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STABILITY OF LAMINATED BEAMS WITH SECOND SOUND

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Abstract. In this manuscript, we prove the well-posedness and exponential stability for a thermoelastic structure given by a laminated Timoshenko beam model consisting of two identical layers, taking into account that an adhesive of the small thickness is bonding these layers and produce a interfacial slip. We consider the action of the temperature difference and heat flux as stabilization mechanisms. We use semigroup approach, Lumer-Phillips theorem for existence and uniqueness of solution and Gearhart-Huang-Prüss theorem to prove the exponential stability.

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1. Introduction

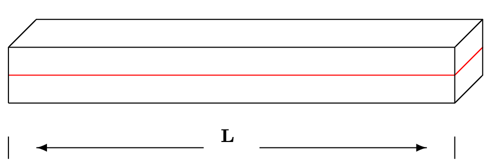
The two-layered beam with structural damping is given by the following system

$$\rho\, u_{tt} + G\,(\psi - u_x)_x = 0, \tag{1.1}$$

$$I_\rho\,(3\,S_{tt} - \psi_{tt}) - G\,(\psi - u_x) - D\,(3\,S_{xx} - \psi_{xx}) = 0, \tag{1.2}$$

$$3\,I_\rho\,S_{tt} + 3\,G(\psi - u_x) + 4\,\delta_0\,S + 4\,\gamma_0\,S_t - 3\,D\,S_{xx} = 0, \tag{1.3}$$

where $(x,\, t) \in (0,\, L) \times (0,\, \infty)$.



The model (1.1)-(1.3) was derived under assumption of the Timoshenko theory in [10], where $u(x,\, t)$ denotes the transverse displacement, $\psi(x, t)$ represents the rotation angle and $S(x,\, t)$ is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x, respectively, and $\rho,\, G,\, I_\rho,\, D,\, \delta_0,\, \gamma_0$ are

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the density of the beams, the shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness and adhesive damping of the beams. From now and on, considering $s(x, t) = 3\,S(x, t)$, $\rho_1 = \rho$, $\rho_2 = I_\rho$, $k = G$, $b = D$, $3\,\delta = \delta_0$, $3\,\gamma = \gamma_0$, we deduce from (1.1)-(1.3) the following system

$$\rho_1\, u_{tt} + k\,(\psi - u_x)_x = 0, \tag{1.4}$$

$$\rho_2\,(s-\psi)_{tt} - b\,(s-\psi)_{xx} - k\,(\psi - u_x) = 0, \tag{1.5}$$

$$\rho_2\, s_{tt} - b\, s_{xx} + 3\,k\,(\psi - u_x) + 4\,\delta\, s + 4\,\gamma\, s_t = 0. \tag{1.6}$$

Combining (1.5) and (1.6) we have

$$\rho_1\, u_{tt} + k\,(\psi - u_x)_x = 0, \tag{1.7}$$

$$\rho_2\,\psi_{tt} - b\,\psi_{xx} + 4\,k\,(\psi - u_x) + 4\,\delta\, s + 4\,\gamma\, s_t = 0. \tag{1.8}$$

Note that for $s = 0$ in (1.8) we obtain the conservative Timoshenko system. There is an extensive bibliography for the thermoelastic Timoshenko system and thermoelasticity with Cattaneo's law (second sound). We cite for instance [3,4,18,19,21,26,30–32]. For laminated beam, there are only a few papers specifically in connection with the system (1.1)-(1.3). Among them, we mention the references [6,9,15,16,27,33]. More recently, dynamics of laminated Timoshenko beams were studied in [7] where the authors established the existence of smooth finite dimensional global attractors for the corresponding solution semigroup. Hybrid laminated Timoshenko beam model was considered in [28] where the beam is fastened securely on the left while on the right it is free and has an attached container. Using the semigroup approach and a result of A. Borichev and Y. Tomilov (see [5]), the authors proved that the solution is polynomially stable.

In [33], the authors proved that the frictional damping created by the interfacial slip alone is not enough to stabilize the system (1.1)-(1.3) exponentially to its equilibrium state, then another mechanism of stabilization is necessary to be introduced to produce exponential decay of solution. With this approach, inspired by the result [29], the exponential stability for a structure with interfacial slip taking into account the frictional damping was proved in [27].

In the theory of thermoelasticity of second sound, the heat equation is of hyperbolic type and so predicts a finite speed of heat propagation. See [19,26,32] for more detail on this theory. About thermoelastic Timoshenko system in the presence of the frictional damping and heat conduction modelled by Cattaneo's law for the system bellow

$$\begin{aligned}
\rho_1\,\varphi_{tt} - \sigma\,(\varphi_x + \psi)_x + \mu\,\varphi_t &= 0,\\
\rho_2\,\psi_{tt} - b\,\psi_{xx} + k\,(\varphi_x + \psi) + \beta\,\theta_x &= 0,\\
\rho_3\,\theta_t + \gamma\, q_x + \delta\,\psi_{tx} &= 0,\\
\tau_0\, q_t + q + \kappa\,\theta_x &= 0,
\end{aligned}$$

several exponential decay results for both linear and nonlinear cases have been established in [18,19].

In the absence of the frictional damping in the work [31], it was shown that the coupling via Cattaneo's law causes loss of the exponential decay usually obtained in the case of coupling via Fourier's law. The authors considered the following system

$$
\begin{aligned}
\rho_1\,\varphi_{tt} - k\,(\varphi_x + \psi)_x &= 0,\\
\rho_2\,\varphi_{tt} - b\,\psi_{xx} + k\,(\varphi_x + \psi) + \delta\,\theta_x &= 0,\\
\rho_3\,\theta_t + q_x + \delta\,\psi_{x\,t} &= 0,\\
\tau\,q_t + \beta\,q + \theta_x &= 0.
\end{aligned}
\tag{1.9}
$$

Later, in [3] the stability number χ_0 given by

$$
\chi_0 = \left(\tau - \frac{\rho_1}{k\rho_3}\right)\left(\rho_2 - \frac{b\rho_1}{k}\right) - \frac{\tau\rho_1\delta^2}{k\rho_3}
$$

was introduced and the authors proved that for (1.9), the exponential stability holds if and only if $\chi_0 = 0$, that is, the velocities of waves propagations are equal. In fact, for all the literature concerning Timoshenko system, when only one dissipation is taking into account the conclusion is that the exponential stability holds if and only if the velocities of waves propagations are equal.

Well-posedness and asymptotic behaviour for a laminated beam system in thermoelasticity with Fourier's law and Cattaneo's law forms the centre of this work. The main difficulty carrying out this paper is the presence of Fourier's law of heat conduction that can produce lack of exponential stability when the wave speed is different for systems partially damped, see [22]. Recent investigations lead that the existence of a second spectrum is an essential element to justify from the physical point of view the imposed condition of equal wave speed. For more information on this subject, we cite [1] and reference therein.

However, if the damping terms are added in all equations (as in [27,29]), the energy of the system decays exponentially without assumption over the coefficients of the system. Motivated by the above results, we will investigate the damped system below, that is, we are interested in the asymptotic behaviour of a thermoelastic structure with interfacial slip and second sound derived from (1.1)-(1.3). More precisely, we deal with the system as follows

$$
\begin{aligned}
\rho_1\,u_{tt} + k\,(\psi - u_x)_x &= 0,\\
\rho_2\,(s-\psi)_{tt} - b\,(s-\psi)_{xx} - k\,(\psi - u_x) + \beta\,(s-\psi)_t + \mu\,\theta_x &= 0,\\
\rho_2\,s_{tt} - b\,s_{xx} + 3\,k\,(\psi - u_x) + 4\,\delta\,s + 4\,\gamma\,s_t &= 0,\\
\theta_t - \theta_{xx} + \nu\,q_x + \mu\,(s-\psi)_{x\,t} &= 0,\\
q_t + q + \nu\,\theta_x &= 0,
\end{aligned}
\tag{1.10}
$$

where $(x,\,t) \in (0,\,L) \times (0,\,\infty)$ with Dirichlet boundary conditions

$$
\begin{aligned}
u(0,\,t) = \psi(0,\,t) = s(0,\,t) = \theta(0,\,t) = q(0,\,t) = 0,\\
u(L,\,t) = \psi(L,\,t) = s(L,\,t) = \theta(L,\,t) = q(L,\,t) = 0
\end{aligned}
$$

and initial data

$$
\begin{aligned}
(u(x,\,0),\,\psi(x,\,0),\,s(x,\,0),\,\theta(x,\,0),\,q(x,\,0)) &= (u_0(x),\,\psi_0(x),\,s_0(x),\,\theta_0(x),\,q_0(x)),\\
(u_t(x,\,0),\,\psi_t(x,\,0),\,s_t(x,\,0),\,\theta_t(x,\,0),\,q_t(x,\,0)) &= (u_1(x),\,\psi_1(x),\,s_1(x),\,\theta_1(x),\,q_1(x)).
\end{aligned}
$$

In this system, $u = u(x,\, t)$ is the transversal displacement, $\psi = \psi(x,\, t)$ represents the rotation angle, $s = s(x,\, t)$ is proportional to the amount of slip along the interface, $\theta = \theta(x,\, t)$ is the temperature difference, $q = q(x,\, t)$ is the heat flux and $\rho_1,\, \rho_2,\, k,\, b,\, \delta,\, \beta,\, \gamma,\, \mu,\, \nu$ are positive constants.

In order to achieve our goal, we use the Sobolev spaces and semigroup theory with its properties as in [2,24]. For asymptotic behaviour, the idea presented by Liu and Zheng [17] for dissipative systems is used. This technique is different from some others in the literature, like as the traditional energy method, see Rivera [20], the direct method, see Kormonik [13,14] and Nakao's method, see [23]. This manuscript is organized as follows. In Section 2, we deal with setting of the semigroup, where we prove the well-posedness of the problem. In Section 3, we show the exponential stability using the result obtained by Gearhart [8] and Huang [11] independently (see also Prüss [25]).

2. Setting of the Semigroup

In this section, for convenience, we introduce another variables ξ and z given by

$$\xi = s - \psi.$$

Then (1.10) change to

$$\rho_1\, u_{tt} + k\,(s - \xi - u_x)_x = 0, \tag{2.1}$$

$$\rho_2\, \xi_{tt} - b\, \xi_{xx} - k\,(s - \xi - u_x) + \beta\, \xi_t + \mu\, \theta_x = 0, \tag{2.2}$$

$$\rho_2\, s_{tt} - b\, s_{xx} + 3\, k\,(s - \xi - u_x) + 4\, \delta\, s + 4\, \gamma\, s_t = 0, \tag{2.3}$$

$$\theta_t - \theta_{xx} + \nu\, q_x + \mu\, \xi_{x\,t} = 0, \tag{2.4}$$

$$q_t + q + \nu\, \theta_x = 0, \tag{2.5}$$

where $(x,\, t) \in (0,\, L) \times (0,\, \infty)$ with boundary conditions

$$\begin{aligned} u(0,\, t) = \xi(0,\, t) = s(0,\, t) = \theta(0,\, t) = q(0,\, t) = 0, \\ u(L,\, t) = \xi(L,\, t) = s(L,\, t) = \theta(L,\, t) = q(L,\, t) = 0 \end{aligned} \tag{2.6}$$

and initial data

$$\begin{aligned} (u(x,\, 0),\, \xi(x,\, 0),\, s(x,\, 0),\, \theta(x,\, 0),\, q(x,\, 0)) = (u_0(x),\, \xi_0(x),\, s_0(x),\, \theta_0(x),\, q_0(x)), \\ (u_t(x,\, 0),\, \xi_t(x,\, 0),\, s_t(x,\, 0),\, \theta_t(x,\, 0),\, q_t(x,\, 0)) = (u_1(x),\, \xi_1(x),\, s_1(x),\, \theta_1(x),\, q_1(x)). \end{aligned} \tag{2.7}$$

For the standard $L^2(0,\, L)$ space, the scalar product and the norm are denoted by

$$\langle \varphi,\, \psi \rangle_{L^2(0,\, L)} = \int_0^L \varphi\, \psi\, dx, \quad \|\psi\|^2_{L^2(0,\, L)} = \int_0^L |\psi|^2\, dx.$$

Introducing the vector function $U = (u,\, u_t,\, \xi,\, \xi_t,\, s,\, s_t,\, \theta,\, q)^T$, the system (2.1)-(2.7) can be written as

$$\left\{ \begin{array}{rcl} U_t - \mathcal{A}U & = & 0, \\ U(0) & = & U_0 = (u_0,\, u_1,\, \xi_0,\, \xi_1,\, s_0,\, s_1,\, \theta_0,\, q_0)^T, \end{array} \right. \tag{2.8}$$

where the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ u_t \\ \xi \\ \xi_t \\ s \\ s_t \\ \theta \\ q \end{pmatrix} = \begin{pmatrix} u_t \\ -\frac{k}{\rho_1}\,(s-\xi-u_x)_x \\ \xi_t \\ \frac{1}{\rho_2}\,[b\,\xi_{xx} + k\,(s-\xi-u_x) - \beta\,\xi_t - \mu\,\theta_x] \\ s_t \\ \frac{1}{\rho_2}\,[b\,s_{xx} - 3\,k\,(s-\xi-u_x) - 4\,\delta\,s - 4\,\gamma\,s_t] \\ \theta_{xx} - \nu\,q_x - \mu\,\xi_{xt} \\ -\,q - \nu\,\theta_x \end{pmatrix}, \tag{2.9}$$

where the space $\mathcal{H}$ is given by

$$\mathcal{H} = \left[H_0^1(0,\,L) \times L^2(0,\,L)\right]^3 \times H_0^1(0,\,L)^2$$

is a Hilbert space equipped with the inner product given by

$$\begin{aligned} \langle U,\,\bar{U}\rangle_{\mathcal{H}} \;=\; & 3\,\rho_1 \int_0^L u_t\,\bar{u}_t\,dx + 3\,k \int_0^L (s-\xi-u_x)\,(\bar{s}-\bar{\xi}-\bar{u}_x)\,dx \\ & + \rho_2 \int_0^L s_t\,\bar{s}_t\,dx + b \int_0^L s_x\,\bar{s}_x\,dx + 4\,\delta \int_0^L s\,\bar{s}\,dx + 3 \int_0^L \theta\,\bar{\theta}\,dx \\ & + 3 \int_0^L q\,\bar{q}\,dx + 3\,\rho_2 \int_0^L \xi_t\,\bar{\xi}_t\,dx + 3\,b \int_0^L \xi_x\,\bar{\xi}_x\,dx \end{aligned} \tag{2.10}$$

and the correspondent norm is

$$\begin{aligned} \|U\|_{\mathcal{H}}^2 \;=\; & 3\,\rho_1 \int_0^L |u_t|^2\,dx + 3\,k \int_0^L |s-\xi-u_x|^2\,dx + \rho_2 \int_0^L |s_t|^2\,dx + b \int_0^L |s_x|^2\,dx \\ & + 4\,\delta \int_0^L |s|^2\,dx + 3 \int_0^L |\theta|^2\,dx + 3 \int_0^L |q|^2\,dx + 3\,\rho_2 \int_0^L |\xi_t|^2\,dx + 3\,b \int_0^L |\xi_x|^2\,dx. \end{aligned}$$

Denoting $\Lambda = (u,\,u_t,\,\xi,\,\xi_t,\,s,\,s_t,\,\theta,\,q)^T$, the domain of $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A}) \;=\; \left\{\Lambda \in \mathcal{H};\;\; u,\xi,s,\theta \in H^2(0,\,L) \cap H_0^1(0,\,L), \quad u_t,\xi_t,s_t,q \in H_0^1(0,\,L)\right\}.$$

Note that $\mathcal{D}(\mathcal{A})$ is independent of time $t > 0$ and clearly, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Next, will prove that the operator $\mathcal{A}$ is dissipative.

Proposition 2.1. *For $U = (u,\,u_t,\,\xi,\,\xi_t,\,s,\,s_t,\,\theta,\,q)^T \in D(\mathcal{A})$ we have*

$$\langle \mathcal{A}U,\,U\rangle_{\mathcal{H}} = -4\gamma \int_0^L |s_t|^2 dx - 3 \int_0^L |\theta_x|^2 dx - 3 \int_0^L |q|^2 dx - 3\beta \int_0^L |\xi_t|^2 dx \le 0. \tag{2.11}$$

Proof. Direct computation, using (2.10), gives

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_{\mathcal{H}} = & -3k\langle (s-\xi-u_x)_x, u_t\rangle_{L^2(0,L)} + 3k\langle (s-\xi-u_x)_t, (s-\xi-u_x)\rangle_{L^2(0,L)} \\ & + \langle bs_{xx} - 3k(s-\xi-u_x) - 4\delta s - 4\gamma s_t, s_t\rangle_{L^2(0,L)} + b\langle s_{xt}, s_x\rangle_{L^2(0,L)} \\ & + 4\gamma\langle s_t, s\rangle_{L^2(0,L)} + 3\langle \theta_{xx} - \nu q_x - \mu\xi_{xt}, \theta\rangle_{L^2(0,L)} - 3\langle q + \nu\theta_x, q\rangle_{L^2(0,L)} \\ & + 3\langle b\xi_{xx} + k(s-\xi-u_x) - \beta\xi_t - \mu\theta_x, \xi_t\rangle_{L^2(0,L)} + 3b\langle \xi_{xt}, \xi_x\rangle_{L^2(0,L)}. \end{aligned}$$

Integrating by parts on $(0, L)$ and simplifying the terms yield (2.11). □

We now consider the following result.

Proposition 2.2. *Let $\varrho(\mathcal{A})$ be the resolvent set of the operator $\mathcal{A}$. We have that $0 \in \varrho(\mathcal{A})$.*

Proof. For $F = (f^1, f^2, f^3, f^4, f^5, f^6, f^7, f^8)^T \in \mathcal{H}$, we must show that there exists a unique $U = (u, u_t, \xi, \xi_t, s, s_t, \theta, q)^T$ in $\mathcal{D}(\mathcal{A})$ such that $\mathcal{A}U = F$, that is,

$$u_t = f^1 \quad \text{in} \quad H_0^1(0, L), \tag{2.12}$$

$$(s - \xi - u_x)_x = -\frac{\rho_1}{k} f^2 \quad \text{in} \quad L^2(0, L), \tag{2.13}$$

$$\xi_t = f^3 \quad \text{in} \quad H_0^1(0, L), \tag{2.14}$$

$$b\,\xi_{xx} + k\,(s - \xi - u_x) - \beta\,\xi_t - \mu\,\theta_x = \rho_2\, f^4 \quad \text{in} \quad L^2(0, L), \tag{2.15}$$

$$s_t = f^5 \quad \text{in} \quad H_0^1(0, L), \tag{2.16}$$

$$b\,s_{xx} - 3\,k\,(s - \xi - u_x) - 4\,\delta\,s - 4\,\gamma\,s_t = \rho_2\, f^6 \quad \text{in} \quad L^2(0, L), \tag{2.17}$$

$$\theta_{xx} - \nu\,q_x - \mu\,\xi_{xt} = f^7 \quad \text{in} \quad H_0^1(0, L), \tag{2.18}$$

$$-\,q - \nu\,\theta_x = f^8 \quad \text{in} \quad H_0^1(0, L). \tag{2.19}$$

Replacing (2.14) in (2.15), (2.16) in (2.17) and (2.14) into (2.18) we obtain

$$(s - \xi - u_x)_x = -\frac{\rho_1}{k} f^2 \quad \text{in} \quad L^2(0, L), \tag{2.20}$$

$$b\,\xi_{xx} + k\,(s - \xi - u_x) - \mu\,\theta_x = \beta\, f^3 + \rho_2\, f^4 \quad \text{in} \quad L^2(0, L), \tag{2.21}$$

$$b\,s_{xx} - 3\,k\,(s - \xi - u_x) - 4\,\delta\,s = 4\,\gamma\, f^5 + \rho_2\, f^6 \quad \text{in} \quad L^2(0, L), \tag{2.22}$$

$$\theta_{xx} - \nu\,q_x = \mu\, f_x^3 + f^7 \quad \text{in} \quad L^2(0, L), \tag{2.23}$$

$$-\,q - \nu\,\theta_x = f^8 \quad \text{in} \quad H_0^1(0, L). \tag{2.24}$$

Using boundary conditions (2.6) and (2.20) we have

$$-\frac{\rho_1}{k}\int_0^x f^2(y)\,dy = s - \xi - u_x \quad \text{in} \quad L^2(0, L). \tag{2.25}$$

From (2.23) and (2.24) we obtain

$$\begin{aligned} -\nu\,q + \theta_x &= \mu\int_0^x f_x^3(y)\,dy + \int_0^x f^7(y)\,dy \quad \text{in} \quad L^2(0, L), \\ \nu\,q + \nu^2\,\theta_x &= -\nu\, f^8 \quad \text{in} \quad L^2(0, L), \end{aligned}$$

then, we get

$$\theta_x = \frac{1}{1+\nu^2}\left[\mu\int_0^x f_x^3(y)\,dy + \int_0^x f^7(y)\,dy - \nu\, f^8\right] \quad \text{in} \quad L^2(0, L) \tag{2.26}$$

and we can write

$$b\,\xi_{xx} = F^1 \quad \text{in} \quad L^2(0, L), \tag{2.27}$$

$$b\,s_{xx} - 4\,\delta\,s = F^2 \quad \text{in} \quad L^2(0, L), \tag{2.28}$$

$$(1+\nu^2)\theta_{xx} = F^3 \quad \text{in} \quad L^2(0, L), \tag{2.29}$$

where

$$F^1 := \rho_1 \int_0^x f^2(y)\,dy + \frac{\mu}{1+\nu^2}\left[\mu \int_0^x f_x^3(y)\,dy + \int_0^x f^7(y)\,dy - \nu\, f^8\right] + \beta\, f^3 + \rho_2\, f^4,$$

$$F^2 := -3\rho_1 \int_0^x f^2(y)\,dy + 4\,\gamma\, f^5 + \rho_2\, f^6,$$

$$F^3 := \mu\, f_x^3 + f^7 - \nu f_x^8.$$

By the standard result of elliptical regularity (see [12], theorem 3.3.3, page 135.), the system (2.27)-(2.29) has a unique solution

$$\xi, s, \theta \in H^2(0,\, L) \cap H_0^1(0,\, L).$$

Then we plug ξ and s just obtained by solving (2.27) and (2.28) into (2.20) and we yields a unique solvability of $u \in H^2(0,\, L) \cap H_0^1(0,\, L)$. Finally, by the regularity of u, ξ, s, z, θ and (2.12),(2.14),(2.16),(2.19), we have that $U \in \mathcal{D}(\mathcal{A})$ is a unique solution of $\mathcal{A}U = F$. It is clear from the regularity of the linear elliptic equations that $\|U\|_{\mathcal{H}} \leq K\,\|F\|_{\mathcal{H}}$, for a positive constant K independent of U. So we conclude that 0 belongs to resolvent set of $\mathcal{A}$. □

The well-posedness of (1.10) is ensured by the following theorem.

Theorem 2.1. *For $U_0 \in \mathcal{H}$, there exists a unique weak solution U of (2.8) satisfying*

$$U \in C((0,\infty); \mathcal{H}). \tag{2.30}$$

Moreover, if $U_0 \in D(\mathcal{A})$, then

$$U \in C((0,\infty); D(\mathcal{A})) \cap C^1((0,\infty); \mathcal{H}). \tag{2.31}$$

Proof. As $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$, $\mathcal{A}$ is dissipative and $0 \in \varrho(\mathcal{A})$, we conclude by the Lumer-Phillips theorem (see [24]) that $\mathcal{A}$ generates a C_0-semigroup of contractions $S(t) = e^{t\mathcal{A}}$ on $\mathcal{H}$. From semigroup theory, $U(t) = e^{t\mathcal{A}}U_0$ is the unique solution of (2.8) satisfying (2.30) and (2.31). □

3. Exponential stability

In this section the first result that we are going to present is about the necessary and sufficient conditions of exponential stability of a C_0-semigroup on a Hilbert space $\mathcal{H}$. This result was obtained by Gearhart [8] and Huang [11] independently (see also Prüss [25]). The following statement is due to Huang.

Theorem 3.1. *Let $\{\mathcal{S}(t)\}_{t\geq 0}$ be a C_0-semigroup of contractions of a linear operators on Hilbert space $\mathcal{H}$ with infinitesimal generator $\mathcal{A}$. Then $\{\mathcal{S}(t)\}_{t\geq 0}$ is exponentially stable if and only if*

$$\{i\,\zeta :\ \zeta \in \mathbb{R}\} \subset \varrho(\mathcal{A})$$

and

$$\limsup_{|\zeta|\to\infty} \|(i\,\zeta\, I - \mathcal{A})^{-1}\| < \infty.$$

Now we introduce our main theorem.

Theorem 3.2 (Main Theorem)**.** *The C_0-semigroup of contractions $\mathcal{S}(t) = e^{\mathcal{A}\,t}$, generated by $\mathcal{A}$, is exponentially stable, that is, there exist positive constants M and w such that*

$$\|S(t)\| \leq M\, e^{-\,w\,t}.$$

Proof. First, by a contradiction argument, we will prove that

$$\{i\,\zeta :\ \zeta \in \mathbb{R}\} \subset \varrho(\mathcal{A}).$$

The proof will be covered by three steps.

(a) Let I be the identity operator. Since 0 is in the resolvent of $\mathcal{A}$, by the contraction mapping theorem, for any real ζ such that $|\zeta| < \|\mathcal{A}^{-1}\|^{-1}$, the operator $i\,\zeta\,I - \mathcal{A} = \mathcal{A}(i\,\zeta\,\mathcal{A}^{-1} - I)$ is invertible. Moreover, $\|(i\,\zeta\,I - \mathcal{A})^{-1}\|$ is a continuous function of ζ in the interval $(-\|\mathcal{A}^{-1}\|^{-1},\ \|\mathcal{A}^{-1}\|^{-1})$.

(b) If $\sup\{(i\,\zeta\,I - \mathcal{A})^{-1}\ :\ |\zeta| < \|\mathcal{A}^{-1}\|^{-1}\} = M < \infty$, then by using the contraction theorem again, the operator

$$i\,\zeta\,I - \mathcal{A} = (i\,\zeta_0\,I - \mathcal{A})\left(I + i\,(\zeta - \zeta_0)(i\,\zeta_0\,I - \mathcal{A})^{-1}\right)$$

is invertible for $|\zeta - \zeta_0| < M^{-1}$. Hence choosing ζ_0 close enough to $\|\mathcal{A}^{-1}\|^{-1}$, the set $\{\zeta :\ |\zeta| < \|\mathcal{A}^{-1}\|^{-1} + M^{-1}\}$ is contained in the resolvent set of $\mathcal{A}$ and $\|(i\,\zeta\,I - \mathcal{A})^{-1}\|$ is a continuous function of ζ in the interval $(-\|\mathcal{A}^{-1}\|^{-1} - M^{-1},\ \|\mathcal{A}^{-1}\|^{-1} + M^{-1})$.

(c) Suppose that the statement of this theorem is not true. Then, there exists a real number $\omega \neq 0$ with $\|\mathcal{A}^{-1}\|^{-1} \leq |\omega| < \infty$ satisfying that the set $\{i\,\zeta\ :\ |\zeta| < |\omega|\}$ is in the resolvent of $\mathcal{A}$ and $\sup\{\|(i\,\zeta\,I - \mathcal{A})^{-1}\|\ :\ |\zeta| < |\omega|\} = \infty$. In this situation we can find a sequence of real numbers $\zeta_n \in \mathbb{R}$ with $\zeta_n \to \omega$, $|\zeta_n| < |\omega|$ and a sequence of complex vector functions $U_n \in \mathcal{D}(\mathcal{A})$ with $\|U_n\| = 1$ such that $\|(i\,\zeta_n\,I - \mathcal{A})U_n\| \to 0$, as $n \to \infty$.

But, $(i\,\zeta_n\,I - \mathcal{A})U_n$ is given by

$$i\,\zeta_n \begin{pmatrix} u_n \\ u_{n,t} \\ \xi_n \\ \xi_{n,t} \\ s_n \\ s_{n,t} \\ \theta_n \\ q_n \end{pmatrix} - \begin{pmatrix} u_{n,t} \\ -\frac{k}{\rho_1}\,(s_n - \xi_n - u_{n,x})_x \\ \xi_{n,t} \\ \frac{1}{\rho_2}\,[b\,\xi_{n,\,xx} + k\,(s_n - \xi_n - u_{n,x}) - \beta\,\xi_{n,t} - \mu\,\theta_{n,\,x}] \\ s_{n,t} \\ \frac{1}{\rho_2}\,[b\,s_{n,\,xx} - 3\,k\,(s_n - \xi_n - u_{n,x}) - 4\,\delta\,s_n - 4\,\gamma\,s_{n,t}] \\ \theta_{n,\,xx} - \nu\,q_{n,\,x} - \mu\,\xi_{n,\,xt} \\ -\,q_n - \nu\,\theta_{n,\,x} \end{pmatrix},$$

then we have

$$i\,\zeta_n\,u_n - u_{n,t} \longrightarrow 0, \tag{3.1}$$

$$i\,\zeta_n\,u_{n,t} + \frac{k}{\rho_1}\,(s_n - \xi_n - u_{n,x})_x \longrightarrow 0, \tag{3.2}$$

$$i\,\zeta_n\,\xi_n - \xi_{n,t} \longrightarrow 0, \tag{3.3}$$

$$i\,\zeta_n\,\xi_{n,t} - \frac{1}{\rho_2}\,[b\,\xi_{n,\,xx} + k\,(s_n - \xi_n - u_{n,x}) - \beta\,\xi_{n,t} - \mu\,\theta_{n,\,x}] \longrightarrow 0, \tag{3.4}$$

$$i\,\zeta_n\,s_n - s_{n,t} \longrightarrow 0, \tag{3.5}$$

$$i\,\zeta_n\,s_{n,t} - \frac{1}{\rho_2}\,[b\,s_{n,\,xx} - 3\,k\,(s_n - \xi_n - u_{n,x}) - 4\,\delta\,s_n - 4\,\gamma\,s_{n,t}] \longrightarrow 0, \tag{3.6}$$

$$i\,\zeta_n\,\theta_n - \theta_{n,\,xx} + \nu\,q_{n,\,x} + \mu\,\xi_{n,\,xt} \longrightarrow 0, \tag{3.7}$$

$$i\,\zeta_n\,q_n + q_n + \nu\,\theta_{n,\,x} \longrightarrow 0. \tag{3.8}$$

Taking the inner product of $(i\,\zeta_n\,I - \mathcal{A})U_n$ with U_n in $\mathcal{H}$ we obtain

$$\langle (i\,\zeta_n\,I - \mathcal{A})U_n,\,U_n\rangle_{\mathcal{H}} = i\,\langle \zeta_n\,U_n,\,U_n\rangle_{\mathcal{H}} - \langle \mathcal{A}\,U_n,\,U_n\rangle_{\mathcal{H}}.$$

Taking the real part and using (2.11) after apply Poincaré inequality we obtain

$$4\,\gamma\int_0^L |s_{n,t}|^2\,dx + \frac{3}{C_P}\int_0^L |\theta_n|^2\,dx + 3\int_0^L |q_n|^2\,dx + 3\,\beta\int_0^L |\xi_{n,t}|^2\,dx \\ \leq Re\,\langle (i\,\zeta_n\,I - \mathcal{A})U_n,\,U_n\rangle_{\mathcal{H}}\,.$$

Noting that $(U_n)_{n\in\mathbb{N}}$ is bounded and $(i\,\zeta_n\,I - \mathcal{A})U_n \to 0$ we obtain

$$4\,\gamma\int_0^L |s_{n,t}|^2\,dx + \frac{3}{C_P}\int_0^L |\theta_n|^2\,dx + 3\int_0^L |q_n|^2\,dx + 3\,\beta\int_0^L |\xi_{n,t}|^2\,dx \\ \leq Re\,\langle (i\,\zeta_n\,I - \mathcal{A})U_n,\,U_n\rangle_{\mathcal{H}} \longrightarrow 0.$$

It follows that

$$s_{n,t} \to 0 \quad \text{as} \quad n \to \infty, \tag{3.9}$$

$$\theta_n \to 0 \quad \text{as} \quad n \to \infty, \tag{3.10}$$

$$q_n \to 0 \quad \text{as} \quad n \to \infty, \tag{3.11}$$

$$\xi_{n,t} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.12}$$

From (3.3), (3.5), (3.9) and (3.12) we obtain

$$\xi_n \to 0 \quad \text{as} \quad n \to \infty, \tag{3.13}$$

$$s_n \to 0 \quad \text{as} \quad n \to \infty. \tag{3.14}$$

Since

$$\|\xi_n\|^2_{H^1_0(0,L)} = |\xi_n|^2_{L^2(0,L)} + |\xi_{n,x}|^2_{L^2(0,L)},$$

we get

$$\xi_{n,x} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.15}$$

Similarly, we have that

(3.16) $$\theta_{n,x} \to 0 \quad \text{as} \quad n \to \infty.$$

Taking the inner product of the expressions (3.2) and (3.4) with $\rho_1 u_n$ and $-\rho_2 \psi_n$, respectively, adding the results and taking the real part, we obtain

$$k\text{Re}\left(\|s_n - \xi_n - u_{n,x}\|^2_{L^2(0,L)}\right) - b\langle \xi_{n,x}, \psi_{n,x}\rangle_{L^2(0,L)}$$
$$- \beta\langle \xi_{n,t}, \psi_n\rangle_{L^2(0,L)} - \mu\langle \theta_{n,x}, \psi_n\rangle_{L^2(0,L)} \longrightarrow 0.$$

Thus, it follows that

$$k\text{Re}\left(\|s_n - \xi_n - u_{n,x}\|^2_{L^2(0,L)}\right) + b\langle |\xi_{n,x}|, |\psi_{n,x}|\rangle_{L^2(0,L)}$$
$$+ \beta\langle |\xi_{n,t}|, |\psi_n|\rangle_{L^2(0,L)} + \mu\langle |\theta_{n,x}|, |\psi_n|\rangle_{L^2(0,L)} \longrightarrow 0$$

and by (3.12), (3.15) and (3.16), we have

(3.17) $$s_n - \xi_n - u_{n,x} \to 0 \quad \text{as} \quad n \to \infty,$$

then as

$$\|s_n - \xi_n - u_{n,x}\|^2_{H^1_0(0,L)} = |s_n - \xi_n - u_{n,x}|^2_{L^2(0,L)} + |(s_n - \xi_n - u_{n,x})_x|^2_{L^2(0,L)},$$

we obtain

(3.18) $$(s_n - \xi_n - u_{n,x})_x \to 0 \quad \text{as} \quad n \to \infty.$$

From (3.2) and (3.18), we get

(3.19) $$u_{n,t} \to 0 \quad \text{as} \quad n \to \infty.$$

Finally, using (3.1) and (3.19), we obtain

(3.20) $$u_n \to 0 \quad \text{as} \quad n \to \infty.$$

Hence from (3.9)-(3.12), (3.13), (3.14), (3.19) and (3.20) contradict $\|U_n\| = 1$, then it is proved that

$$\{i\,\zeta : \ \zeta \in \mathbb{R}\} \subset \varrho(\mathcal{A}).$$

Again, by a contradiction argument we will prove the

(3.21) $$\limsup_{|\zeta|\to\infty} \|(i\,\zeta\, I - \mathcal{A})^{-1}\| < \infty.$$

If (3.21) is not true, then there exists a sequence $(V_n)_{n\in\mathbb{N}}$ such that

$$\frac{\|(\lambda_n\, I - \mathcal{A})^{-1} V_n\|}{\|V_n\|} \geq n, \quad \text{where} \quad \lambda_n = i\,\zeta_n.$$

Hence $\|(\lambda_n\, I - \mathcal{A})^{-1} V_n\| \geq n\,\|V_n\|$. Since $\lambda_n \in \varrho(\mathcal{A})$ it follows that there exists a unique sequence $(U_n)_{n\in\mathbb{N}}$ such that $\lambda_n\, U_n - \mathcal{A}U_n = V_n$ with $\|U_n\| = 1$ and then

$$U_n = (\lambda_n\, I - \mathcal{A})^{-1} V_n,$$

with

$$\|U_n\| \geq n\,\|\lambda_n\,U_n - \mathcal{A}U_n\|.$$

We define $F_n = \lambda_n\,U_n - \mathcal{A}U_n$ and we have

$$\|F_n\| \leq \frac{1}{n},$$

where it is follows that $F_n \to 0$ strong in $\mathcal{H}$ as $n \to \infty$.

Now taking the inner product of F_n with U_n we have

$$\lambda_n\langle U_n,\,U_n\rangle_{\mathcal{H}} - \langle \mathcal{A}U_n,\,U_n\rangle_{\mathcal{H}} = \langle F_n,\,U_n\rangle_{\mathcal{H}}, \quad \lambda_n = i\,\zeta_n. \tag{3.22}$$

Taking the real part of (3.22) we have

$$Re\langle F_n,\,U_n\rangle_{\mathcal{H}} = -\,\langle \mathcal{A}U_n,\,U_n\rangle_{\mathcal{H}}.$$

Using that $\|U_n\|$ is bounded, $F_n \to 0$ and (2.11), we have

$$4\,\gamma\int_0^L |s_{n,t}|^2\,dx + \frac{3}{C_P}\int_0^L |\theta_n|^2\,dx + 3\int_0^L |q_n|^2\,dx + 3\,\beta\int_0^L |\xi_{n,t}|^2\,dx \\ \leq Re\langle F_n,\,U_n\rangle_{\mathcal{H}} \longrightarrow 0.$$

Thus

$$s_{n,t} \to 0 \quad \text{as} \quad n \to \infty, \tag{3.23}$$

$$\theta_n \to 0 \quad \text{as} \quad n \to \infty, \tag{3.24}$$

$$q_n \to 0 \quad \text{as} \quad n \to \infty, \tag{3.25}$$

$$\xi_{n,t} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.26}$$

From $F_n = \lambda_n\,U_n - \mathcal{A}U_n$ we have

$$\lambda_n\,u_n - u_{n,t} = f_{1n}, \tag{3.27}$$

$$\lambda_n\,u_{n,t} + \frac{k}{\rho_1}\,(s_n - \xi_n - u_{n,x})_x = f_{2n}, \tag{3.28}$$

$$\lambda_n\,\xi_n - \xi_{n,t} = f_{3n}, \tag{3.29}$$

$$\lambda_n\,\xi_{n,t} - \frac{1}{\rho_2}\,[b\,\xi_{n,\,xx} + k\,(s_n - \xi_n - u_{n,x}) - \beta\,\xi_{n,t} - \mu\,\theta_{n,\,x}] = f_{4n}, \tag{3.30}$$

$$\lambda_n\,s_n - s_{n,t} = f_{5n}, \tag{3.31}$$

$$\lambda_n\,s_{n,t} - \frac{1}{\rho_2}\,[b\,s_{n,\,xx} - 3\,k\,(s_n - \xi_n - u_{n,x}) - 4\,\delta\,s_n - 4\,\gamma\,s_{n,t}] = f_{6n}, \tag{3.32}$$

$$\lambda_n\,\theta_n - \theta_{n,\,xx} + \nu\,q_{n,\,x} + \mu\,\xi_{n,\,xt} = f_{7n}, \tag{3.33}$$

$$\lambda_n\,q_n + q_n + \nu\,\theta_{n,\,x} = f_{8n}, \tag{3.34}$$

where

$$f_{in} \to 0 \quad \text{in} \quad L^2(0,L), \quad \text{for} \quad i = 1,\ldots,8. \tag{3.35}$$

From (3.23), (3.26), (3.29), (3.31) and (3.35), we obtain

$$\xi_n \to 0 \quad \text{as} \quad n \to \infty, \tag{3.36}$$

$$s_n \to 0 \quad \text{as} \quad n \to \infty. \tag{3.37}$$

For previously used arguments we have to

$$\xi_{n,x} \to 0 \quad \text{as} \quad n \to \infty, \tag{3.38}$$

$$\theta_{n,x} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.39}$$

Taking the inner product of the expressions (3.28) and (3.30) with $\rho_1 u_n$ and $\rho_2 \psi_n$, respectively, adding the results and taking the real part, we obtain

$$\begin{aligned} k\mathrm{Re}\left(\|s_n - \xi_n - u_{n,x}\|^2_{L^2(0,L)}\right) &= \rho_1 \langle f_{2,n}, u_n\rangle_{L^2(0,L)} - \rho_2 \langle f_{4,n}, \psi_n\rangle_{L^2(0,L)} \\ &\quad + b\langle \xi_{n,x}, \psi_{n,x}\rangle_{L^2(0,L)} + \beta \langle \xi_{n,t}, \psi_n\rangle_{L^2(0,L)} \\ &\quad + \mu \langle \theta_{n,x}, \psi_n\rangle_{L^2(0,L)}. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} k\mathrm{Re}\left(\|s_n - \xi_n - u_{n,x}\|^2_{L^2(0,L)}\right) &= \rho_1 \langle |f_{2,n}|, |u_n|\rangle_{L^2(0,L)} + \rho_2 \langle |f_{4,n}|, |\psi_n|\rangle_{L^2(0,L)} \\ &\quad + b\langle |\xi_{n,x}|, |\psi_{n,x}|\rangle_{L^2(0,L)} + \beta \langle |\xi_{n,t}|, |\psi_n|\rangle_{L^2(0,L)} \\ &\quad + \mu \langle |\theta_{n,x}|, |\psi_n|\rangle_{L^2(0,L)}, \end{aligned}$$

and by (3.26), (3.35), (3.38) and (3.39), we have

$$s_n - \xi_n - u_{n,x} \to 0 \quad \text{as} \quad n \to \infty, \tag{3.40}$$

so

$$(s_n - \xi_n - u_{n,x})_x \to 0 \quad \text{as} \quad n \to \infty. \tag{3.41}$$

From (3.28), (3.35) and (3.41), we get

$$u_{n,t} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.42}$$

Finally, by (3.27), (3.35) and (3.42), we obtain

$$u_n \to 0 \quad \text{as} \quad n \to \infty. \tag{3.43}$$

Then from (3.23)-(3.26), (3.36), (3.37), (3.42) and (3.43) we conclude that $U_n \to 0$ that is a contradiction with $\|U_n\| = 1$. The proof of the theorem is complete. □

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