{-\infty}^{0}\hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(z))\lim_{r\to\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_t^{t+K}|Z_l(s+z)|_{\mathbb{H}}^q\Delta s\right)^{\frac{1}{q}}\right.\\
&\left.\times\left(\frac{1}{K}\int_t^{t+K}|Z_l(s+z)|_{\mathbb{H}}^q\Delta s\right)^{\frac{p-1}{q}}\Delta t\Delta z\right]^{\frac{1}{p}},\\
&\leq r_l^{\frac{1}{q}}(q)\,|Z_l|_\infty^{\frac{p-1}{p}}\left\{\int_{-\infty}^{0}\hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(z))\right.\\
&\left.\times\left[\lim_{r\to\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_t^{t+K}|Z_l(s+z)|_{\mathbb{H}}^q\Delta s\right)^{\frac{1}{q}}\Delta t\right]\Delta z\right\}^{\frac{1}{p}}=0.
\end{aligned}$$

Which yields that $(G_\psi)_l(.)\in S^pPAA_0(\mathbb{T},\mathbb{H})$. This completes the proof of theorem. $\square$

Theorem 3.2. Assume that the conditions (S_1)-(S_5) are satisfied. Then, system (2.1) has a unique S^p-pseudo almost automorphic solution in the region $\mathbb{B}=\{\psi:\psi\in S^pPAA(\mathbb{T},\mathbb{H}),\|\psi-\psi_0\|_{S^p}\leq r^*\}$.

Proof. **First step :** At first, we show that $(\Pi_\psi)_l$ is a self-mapping from $\mathbb{B}$ to $\mathbb{B}$. Let $\psi \in \mathbb{B}$, by using Hölder's and Minkowski's inequality we can obtain

$$\begin{aligned}
&\mid (\Lambda_\psi)_l(t)-\psi_0(t)\mid_{\mathbb{H}}= \Bigg|\int_{-\infty}^{t}\hat{e}_{\ominus a_l}(t,\sigma(z))\times\Bigg[\sum_{m=1}^{n} b_{lm}(z)f_m(\psi_m(z))\\
&+\sum_{m=1}^{n} c_{lm}(z)g_m(\psi_m(z-\xi_m(z)))+\sum_{m=1}^{n} d_{lm}(z)\int_{-\infty}^{z}N_{lm}(z-s)h_m(\psi_m(s))\Delta s\Bigg]\Delta z\Bigg|_{\mathbb{H}},\\
&\leq r_l^{\frac{1}{q}}(q)\times\Bigg[\int_{-\infty}^{0}\hat{e}_{\ominus(\frac{\bar{a}_l p}{2})}(0,\sigma(z))\Bigg|\sum_{m=1}^{n} b_{lm}(z+t)f_m(\psi_m(z+t))\\
&+\sum_{m=1}^{n} c_{lm}(z+t)g_m(\psi_m(z+t-\xi_m(z+t)))\\
&+\sum_{m=1}^{n} d_{lm}(z+t)\int_{-\infty}^{z+t}N_{lm}(z+t-s)h_m(\psi_m(s))\Delta s\Bigg|_{\mathbb{H}}^{p}\Delta z\Bigg]^{\frac{1}{p}}.
\end{aligned}$$

Furthermore

$$\begin{aligned}
&\parallel (\Pi_\psi)_l(t)-\psi_0(t)\parallel_{S^p}=\sup_{t_1\in\mathbb{T}}\Bigg[\frac{1}{K}\int_{t_1}^{t_1+K}\Bigg|\int_{-\infty}^{t}\hat{e}_{\ominus a_l}(t,\sigma(z))\\
&\times\Bigg(\sum_{m=1}^{n} b_{lm}(z)f_m(\psi_m(z))+\sum_{m=1}^{n} c_{lm}(z)g_m(\psi_m(z-\xi_m(z)))\\
&+\sum_{m=1}^{n} d_{lm}(z)\int_{-\infty}^{z}N_{lm}(z-s)h_m(\psi_m(s))\Delta s\Bigg)\Delta z\Bigg|_{\mathbb{H}}^{p}\Delta t\Bigg]^{\frac{1}{p}},\\
&\leq r_l^{\frac{1}{q}}(q).\sup_{t_1\in\mathbb{T}}\Bigg[\int_{-\infty}^{0}\hat{e}_{\ominus(\frac{\bar{a}_l p}{2})}(0,\sigma(z))\times\frac{1}{K}\int_{t_1}^{t_1+K}\Bigg|\sum_{m=1}^{n} b_{lm}(z+t)f_m(\psi_m(z+t))\\
&+\sum_{m=1}^{n} c_{lm}(z+t)g_m(\psi_m(z+t-\xi_m(z+t)))\\
&+\sum_{m=1}^{n} d_{lm}(z+t)\int_{-\infty}^{z+t}N_{lm}(z+t-s)h_m(\psi_m(s))\Delta s\Bigg|_{\mathbb{H}}^{p}\Delta(z+t)\Delta z\Bigg]^{\frac{1}{p}},\\
&\leq r_l^{\frac{1}{q}}(q).\sup_{t_2\in\mathbb{T}}\Bigg[\int_{-\infty}^{0}\hat{e}_{\ominus(\frac{\bar{a}_l p}{2})}(0,\sigma(z))\times\frac{1}{K}\int_{t_2}^{t_2+K}\Bigg|\sum_{m=1}^{n} b_{lm}(\hat{t})f_m(\psi_m(\hat{t}))\\
&+\sum_{m=1}^{n} c_{lm}(\hat{t})g_m(\psi_m(\hat{t}-\xi_m(\hat{t})))\\
&+\sum_{m=1}^{n} d_{lm}(\hat{t})\int_{-\infty}^{\hat{t}}N_{lm}(\hat{t}-s)h_m(\psi_m(s))\Delta s\Bigg|_{\mathbb{H}}^{p}\Delta\hat{t}\Delta z\Bigg]^{\frac{1}{p}}.
\end{aligned}$$

$$\leq r_l^{\frac{1}{q}}(q).r_l^{\frac{1}{p}}(p) \times \left(\sup_{t_2\in\mathbb{T}} \left[\frac{1}{K} \int_{t_2}^{t_2+K} \sum_{m=1}^{n} \mid b_{lm}(\hat{t}) \mid_{\mathbb{H}}^{p} \mid f_m(\psi_m(\hat{t})) \mid_{\mathbb{H}}^{p} \Delta\hat{t} \right]^{\frac{1}{p}} \right.$$

$$+ \sup_{t_2\in\mathbb{T}} \left[\frac{1}{K} \int_{t_2}^{t_2+K} \sum_{m=1}^{n} \mid c_{lm}(\hat{t}) \mid_{\mathbb{H}}^{p} \mid g_m(\psi_m(\hat{t} - \xi_m(\hat{t}))) \mid_{\mathbb{H}}^{p} \Delta\hat{t} \right]^{\frac{1}{p}}$$

$$\left. + \sup_{t_2\in\mathbb{T}} \left[\frac{1}{K} \int_{t_2}^{t_2+K} \sum_{m=1}^{n} \mid d_{lm}(\hat{t}) \mid_{\mathbb{H}}^{p} \int_{-\infty}^{\hat{t}} \mid N_{lm}(\hat{t} - s) h_m(\psi_m(s)) \Delta s \mid_{\mathbb{H}}^{p} \Delta\hat{t} \right]^{\frac{1}{p}} \right),$$

$$\leq \max_{1\leq l\leq n} \left\{ r_l^{\frac{1}{q}}(q).r_l^{\frac{1}{p}}(p) \left[\sum_{m=1}^{n} b_{lm}^{*} L_m^{f} + \sum_{m=1}^{n} \frac{c_{lm}^{*} L_m^{g}}{(1-\xi_m^{*})^{\frac{1}{p}}} + \sum_{m=1}^{n} d_{lm}^{*} L_m^{h} N^{*} \right] \right\} \times \parallel \psi \parallel_{S^p},$$

$$\leq \varpi. \parallel \psi \parallel_{S^p}.$$

On the other hand,

$$\parallel \psi_0(t) \parallel_{S^p} = \sup_{t_1\in\mathbb{T}} \left(\frac{1}{K} \int_{t_1}^{t_1+K} \left| \int_{-\infty}^{t} \hat{e}_{\ominus a_l}(t, \sigma(z)) I_l(z) \right|_{\mathbb{H}}^{p} \Delta z \Delta t \right)^{\frac{1}{p}},$$

$$\leq r_l^{\frac{1}{q}}(q) \sup_{t_1\in\mathbb{T}} \left(\int_{-\infty}^{0} \hat{e}_{\ominus(\frac{\bar{a}_l p}{2})}(0, \sigma(z)) \frac{1}{K} \int_{t_1}^{t_1+K} \mid I_l(z+t) \mid_{\mathbb{H}}^{p} \Delta t \Delta z \right)^{\frac{1}{p}},$$

$$\leq r_l^{\frac{1}{q}}(q) \sup_{t_2\in\mathbb{T}} \left(\int_{-\infty}^{0} \hat{e}_{\ominus(\frac{\bar{a}_l p}{2})}(0, \sigma(z)) \frac{1}{K} \int_{t_2}^{t_2+K} \mid I_l(\bar{t}) \mid_{\mathbb{H}}^{p} \Delta\bar{t} \Delta z \right)^{\frac{1}{p}},$$

$$\leq r_l^{\frac{1}{q}}(q).r_l^{\frac{1}{p}}(p). \parallel I \parallel_{S^p} < \rho_I. \quad \text{(Hölder's inequality)}$$

So, for any $\psi \in \mathbb{B}$, we have

$$\parallel \psi \parallel_{S^p} \leq \parallel \psi - \psi_0 \parallel_{S^p} + \parallel \psi_0 \parallel_{S^p} \leq \frac{\varpi \rho_I}{1-\varpi} + \rho_I = \frac{\rho_I}{1-\varpi}.$$

Hence,

$$\parallel (\Pi_\psi)_l(t) - \psi_0(t) \parallel_{S^p} \leq \frac{\varpi \rho_I}{1-\varpi},$$

which implies that $(\Pi_\psi)_l \in \mathbb{B}$.

Second step : we shall prove that $(\Pi_\psi)_l$ is a contraction mapping. In fact, For $\psi, \phi \in \mathbb{B}$, we get

$$
\begin{aligned}
&\| (\Pi_\psi)_l(t) - (\Pi_\phi)_l(t) \|_{S^p} = \\
&\sup_{t_1\in\mathbb{T}} \left[\frac{1}{K} \int_{t_1}^{t_1+K} \left| \int_{-\infty}^{t} \hat{e}_{\ominus a_l}(t,\sigma(z)) \Bigg(\sum_{m=1}^{n} b_{lm}(z)(f_m(\psi_m(z)) \right.\right. \\
&- f_m(\phi_m(z))) + \sum_{m=1}^{n} c_{lm}(z)\left(g_m(\psi_m(z-\xi_m(z))) - g_m(\phi_m(z-\xi_m(z)))\right) \\
&\left.\left. + \sum_{m=1}^{n} d_{lm}(z) \int_{-\infty}^{z} N_{lm}(z-s)\left(h_m(\psi_m(s)) - h_m(\phi_m(s))\right)\Delta s \Bigg) \Delta z \right|_{\mathbb{H}}^{p} \Delta t \right]^{\frac{1}{p}}, \\
&\leq \max_{1\leq l\leq n} \left\{ r_l^{\frac{1}{q}}(q) r_l^{\frac{1}{p}}(p) \left[\sum_{m=1}^{n} b_{lm}^* L_m^f + \sum_{m=1}^{n} \frac{c_{lm}^* L_m^g}{(1-\xi_m^*)^{\frac{1}{p}}} + \sum_{m=1}^{n} d_{lm}^* L_m^h N^* \right] \right\} \\
&\times \| \psi - \phi \|_{S^p} \leq \varpi . \| \psi - \phi \|_{S^p} < 1.
\end{aligned}
$$

Hence, we obtain that $(\Pi_\psi)_l$ is a contraction mapping. Then, system (2.1) has a unique S^p-pseudo almost automorphic solution in the region $\mathbb{B}$. The proof is complete. □

3.2. **Global S^p-exponential stability.**

Lemma 3.4 ([31]). For any $u, v \in \mathbb{H}$, if $P \in \mathbb{H}^{n\times n}$ is a positive-definite Hermitian matrix, then

$$\bar{u}v + \bar{v}u \leq \bar{u}Pu + \bar{v}P^{-1}v.$$

Definition 3.1. The dynamical networks (2.1) is said to be S^p-globally exponentially stable, if there exist positive constants λ with $\ominus\lambda \in \mathfrak{R}^+$ and $M > 0$ such that

$$\| v(t) - u(t) \|_{S^p} \leq M\hat{e}_{\ominus\lambda}(t,0), \quad \forall t \in (0,\infty)_{\mathbb{T}}.$$

Where $u(.) = (u_1(.), u_2(.), ..., u_n(.))$ is a Stepanov-like pseudo almost automorphic solution of QVFRNNs (2.1) on $\mathbb{T}$ and $v(.) = (v_1(.), v_2(.), ..., v_n(.))$ is an arbitrary solution of QVRNNs (2.1) on $\mathbb{T}$.

Theorem 3.3. Suppose that assumptions (S_1)-(S_5) hold, and $\varpi < 1$. Then the unique Stepanov-like pseudo almost automorphic solution of system (2.1) is S^p-globally exponentially stable on $\mathbb{T}$ whenever

$$
\begin{aligned}
(S_6)\ \frac{1}{a_l^*} - \check{a}_l < &- \sum_{m=1}^{n} a_l^* \left(b_{lm}^* L_l^f\right)^2 - \sum_{m=1}^{n} \frac{a_l^* \left(c_{lm}^* L_l^g\right)^2}{1-\xi^*} \\
&- \sum_{m=1}^{n} a_l^* \left(d_{lm}^* L_l^h\right)^2 \int_0^\infty N_{lm}^2(z)\Delta z.
\end{aligned}
$$

Proof. Let $u(.)$ be the S^p-pseudo almost automorphic solution on $\mathbb{T}$ and let $y(.)$ be an arbitrary solution of sys.(2.1), $X(.) = u(.) - v(.)$, $F(X(.)) = f(u(.)) - f(v(.))$, $G(X(.)) = g(u(.)) - g(v(.))$, and $H(X(.)) = h(u(.)) - h(v(.))$. Let $x \in [0,\infty)$, we define the function $x \mapsto D_l(x)$ as follows:

$$D_l(x) = x + \frac{1}{a_l^*} - \breve{a}_l + \sum_{m=1}^{n} a_l^* \left(b_{lm}^* L_l^f\right)^2 + \frac{\exp(x(\bar{\mu}+\bar{\xi}))}{1-\xi^*} \sum_{m=1}^{n} a_l^* \left(c_{lm}^* L_l^g\right)^2$$
$$+ \sum_{m=1}^{n} a_l^* \left(d_{lm}^* L_l^h\right)^2 \int_0^\infty N_{lm}^2(z) \hat{e}_x(\sigma(z),0)\Delta z < 0.$$

By (S_6), we have $D_l(0) < 0$. Since the function $D_l(.)$ is continuous on $[0,\infty)$, there exist $0 < \lambda < \min_{1\le l\le n} \overline{a}_l$, such that $\Theta_l(\lambda) < 0$.
Now, consider the Lyapunov function as follows :

$$\begin{aligned}
V(t) &= \sum_{l=1}^{n} \mid X_l(t) \mid_{\mathbb{H}}^2 \hat{e}_\lambda(t,0) \\
&+ \frac{\exp(\lambda(\bar{\mu}+\bar{\xi}))}{1-\xi^*} \sum_{l=1}^{n}\sum_{m=1}^{n} a_l^* \left(c_{lm}^* L_l^g\right)^2 \int_{t-\xi(t)}^{t} \mid X_l(z) \mid_{\mathbb{H}}^2 \hat{e}_\lambda(\sigma(z),0)\Delta z \\
&+ \sum_{l=1}^{n}\sum_{m=1}^{n} \left(d_{lm}^* L_l^h\right)^2 \int_0^\infty N_{lm}^2(z) \int_{t-z}^{t} \mid X_l(s) \mid_{\mathbb{H}}^2 \hat{e}_\lambda(\sigma(z),0)\hat{e}_\lambda(\sigma(s),0)\Delta s\Delta z.
\end{aligned}$$

Consequently,

$$\begin{aligned}
V^\Delta(t) &= \lambda \hat{e}_\lambda(t,0) \sum_{l=1}^{n} \mid X_l(t) \mid_{\mathbb{H}}^2 + \hat{e}_\lambda(t,0) \sum_{l=1}^{n} X_l^\Delta(t)\overline{X_l(t)} \\
&+ \frac{\exp(\lambda(\bar{\mu}+\bar{\xi}))}{1-\xi^*} \sum_{l=1}^{n}\sum_{m=1}^{n} a_l^* \left(c_{lm}^* L_l^g\right)^2 \\
&\times \left[\hat{e}_\lambda(t,0) \mid X_l(t) \mid_{\mathbb{H}}^2 - (1-\xi_m^\Delta(t))\exp(-\lambda\xi_m(t))\hat{e}_\lambda(t,0) \mid X_l(t-\xi_m^\Delta(t)) \mid_{\mathbb{H}}^2\right] \\
&+ \sum_{l=1}^{n}\sum_{m=1}^{n} a_l^* \left(d_{lm}^* L_l^h\right)^2 \int_0^\infty N_{lm}^2(z) \left[\hat{e}_\lambda(t,0)\hat{e}_\lambda(\sigma(z),0) \mid X_l(t) \mid_{\mathbb{H}}^2 - \hat{e}_\lambda(t,0) \mid X_l(t-z) \mid_{\mathbb{H}}^2\right] \Delta z
\end{aligned}$$

According to lemma (3.4) and by using (S_1), $\exp(\lambda\bar{\xi}).\exp(-\lambda\xi_m(t)) > 1$, we have

$$
\begin{aligned}
V^{\Delta}(t) &\leq \lambda \hat{e}_{\lambda}(t,0)\sum_{l=1}^{n} \mid X_l(t) \mid_{\mathbb{H}}^2 - \hat{e}_{\lambda}(t,0)\sum_{l=1}^{n} \breve{a}_l \mid X_l(t) \mid_{\mathbb{H}}^2 \\
&+ \hat{e}_{\lambda}(t,0)\sum_{l=1}^{n}\left(\sum_{m=1}^{n} b_{lm}(t)F_m(X_m(t)).\overline{b_{lm}(t)F_m(X_m(t))}.a_l^* + \frac{X_l(t)\overline{X_l(t)}}{a_l^*}\right) \\
&+ \hat{e}_{\lambda}(t,0)\sum_{l=1}^{n}\left(\sum_{m=1}^{n} \overline{c_{lm}(t)G_m(X_m(t-\xi_m(t)))}.c_{lm}(t)G_m(X_m(t-\xi_m(t))).a_l^*\right. \\
&\left.+ \frac{X_l(t)\overline{X_l(t)}}{a_l^*}\right) + \hat{e}_{\lambda}(t,0)\sum_{l=1}^{n}\left(\sum_{m=1}^{n} \overline{d_{lm}(t)\int_{-\infty}^{t} N_{lm}(t-z)H_m(X_m(t))\Delta z}.d_{lm}(t)\right. \\
&\left.\times \int_{-\infty}^{t} N_{lm}(t-z)H_m(X_m(t))\Delta z.a_l^* + \frac{X_l(t)\overline{X_l(t)}}{a_l^*}\right) + \frac{\exp(\lambda\bar{\mu})}{1-\xi^*}\sum_{l=1}^{n}\sum_{m=1}^{n} a_l^*\left(c_{lm}^* L_l^g\right)^2 \\
&\times \left(\exp(\lambda\bar{\xi})\hat{e}_{\lambda}(t,0) \mid X_l(t) \mid_{\mathbb{H}}^2 - (1-\xi_m^{\Delta}(t))\hat{e}_{\lambda}(t,0) \mid X_l(t-\xi_m^{\Delta}(t)) \mid_{\mathbb{H}}^2\right) \\
&+ \sum_{l=1}^{n}\sum_{m=1}^{n} a_l^*\left(d_{lm}^* L_l^h\right)^2 \int_0^{\infty} N_{lm}^2(z)\left(\hat{e}_{\lambda}(t,0)\hat{e}_{\lambda}(\sigma(z),0) \mid X_l(t) \mid_{\mathbb{H}}^2 - \hat{e}_{\lambda}(t,0) \mid X_l(t-z) \mid_{\mathbb{H}}^2\right)\Delta z, \\
&\leq \hat{e}_{\lambda}(t,0)\sum_{l=1}^{n}\left(\lambda + \frac{1}{a_l^*} - \breve{a}_l + \sum_{m=1}^{n} a_l^*\left(b_{lm}^* L_l^f\right)^2 + \frac{\exp(\lambda(\bar{\mu}+\bar{\xi}))}{1-\xi^*}\sum_{m=1}^{n} a_l^*\left(c_{lm}^* L_l^g\right)^2\right. \\
&\left.+ \sum_{m=1}^{n} a_l^*\left(d_{lm}^* L_l^h\right)^2 \int_0^{\infty} N_{lm}^2(z)\hat{e}_{\lambda}(\sigma(z),0)\Delta z\right) \mid X_l(t) \mid_{\mathbb{H}}^2, \\
&< 0.
\end{aligned}
$$

For this reason, $\sum_{l=1}^{n} \mid X_l(t) \mid_{\mathbb{H}}^2 \leq \hat{e}_{\ominus\lambda}(t,0)V(0)$, with $\ominus\lambda \in \mathfrak{R}^+$, $l = 1,\ldots,n.$
It follows that $\sum_{l=1}^{n} \mid X_l(t) \mid_{\mathbb{H}}^p \leq \hat{e}_{\ominus p\lambda}(t,0)V(0)^p$, $p \geq 2$. Thus,

$$
\begin{aligned}
\sum_{l=1}^{n}\frac{1}{K}\int_{t-\bar{\xi}}^{t-\bar{\xi}+K} \mid X_l(z) \mid_{\mathbb{H}}^p \Delta z &\leq \sum_{l=1}^{n}\frac{1}{K}\int_{t}^{t+K} \mid X_l(z) \mid_{\mathbb{H}}^p \Delta z \\
&\leq \int_t^{t+K} \frac{\hat{e}_{\ominus p\lambda}(t,0)V(0)^p}{K}.
\end{aligned}
$$

Furthermore,

$$
\sum_{l=1}^{n}\frac{1}{K}\int_{t}^{t+K} \mid X_l(z) \mid_{\mathbb{H}}^p \Delta z \leq \frac{V(0)^p\hat{e}_{\ominus\lambda p}(t,0)}{K}\,\frac{(\exp(-\lambda pK)-1)}{\ominus\lambda p}.
$$

Besides,

$$
\max_{l=1,\ldots,n}\sup_{t_1\in\mathbb{T}}\left(\frac{1}{K}\int_{t_1}^{t_1+K} \mid X_l(z) \mid_{\mathbb{H}}^p \Delta z\right)^{\frac{1}{p}} \leq \frac{V(0)\hat{e}_{\ominus\lambda}(t,0)}{K^{\frac{1}{p}}}\left(\frac{(\exp(-\lambda pK)-1)}{\ominus\lambda p}\right)^{\frac{1}{p}}.
$$

We claim that

$$\| X \|_{S^p} \leq \frac{V(0)\hat{e}_{\ominus\lambda}(t,0)}{K^{\frac{1}{p}}}\left(\frac{(\exp(-\lambda pK)-1)}{\ominus\lambda p}\right)^{\frac{1}{p}} \leq M\hat{e}_{\ominus\lambda}(t,0).$$

Where $M = \frac{V(0)}{K^{\frac{1}{p}}}\left(\frac{(\exp(-\lambda pK)-1)}{\ominus\lambda p}\right)^{\frac{1}{p}}$. Therefore, the unique Stepanov-like pseudo almost automorphic solution of QVRNNs (2.1) is S^p-globally exponentially stable on time scales. This completes the proof.

Remark 3.1. *To our knowledge, there have been no results concentrated on the Stepanov-like almost pariodic, Stepanov-like pseudo almost periodic, and weighted Stepanov-like pseudo-almost periodic solution for QVFRNNs with Mixed Delays on time scales. As a consequence, the obtained results in this work are essentially new and the methods used in this paper can also be applied to study the Stepanov-like pseudo almost automorphic solution on time scales for some other models of dynamical NNs.*

□

Corollary 3.1. *If $\mathbb{H} = \mathbb{R}$, then sys (2.1) is a real-valued RNNs on $\mathbb{T}$. Assume that the following conditions hold*

(F_1) : *For all $1 \leq l \leq n$, the function $\xi_m(.) \in S^pAA(\mathbb{T},\mathbb{R}) \cap C^1_{rd}(\mathbb{T},\mathbb{R})$, such that*

$$0 \leq \xi_m(.) \leq \overline{\xi}, \quad 0 \leq \xi^* - \xi^{\Delta}_m(.) < 1 - \xi^{\Delta}_m(.).$$

(F_2) : *For all $1 \leq l,m \leq n$, the functions $a_{lm}(.)$, $b_{lm}(.)$, $c_{lm}(.)$, $d_{lm}(.) \in S^pPAA(\mathbb{T},\mathbb{R})$.*

(F_3) : *There exist positive constants L^f_l, L^g_l, L_{h_l} such that for any u, $v \in \mathbb{R}$, the activity functions f_l, g_l, $h_l \in C_{rd}(\mathbb{R},\mathbb{R})$ satisfying*

$$| f_l(u) - f_l(v) | \leq L^f_l \mid u - v \mid,$$

$$| g_l(u) - g_l(v) | \leq L^g_l \mid u - v \mid,$$

$$| h_l(u) - h_l(v) | \leq L^h_l \mid u - v \mid.$$

Furthermore, we suppose that $f_l(0) = g_l(0) = h_l(0) = 0$.

(F_4) : *For all $1 \leq l,m \leq n$, the delay kernel $N_{lm} : [0,+\infty)_{\mathbb{T}} \to [0,+\infty)$ is rd-continuous, $\Delta-$integrable and there exists $\lambda > 0$ such that $\int_0^{+\infty} N_{lm}(z)\Delta z = N^*$.*

$(F_5):$ $\varpi < 1$, *and*

$$\frac{1}{a_l^*} - \check{a}_l < -\sum_{m=1}^{n} a_l^* \left(b_{lm}^* L_l^f\right)^2 - \sum_{m=1}^{n} \frac{a_l^* \left(c_{lm}^* L_l^g\right)^2}{1-\xi^*} - \sum_{m=1}^{n} a_l^* \left(d_{lm}^* L_l^h\right)^2 \int_0^\infty N_{lm}^2(z)\Delta z.$$

Then, sys.(2.1) has a unique S^p-pseudo almost automorphic solution on time scales and this solution is S^p-globally exponentially stable.

Corollary 3.2. *If $\mathbb{H} = \mathbb{C}$, then sys (2.1) is a complex-valued RNNs on $\mathbb{T}$. Assume that the following conditions hold*

$(A_1):$ *For all $1 \le l \le n$, the function $\xi_m(.) \in S^pAA(\mathbb{T},\mathbb{C}) \cap C_{rd}^1(\mathbb{T},\mathbb{C})$, such that*

$$0 \le \xi_m(.) \le \overline{\xi}, \quad 0 \le \xi^* - \xi_m^\Delta(.) < 1 - \xi_m^\Delta(.).$$

$(A_2):$ *For all $1 \le l, m \le n$, the functions $a_{lm}(.)$, $b_{lm}(.)$, $c_{lm}(.)$, $d_{lm}(.) \in S^pPAA(\mathbb{T},\mathbb{C})$.*

$(A_3):$ *There exist positive constants L_l^f, L_l^g, L_{h_l} such that for any u, $v \in \mathbb{C}$, the activity functions f_l, g_l, $h_l \in C_{rd}(\mathbb{C},\mathbb{C})$ satisfying*

$$|\, f_l(u) - f_l(v) \,|_{\mathbb{C}} \le L_l^f \,|\, u - v \,|_{\mathbb{C}},$$

$$|\, g_l(u) - g_l(v) \,|_{\mathbb{C}} \le L_l^g \,|\, u - v \,|_{\mathbb{C}},$$

$$|\, h_l(u) - h_l(v) \,|_{\mathbb{C}} \le L_l^h \,|\, u - v \,|_{\mathbb{C}}\,.$$

And $f_l(0) = g_l(0) = h_l(0) = 0$.

$(A_4):$ *For all $1 \le l, m \le n$, the delay kernel $N_{lm} : [0,+\infty)_{\mathbb{T}} \to [0,+\infty)$ is rd-continuous, $\Delta-$integrable and there exists $\lambda > 0$ such that $\int_0^{+\infty} N_{lm}(z)\Delta z = N^*$.*

$(A_5):$ $\varpi < 1$, *and*

$$1a_l^* - \check{a}_l < -\sum_{m=1}^{n} a_l^* \left(b_{lm}^* L_l^f\right)^2 - \sum_{m=1}^{n} \frac{a_l^* \left(c_{lm}^* L_l^g\right)^2}{1-\xi^*} - \sum_{m=1}^{n} a_l^* \left(d_{lm}^* L_l^h\right)^2 \int_0^\infty N_{lm}^2(z)\Delta z.$$

Where

$$b_{lm}^* = \sup_{t\in\mathbb{T}} |\, b_{lm}(t) \,|_{\mathbb{C}}, \quad c_{lm}^* = \sup_{t\in\mathbb{T}} |\, c_{lm}(t) \,|_{\mathbb{C}}, \quad d_{lm}^* = \sup_{t\in\mathbb{T}} |\, d_{lm}(t) \,|_{\mathbb{C}}\,.$$

Therefore, sys.(2.1) has a unique S^p-pseudo almost automorphic solution on time scales and this solution is S^p-globally exponentially stable.

4. Numerical Example

In this section, we give two examples to illustrate the feasibility and effectiveness of our results derived in the previous sections.

Example 4.1. *Let $n = l = m = 1$, $p = q = 2$, $\alpha = 1$ and the coefficients are taken as follows:*

$$f_1(x_1) = g_1(x_1) = \frac{1}{8}\sin^2(t),\ h_1(x_1) = \frac{1}{8}(t-1)j,\ \xi_1 = 1,\ \ a_1 = 1 + j + i + k,$$

$$b_{11}(t) = \exp(-t) + i\cos(t),\ c_{11}(t) = j\sin(t),\ d_{11}(t) = i(\sin(t)+1),$$

$$I_{11} = \exp(-t)\cos(t) + i\sin(t) + j\exp(-t)\cos(t) + 2k\cos(t),\ N_{lm} = \exp(-t).$$

If $\mathbb{T} = \mathbb{R}$ ***:*** *We have* $\mu(t) = \sigma(t) - t = t - t = 0 \quad \forall t \in \mathbb{T}$, *then* $\bar{\mu} = 0$

we have

$$L_1^f = L_1^g = L_1^h = \frac{1}{8},\ \ \xi_1^* = 0, \quad N^* = 1,$$

$$b_{11}^+ = d_{11}^+ = 1 \quad c_{11}^+ = 2, \quad \overline{a}_1 = a_1^* = 4 \quad \check{a}_1 = 8.$$

Moroever, $L^* \simeq 0.156 < 1$.

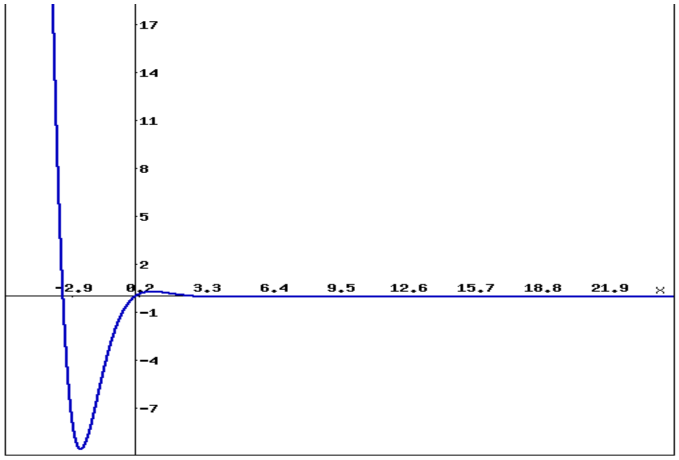

FIGURE 1. Behavior of the state variable x_1^R of QVFRNNs (2.1) $\mathbb{T} = \mathbb{R}$.

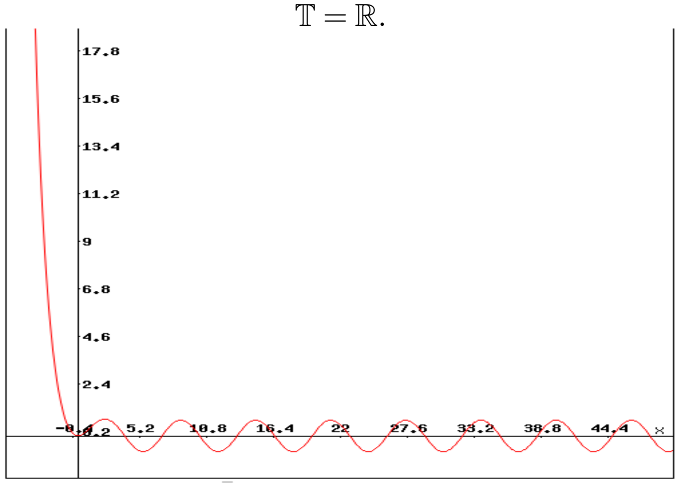

FIGURE 2. Behavior of the state variable x_1^I of QVFRNNs (2.1) $\mathbb{T} = \mathbb{R}$.

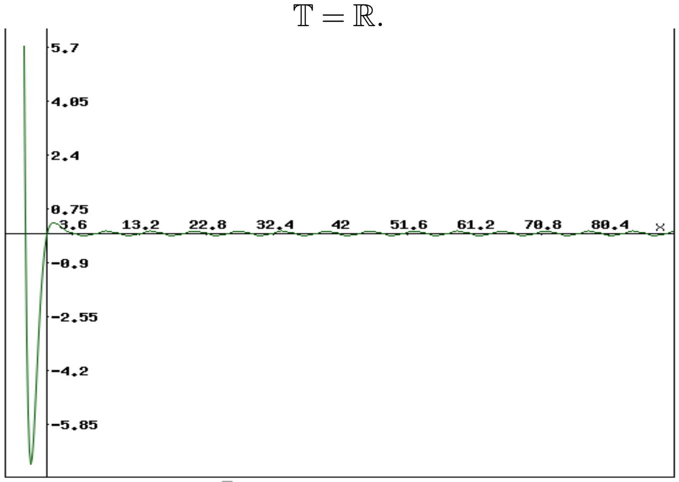

FIGURE 3. Behavior of the state variable x_1^J of QVFRNNs (2.1) $\mathbb{T} = \mathbb{R}$.

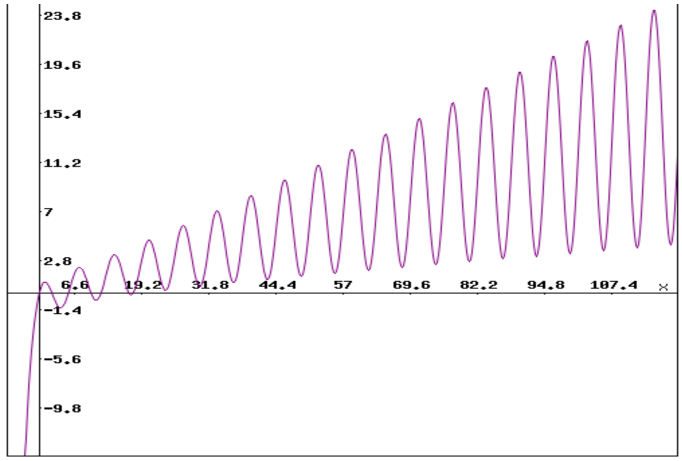

FIGURE 4. Behavior of the state variable x_1^K of QVFRNNs (2.1)
$\mathbb{T} = \mathbb{R}$

If $\mathbb{T} = \mathbb{Z}$ ***:*** *We have* $\mu(t) = \sigma(t) - t = t + 1 - t = 1 \quad \forall t \in \mathbb{T}$, *then* $\bar{\mu} = 1$ *Hence,* $L^* \simeq 0.781 < 1$, *and* $\frac{1}{a_l^*} - \breve{a}_l \simeq -7.75 < -0.43$. *Then, whether* $\mathbb{T} = \mathbb{R}$ *or* $\mathbb{T} = \mathbb{Z}$, *all the conditions of Theorems (3.2) and (3.3) are satisfied. Thus, we know that system (2.1) has a unique Stepanov-like pseudo almost automorphic solution, which is* S^p *globally exponentially stable. So, the discrete-time neural network and its continuous-time analogue have the same dynamical behaviors for the* S^p*-pseudo almost automorphy.*

Example 4.2. Let $n = 2$, $l = 1;2$, $p = q = 2$, $\lambda = 1$ and the coefficients are taken as follows:

$$
\begin{aligned}
& f_m(x_m(.)) = g_m(x_m(.)) = h_m(x_m(.)) \\
& = \frac{1}{5}\left(\cos(x_m^I(.)) + j\tan(x_m^J(.)) + k\sin(x_m^J(.) - x_m^R(.))\right), \\
& a_1(.) = 10 + 3i\sin(\sqrt{2}t), \;\; a_2(.) = 6 + k\cos(\pi t), \\
& I_l(t) = \frac{2 + \exp(it) + \exp(i\sqrt{2}t)}{\mid 2 + \exp(it) + \exp(i\sqrt{2}t) \mid} + \exp(-t),
\end{aligned}
$$

$$
N_{lm} = \exp(-s), \;\; b_{lm} = \tfrac{1}{20}\begin{bmatrix} \cos\left(\frac{1}{2+\sin(t)+\sin(\sqrt{2}t)}\right) + \exp(-t) & \frac{1}{1+t^2} + j\cos(\sqrt{2}t) \\ k\cos(\sqrt{3}t) + j\cos(\sqrt{5}t) & \sin(t) + \exp(-2t) \end{bmatrix},
$$

$$
c_{lm} = \frac{1}{20}\begin{bmatrix} j\cos(\sqrt{5}t) + \exp(-t\cos^2(t)) & k\sin\left(\frac{1}{2+\sin(t)+\sin(\sqrt{5}t)}\right) + \exp(-t^4) \\ i\sin\left(\frac{1}{2+\sin(t)+\sin(\sqrt{5}t)}\right) + i\exp(-t^4) & \cos(\pi t) + k\exp(-t^4) \end{bmatrix},
$$

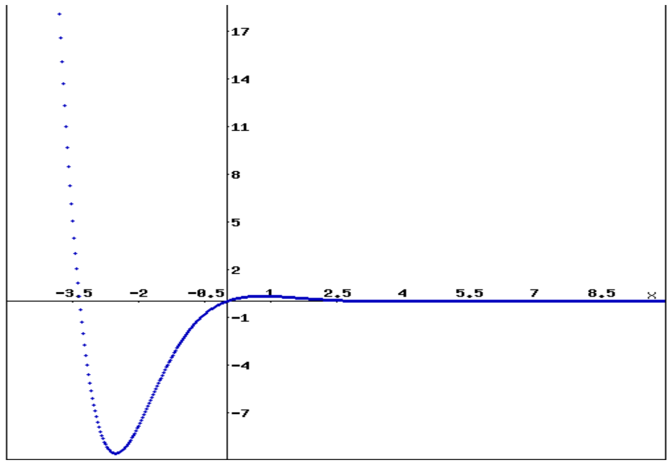

FIGURE 5. Behavior of the state variable x_1^R of QVFRNNs (2.1)

$\mathbb{T} = \mathbb{Z}$

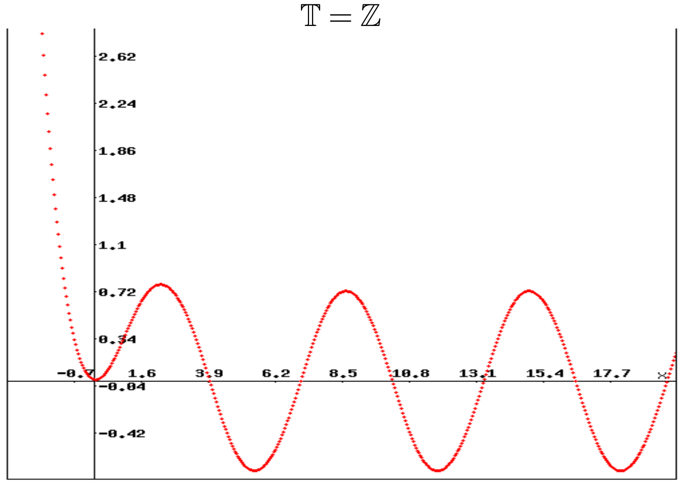

FIGURE 6. Behavior of the state variable x_1^I of QVFRNNs (2.1)

$\mathbb{T} = \mathbb{Z}$

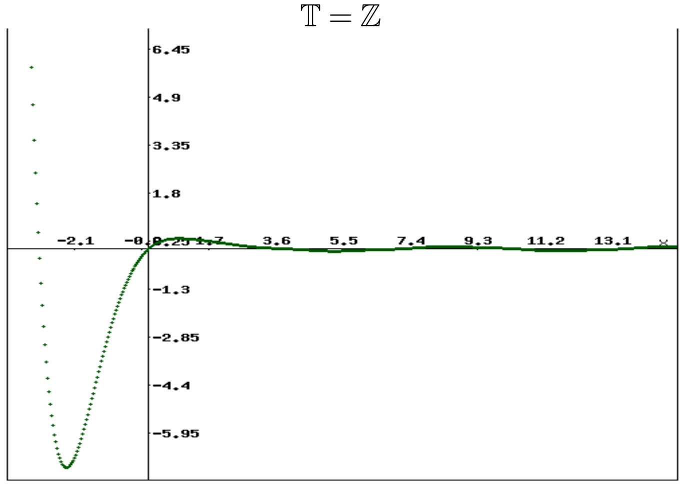

FIGURE 7. Behavior of the state variable x_1^J of QVFRNNs (2.1)

$\mathbb{T} = \mathbb{Z}$

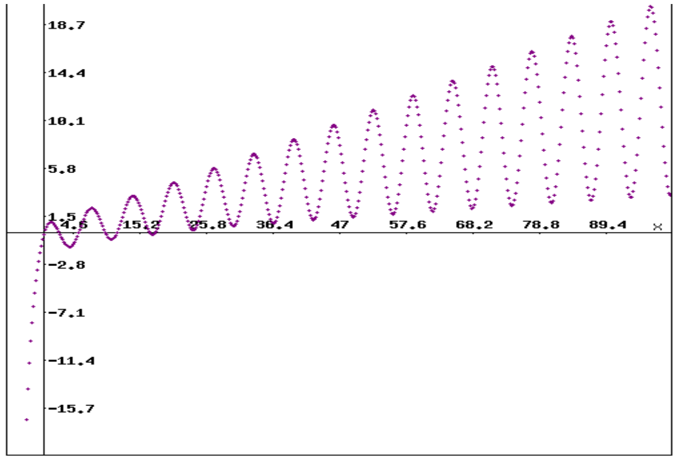

FIGURE 8. Behavior of the state variable x_1^K of QVFRNNs (2.1)

$\mathbb{T} = \mathbb{Z}$

$$d_{lm} = \frac{1}{20}\begin{bmatrix} i\sin(\sqrt{6}t) + \exp(-t^2\sin^2(t)) & \frac{1}{1+t^2} + j\sin(\sqrt{8}t) \\ j\sin(\sqrt{8}t) + k\cos(\pi t) + \exp(-t^4\cos^4(t)) & \sin(t) + i\cos(t) + \exp(-t^4\cos^2(t)) \end{bmatrix}.$$

Then, we have

$$\overline{a}_1 = 7,\ \ \overline{a}_2 = 5,\ \ \check{a}_1 = 12,\ \ \check{a}_2 = 10,\ \ a_1^* = 13,\ \ a_2^* = 7,\ \ L_m^f = L_m^g = L_m^h = \frac{1}{5},\ \ N^* = 1,$$

$b_{lm}^* = c_{lm}^* = \frac{1}{20}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $d_{lm}^* = \frac{1}{20}\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$,

If $\mathbb{T} = \mathbb{R}$: Take $\xi_1(t) = \xi_2(t) = 1$, then $\xi_1^* = \xi_2^* = 0$. Hence, $\varpi \simeq \max\{0.017;\ 0.028\} = 0.017 < 1$.

If $\mathbb{T} = \mathbb{Z}$: Take $\xi_1(t) = \cos^2(\pi t + \frac{\pi}{2})$, $\xi_2(t) = 1$, then $\xi_1^* = \xi_1^* = 0$. Moreover, $\varpi \simeq \max\{0.137;\ 0.168\} = 0.168 < 1$.

Besides, $\frac{1}{a_1^*} - \check{a}_1 \simeq -11.92 < -0.036$ and $\frac{1}{a_2^*} - \check{a}_2 \simeq -9.85 < -0.013$.

Therefore, all of the conditions of theorem (3.2) and (3.3) are satisfied. So, system (2.1) has a unique S^p-pseudo almost automorphic solution which is S^p-globally exponentially stable.

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