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A CLASS OF QTAG-MODULES AND RELATED CONCEPTS

RAFIQUDDIN[1,*], AYAZUL HASAN[2]

[1]Department of Applied Mathematics, Faculty of Engineering and Technology,
Aligarh Muslim University, Aligarh-202002, India
[2]College of Applied Industrial Technology,
Jazan University, Jazan, Kingdom of Saudi Arabia
[*]Corresponding author: rafiqamt786@rediffmail.com

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Abstract. The purpose of this paper is essentially to study α-modules that depend on the notions of summability, purity, basic submodules, projectivity and injectivity. We call a $QTAG$-module an α-closed module if it is the maximal closed submodule of its closure in the α-topology. It is found that an α-closed α-module is an α-injective.

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1. Introduction and Background Material

Modules are the natural generalizations of abelian groups. The results for abelian groups can be generalized for modules after imposing some conditions on modules/rings. In 1976 Singh [12] started the study of TAG-modules satisfying the following two conditions while the rings were associative with unity.

(I) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any non-zero homomorphism $f : W \to V$ can be extended to a homomorphism $g : U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

Later on Benabdallah, Singh, Khan etc. contributed a lot to the study of TAG-modules [7,14]. In 1987 Singh made an improvement and studied the modules satisfying only the condition (I)

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and called them $QTAG$-modules. The study of $QTAG$-modules and their structure began with work of Singh in [13]. This work, executed by many authors, clearly parallels the earlier work on torsion abelian groups. They studied different notions and structures on $QTAG$-modules and developed the theory of these modules by introducing different notions and characterizing different submodules of $QTAG$-modules. Yet there is much to explore.

All the rings R considered here are associative with unity $(1 \neq 0)$ and modules M are unital $QTAG$-modules. A module M in which the lattice of its submodule is totally ordered is called a serial module; in addition, if it has finite composition length, it is called a uniserial module. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module, and for any R-module M with a unique decomposition series, $d(M)$ denotes its decomposition length. For a uniform element $x \in M,\ e(x) = d(xR)$ and $H_M(x) = \sup\left\{d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. Let us denote by M^1, the submodule of M, containing elements of infinite height. The module M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$. The module M is h-reduced if it does not contain any h-divisible submodule. In other words, it is free from the elements of infinite height. The module M is said to be bounded, if there exists an integer n such that $H_M(x) \leq n$ for every uniform element $x \in M$.

The sum of all simple submodules of M is called the socle of M, denoted by $Soc(M)$ and a submodule S of $Soc(M)$ is called a subsocle of M. The cardinality of the minimal generating set of M is denoted by $g(M)$. For all ordinals α, $f_M(\alpha)$ is the α^{th}-Ulm invariant of M and it is equal to $g\big(Soc(H_\alpha(M))/Soc(H_{\alpha+1}(M))\big)$.

A submodule N of M is h-pure in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. For an ordinal α, a submodule $N \subseteq M$ is an α-high submodule of M if N is maximal among the submodules of M that intersect $H_\alpha(M)$ trivially.

For an ordinal α, a submodule N of M is said to be an α-pure, if $H_\beta(M) \cap N = H_\beta(N)$ for all $\beta \leq \alpha$ and a submodule N of M is said to be isotype in M, if it is α-pure for every ordinal α [6]. A submodule $B \subseteq M$ is a basic submodule [10] of M, if B is h-pure in M, $B = \oplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is h-divisible.

Imitating [4], the submodules $H_k(M), k \geq 0$ form a neighborhood system of zero, thus a topology known as h-topology arises. Closed modules are also closed with respect to this topology. Thus, the closure of $N \subseteq M$ is defined as $\overline{N} = \bigcap_{k=0}^{\infty}(N + H_k(M))$. Therefore the submodule $N \subseteq M$ is closed with respect to h-topology if $\overline{N} = N$.

An h-reduced $QTAG$-module M is summable [11] if $Soc(M) = \oplus_{\beta<\alpha} S_\beta$, where S_β is the set of all elements of $H_\beta(M)$ which are not in $H_{\beta+1}(M)$, where α is the length of M. Moreover, M is called totally projective [3], if $H_\alpha(Ext(M/H_\alpha(M), M')) = 0$ for all ordinal α and $QTAG$-modules M'.

It is interesting to note that almost all the results which hold for TAG-modules are also valid for $QTAG$-modules [6]. Many results of this paper are the generalization of [5]. Our notations and terminology generally agree with those in [8] and [9].

2. Chief Results

For facilitating the exposition and for the convenience of the readers, we recall the definition of α-modules from [2].

Definition 2.1. *Let α denote the class of all $QTAG$-modules M such that $M/H_\beta(M)$ is totally projective for all ordinals $\beta < \alpha$, a limit ordinal. These modules are called α-modules.*

To develop the study, we need to prove some results, and we start with the following.

Proposition 2.1. *If N is an α-pure submodule of an α-module M, then N is itself an α-module.*

Proof. We actually only need that $N \cap H_\gamma(M) = H_\gamma(N)$ for all $\gamma < \alpha$. For then it is a simple calculation to show that $N + H_\beta(M)/H_\beta(M)$ is isotype in $M/H_\beta(M)$ for each $\beta < \alpha$. And therefore, $N + H_\beta(M)/H_\beta(M) \cong N/H_\beta(N)$ is totally projective for all $\beta < \alpha$. □

As generalized the notion of a basic submodule in [2], by defining B to be an α-basic submodule of an α-module M if B is totally projective of length at most α, B is α-pure submodule of M, and M/B is h-divisible.

In order to establish the existence of α-basic submodules we require the following notion for technical convenience.

Definition 2.2. *Let α be a limit ordinal and M a $QTAG$-module. An α-high tower of M is a well-ordered ascending chain $\{M_\beta\}_{\beta<\alpha}$ of submodules of M such that, for each β, M_β is a β-high submodule of M.*

Now we need to prove the following lemma.

Lemma 2.1. *Let α be a limit ordinal and $\{M_\beta\}_{\beta<\alpha}$ an α-high tower of a $QTAG$-module M. If each M_β is summable, then $N = \bigcup\limits_{\beta<\alpha} M_\beta$ is summable.*

Proof. As α is a limit ordinal, we may choose a strictly increasing sequence $\beta_1 < \beta_2 < \cdots < \beta_n < \ldots$ of ordinals having α as its limit. Then $N = \bigcup\limits_{n<\omega} M_{\beta_n}$. Set $T_0 = Soc(M_{\beta_1})$ and, for $n > 1$, let T_n be such that $Soc(H_{\beta_n}(M)) = T_n \oplus Soc(H_{\beta_{n+1}}(M))$ with $T_n \subseteq M_{\beta_{n+1}}$. Then we have a direct-sum decomposition $Soc(N) = \bigoplus\limits_{n<\omega} T_n$ which is normal in the sense that $H_M(t_1 + \cdots + t_n) = \min[H_M(t_1), \ldots, H_M(t_n)]$ provided $t_i \in T_i$ for $i = 1, \ldots, n$. Now each M_β is isotype, summable, and of countable length. Therefore, each subsocle of M_β is a summable subsocle of M. In particular, each T_n is a summable subsocle of M. Since the decomposition $Soc(N) = \bigoplus\limits_{n<\omega} T_n$ is normal, it follows that $Soc(N)$ is a summable subsocle of M. Since each M_β is isotype, N is itself an isotype submodule of M and consequently N is summable. □

We continue the study with the following corollary.

Corollary 2.1. *Let α be a limit ordinal and $\{M_\beta\}_{\beta<\alpha}$ an α-high tower of a $QTAG$-module M, where each M_β is totally projective, then $N = \bigcup\limits_{\beta<\alpha} M_\beta$ is totally projective of length at most α.*

Proof. As noted above, N is an isotype submodule of M and clearly N has a length at most α. Thus M_β is also a β-high submodule of N for each $\beta < \alpha$. Since N is summable by Lemma 2.1 implies that N is totally projective. □

Now we prove the following.

Theorem 2.1. *Let M be a $QTAG$-module. Then M contains an α-basic submodule if and only if M is an α-module.*

Proof. If B is an α-pure submodule of M and if M/B is h-divisible, then it follows that $M/H_\beta(M) \cong B/H_\beta(B)$ for all $\beta < \alpha$. Consequently, only α-modules can have α-basic submodules (see [2]). Suppose now that M is an α-module and select an α-high tower $\{M_\beta\}_{\beta<\alpha}$. Now $M_\beta \cong M_\beta + H_\beta(M)/H_\beta(M)$, and since M_β is isotype in M, $M_\beta + H_\beta(M)/H_\beta(M)$ is isotype in $M/H_\beta(M)$. By Corollary 2.1, $B = \bigcup\limits_{\beta<\alpha} M_\beta$ is totally projective. It is easily seen that $Soc(M) \subseteq Soc(B) + H_\beta(M)$ for each $\beta < \alpha$, and therefore B is α-pure in M. Moreover, $B \cap H_1(M) = H_1(B)$ and $Soc(M) \subseteq Soc(B) + H_\beta(M)$ for $\beta < \omega$ imply that M/B is h-divisible. Thus, B is the required α-basic submodule of M. □

Lemma 2.2. *Suppose N is an isotype submodule of a QTAG-module M and that $\{N_\beta\}_{\beta<\alpha}$ is an α-high tower of N, then there exists an α-high tower $\{M_\beta\}_{\beta<\alpha}$ of M such that, for each β, $N_\beta \subseteq M_\beta$ and $N_\beta = N \cap M_\beta$.*

Proof. Let us first note that $N_\beta = N \cap M_\beta$ is a consequence of $N_\beta \subseteq M_\beta$. Indeed, $N_\beta \subseteq M_\beta$ implies $N_\beta \subseteq N \cap M_\beta$ and $(N \cap M_\beta) \cap H_\beta(N) = (N \cap M_\beta) \cap H_\beta(M) = 0$. The maximality of a β-high submodule then yields the equality. Assume now that $\beta < \alpha$ and that for each $\gamma < \beta$ we have a γ-high submodule M_γ of M such that $N_\gamma \subseteq M_\gamma$ and $M_\eta \subseteq M_\gamma$ for all $\eta < \gamma$. In order to be able to choose the desired M_β, it suffices to show that $(N_\beta + \cup_{\gamma<\beta} M_\gamma) \cap Soc(H_\beta(M)) = 0$. Suppose $x + y \in Soc(H_\beta(M))$ where $x \in N_\beta$ and $y \in M_\gamma$ for some $\gamma < \beta$. Then $H(x') = -H(y') \in H_1(M) \cap N \cap M_\gamma = H_1(M) \cap N_\gamma = H_1(N_\gamma)$, where $d\left(\frac{xR}{x'R}\right) = d\left(\frac{yR}{y'R}\right) = 1$, and hence there is $u \in N_\gamma$ such that $x - u \in Soc(N) = Soc(N_\gamma) \oplus Soc(H_\gamma(N))$. Thus we can write $x = u + v + z$ where $v \in Soc(N_\gamma)$ and $z \in Soc(H_\gamma(N))$. Then $u + v + y = x + y - z \in H_\gamma(M) \cap M_\gamma = 0$ and $x + y = z \in N$. Therefore $y \in N \cap M_\gamma = N_\gamma \subseteq N_\beta$ and, consequently, $x + y \in N_\beta \cap H_\beta(M) = N_\beta \cap H_\beta(N) = 0$ as desired. □

Lemma 2.3. *Let M be a totally projective QTAG-module such that $M = \bigcup_{\beta<\alpha} M_\beta$ where $\{M_\beta\}_{\beta<\alpha}$ is an α-high tower. If N is an α-pure submodule of M such that for each β, $N \cap M_\beta$ is a β-high submodule of N, then N is a direct summand of M.*

Proof. We need only show that M/N is totally projective having length at most α. Since $N \cap M_\beta$ is $(\beta+1)$-pure in N and N is α-pure in M, $N \cap M_\beta$ is $(\beta+1)$-pure in M and, a fortiori, $(\beta+1)$-pure in M_β. Since M_β is totally projective, M_β is β-projective. Therefore, there is direct decomposition $M_\beta = (N \cap M_\beta) \oplus K_\beta$ for each $\beta < \alpha$. Now $M/N = \bigcup_{\beta<\alpha} M_\beta + N/N$ and $M_\beta + N/N \cong M_\beta/(M_\beta \cap N) \cong K_\beta$ is totally projective for each β. By Corollary 2.1, it is enough to show that $M_\beta + N/N$ is a β-high submodule of M/N whenever $\omega \le \beta < \alpha$. Since N is α-pure in M, we have $Soc(H_\beta(M/N)) = Soc(H_\beta(M)) + N/N$ for $\beta < \alpha$ and it then easily follows that $Soc(M/N) = Soc(M_\beta + N/N) \oplus Soc(H_\beta(M/N))$. Because of this direct decomposition, it is enough to show that $M_\beta + N/N$ is an h-pure submodule of M/N for $\beta \ge \omega$.

Now

$$\begin{aligned} Soc(M_\beta + N) &= Soc(K_\beta \oplus N) \\ &= Soc(K_\beta) \oplus Soc(N) \\ &= Soc(K_\beta) \oplus Soc(N \cap M_\beta) \oplus Soc(H_\beta(N)) \\ &= Soc(M_\beta) \oplus Soc(H_\beta(N)). \end{aligned}$$

If $\beta \geq \omega$ and if $x \in Soc(M_\beta + N)$, then we can write $x = y + z$ where $y \in Soc(M_\beta)$ and $z \in Soc(H_\beta(N)) \subseteq H_\omega(N)$. If x has finite height in M, then this height is just the height of y in M (= height of y in M_β) and thus just the height of $x = y + z$ in $M_\beta + N$. On the other hand, if x has infinite height in M, then y has infinite height in M_β and $x = y + z$ has infinite height in $M_\beta + N$, it follows that $M_\beta + N$ is an h-pure submodule of M. Thus $M_\beta + N/N$ is h-pure in M/N. □

Proposition 2.2. *Let N be an α-pure submodule of an α-module M such that N is totally projective of length at most α. Then there exists a submodule K of M such that $N \oplus K$ is an α-basic submodule of M.*

Proof. Since N is totally projective of length $\leq \alpha$, N is the union of an α-high tower $\{N_\beta\}_{\beta<\alpha}$ of itself. By Lemma 2.2, there exists an α-high tower $\{M_\beta\}_{\beta<\alpha}$ of M such that $N_\beta = N \cap M_\beta$ for each β. Let $B = \bigcup\limits_{\beta<\alpha} M_\beta$. By the proof of Theorem 2.1, B is an α-basic submodule of M. But $\{M_\beta\}_{\beta<\alpha}$ is also an α-high tower of B, and by Lemma 2.3 we have the required direct decomposition $B = N \oplus K$. □

Now we prove the following result.

Theorem 2.2. *If N is an α-pure submodule of an α-module M, then M/N is an α-module.*

Proof. Let B be an α-basic submodule of N and choose K such that $B \oplus K$ is an α-basic submodule of M. Now if $x \in Soc(N \cap K)$, we can write for each $\beta < \alpha$, $x = y_\beta + z_\beta$, where $y_\beta \in Soc(N)$ and $z_\beta \in H_\beta(N)$. Thus $-y_\beta + x \in H_\beta(B \oplus K) = H_\beta(B) \oplus H_\beta(K)$ and $x \in \bigcap\limits_{\beta<\alpha} H_\beta(K) = H_\alpha(K) = 0$. We then have a direct decomposition $N \oplus K$. If $H_1(a') \in N \oplus K$, then $H_1(a') = y + H_1(b') + c$, where $d\left(\frac{aR}{a'R}\right) = d\left(\frac{bR}{b'R}\right) = 1$, $y \in B$, $b \in N$ and $c \in K$. Since $H_1(M) \cap (B \oplus K) = H_1(B \oplus K)$, we conclude that $H_1(M) \cap (N \oplus K) = H_1(N \oplus K)$. Now $Soc(M) \subseteq Soc(B \oplus K) + H_\beta(M) \subseteq Soc(N \oplus K) + H_\beta(M)$ for all $\beta < \alpha$, and therefore $N \oplus K$ is an α-pure submodule of M. Consequently, $N \oplus K/N$ is α-pure in M/N. Also $N \oplus K/N \cong K$ and $(M/N)/(N \oplus K/N) \cong (M/B \oplus K)/[(N \oplus K)/(B \oplus K)]$ is h-divisible. We have constructed an α-basic submodule of M/N and we conclude that M/N is indeed an α-module. □

As a consequence of the above theorem, we have the following striking analog of a familiar property of h-pure submodules.

Corollary 2.2. *Let N be a submodule of an α-module M. Then N is an α-pure submodule of M if and only if $N + H_\beta(M)/H_\beta(M)$ is a direct summand of $M/H_\beta(M)$ for all $\beta < \alpha$.*

Proof. $N+H_\beta(M)/H_\beta(M)$ being a direct summand of $M/H_\beta(M)$ implies that $N+H_\beta(M)/H_\beta(M)$ is β-pure in $M/H_\beta(M)$, which is equivalent to N being β-pure in M. Since α is a limit ordinal, N is α-pure in M if and only if N is β-pure in M for all $\beta < \alpha$.

Conversely, assume that N is α-pure in M. Then M/N is an α-module and therefore, for $\beta < \alpha$,

$$(M/N)/H_\beta(M/N) = (M/N)/(H_\beta(M) + N/N) \cong (M/H_\beta(M))/(N + H_\beta(M)/H_\beta(M))$$

is totally projective of length at most β. Since $N + H_\beta(M)/H_\beta(M)$ is β-pure in $M/H_\beta(M)$, $N + H_\beta(M)/H_\beta(M)$ is a direct summand of $M/H_\beta(M)$. □

Proposition 2.3. *If N is an α-pure submodule of an α-module M, and if $H_\beta(N)$ is a direct summand of $H_\beta(M)$ for some $\beta < \alpha$, then N is a direct summand of M.*

Proof. Assuming the conditions of the Theorem 2.2, we have for some $\beta < \alpha$:

(i) $(M/N)/H_\beta(M/N)$ is totally projective;

(ii) $N \cap H_\beta(M) = H_\beta(N)$;

(iii) $N + H_\beta(M)/H_\beta(M)$ is a direct summand of $M/H_\beta(M)$; and

(iv) $H_\beta(M) = H_\beta(N) \oplus K$.

It follows that $M = N \oplus L$ where $L \supseteq K$. □

As a corollary, we have the following generalization of the well-known fact that bounded h-pure submodules are direct summands.

Corollary 2.3. *If N is an α-pure submodule of an α-module M and if $H_\beta(N) = 0$ for some $\beta < \alpha$, then N is a direct summand of M.*

As defined in [3], a $QTAG$-module M is fully transitive if for every pair of uniform elements $x, y \in M$, $H_M(x_i) \leq H_M(y_i)$ for all $i \geq 0$ implies that there exists an endomorphism of M that maps x onto y. Here $d\left(\dfrac{xR}{x_iR}\right) = d\left(\dfrac{yR}{y_iR}\right) = i$.

The next corollary tells us that α-modules of length α are fully transitive (see [1]). This, of course, is merely a reflection of the fact that modules of length $\leq \alpha$ behave in the α context exactly as modules without elements of infinite height in the classical situations.

Corollary 2.4. *If M is an α-module of length α, then every finite subset of M is contained in a countably generated direct summand.*

Proof. Let S be a finite subset of M. Then $S \subseteq T$ for some countably generated, α-pure submodule T of M. We may assume that T has length α. Then T is a direct sum of modules of length less than α. Consequently, T is contained in a direct summand K of T having length less than α. By the preceding corollary, K is a direct summand of M. □

For a limit ordinal α, an α-module M is an α-projective if $H_\alpha(Ext(M, M')) = 0$ for all α-modules M', that is, there exists a submodule N bounded by α such that M/N is totally projective, and an α-module M is an α-injective if $H_\alpha(Ext(M', M)) = 0$ for all α-modules M', that is, it is a direct summand of every α-module in which it occurs as an α-pure submodule.

To characterize the α-injective modules we must generalize the notion of a closed module. Mimicking [2], for any $QTAG$-module M, the submodules $\{H_k(M)\}_k$, $k = 0, 1, 2, \ldots, \infty$ from a neighborhood system of zero, giving rise to h-topology. If k is replaced by an arbitrary limit ordinal less than or equal to α, then h-topology may be extended to α-topology, and all the definitions and results which hold for h-topology may be extended for α-topology. In α-topology, for any submodule N of M, the closure of N as $\bigcap_{\beta<\alpha}(N + H_\beta(M))$ denoted by $\overline{N}$.

Definition 2.3. *We call a $QTAG$-module an α-closed module if it is the maximal closed submodule of its closure in the α-topology.*

With the help of the above discussion, we are able to infer the following.

Proposition 2.4. *Let M be an α-closed α-module. Then M is an α-injective.*

Proof. We first show that $H_\alpha(Ext(T, M)) = 0$ for all α-modules T. Assume that M is an α-pure submodule of M' with $M'/M \cong T$ for all α-modules M'. Since α is a limit ordinal, it follows that $M' = H_\beta(M') + M$ for all $\beta < \alpha$. Therefore, if $y \in M'$, we can find for each $\beta < \alpha$ a $x_\beta \in M$ such that $y - x_\beta \in H_\beta(M')$. Moreover, we can assume that the exponent of x_β does not exceed that of y. Indeed, if y has exponent n, then $H_n(x'_\beta) \in H_{\beta+n}(M') \cap M = H_{\beta+n}(M)$, where $d\left(\frac{x_\beta R}{x'_\beta R}\right) = n$ and $H_n(x'_\beta) = H_n(z'_\beta)$, where $d\left(\frac{x_\beta R}{x'_\beta R}\right) = d\left(\frac{z_\beta R}{z'_\beta R}\right) = n$ for some $z_\beta \in H_\beta(M)$. Then $\overline{x}_\beta = x_\beta - z_\beta$ has an exponent at most n and $y - \overline{x}_\beta \in H_\beta(M')$. But $\{x_\beta : \beta < \alpha\}$ is a chain in M with elements uniformly bounded in exponent and, therefore, converges to some $x \in M$. Hence $y - x \in \bigcap_{\beta<\alpha} H_\beta(M') = H_\alpha(M')$. We conclude that $M' = M \oplus H_\alpha(M')$.

Now let M' be an arbitrary α-module and let B be an α-basic submodule of M'. We then have the exact sequence

$$H_\alpha(Ext(M'/B, M)) \longrightarrow H_\alpha(Ext(M', M)) \longrightarrow H_\alpha(Ext(B, M)).$$

The left-hand term of the above sequence vanishes since M'/B is isomorphic to a direct sum of copies of T and the right-hand term vanishes since B is an α-projective. Thus, $H_\alpha(Ext(M', M)) = 0$ and we conclude that M is an α-injective. □

We can now show that there are enough α-injective modules and that an α-injective module is the sum of an α-closed module and an h-divisible module.

Theorem 2.3. *Let M be an α-module. Then M is an α-pure submodule of an α-injective module and M is an α-injective module if and only if M is the direct sum of an h-divisible module and an α-closed α-module.*

Proof. It is evident from Proposition 2.4 that the direct sum of an h-divisible module and an α-closed α-module is necessarily an α-injective. Next, we need the observation that every α-module M of length at most α can be imbedded as an α-pure submodule of an α-closed module $T_M(\alpha)$ such that $T_M(\alpha)/M$ is h-divisible. Indeed, $T_M(\alpha)$ may be taken as the maximal closed submodule of the closure of M in the α-topology. It follows, by the same reasoning as in the proof of Theorem 2.1, that $T_M(\alpha)/H_\beta(T_M(\alpha)) \cong M/H_\beta(M)$ for all $\beta < \alpha$, and therefore that $T_M(\alpha)$ is an α-module.

Now let M be an arbitrary α-module. Let D be a minimal h-divisible module containing $H_\alpha(M)$. Take P to be the amalgamated sum of M and D over $H_\alpha(M)$. Then $P = M' \oplus D$ where $M' \cong M/H_\alpha(M)$ and $M' \cap M$ is an α-high submodule of M. Also, P/M is h-divisible and $Soc(P) \subseteq Soc(M) + H_\beta(P)$ for all $\beta < \alpha$. It follows that M is an α-pure submodule of P. By the transitivity of α-purity, M is an α-pure in the α-injective $T_{M'}(\alpha) \oplus D$.

Finally, assume that M is itself an α-injective and that we have it imbedded, as above, as an α-pure submodule of $\overline{P} = T_{M'}(\alpha) \oplus D$. Since M is an α-injective, $\overline{P} = M \oplus Q$ where $Q \cong \overline{P}/M$ is obviously h-divisible, since both P/M and $\overline{P}/P$ are h-divisible. But then $Q \subseteq D$, and since $Soc(D) \subseteq H_\alpha(M)$, we conclude that $Q = 0$ and $M = T_{M'}(\alpha) \oplus D$. □

Now we are in a position to prove the following result.

Theorem 2.4. *If M and M' are α-closed α-modules with the same Ulm invariants, then $M \cong M'$.*

Proof. Take B and B' to be α-basic submodules of M and M', respectively. It is easily seen that B and B' have the same Ulm invariants as M and M'. Therefore, there is an isomorphism f of B onto B'. Since B is an α-pure submodule of M, we have the exact sequence

$$Hom(M, M') \longrightarrow Hom(B, M') \longrightarrow H_\alpha(Ext(M/B, M')) = 0$$

Thus, there is a homomorphism $f' : M \longrightarrow M'$ that extends f. Let $x \in \text{Ker}f'$ and assume that $x \neq 0$. Then x has some height $\beta < \alpha$ and we can write $x = y + z$ where $y \in B$ and $z \in H_{\beta+1}(M)$. But then x has height β and $f(y) = f'(y) = -f'(z)$ has height at least $\beta + 1$. This, however, is a contradiction, since f is an isomorphism of B onto B' and B' is an isotype submodule of M'. We conclude that $\text{Ker}f' = 0$. Then $f'(M)/B' = f'(M)/f'(B) \cong M/B$ is h-divisible. Hence $f'(M)/B'$ is a direct summand of M'/B', and since B' is an α-pure submodule of M', it follows that $f'(M)$ is an α-pure submodule of M'. Since $f'(M) \cong M$ is an α-injective, we have a direct decomposition $M' = f'(M)) \oplus L$ where $L \cong M'/f'(M)$ is h-divisible. But M' is h-reduced and therefore $L = 0$ and $f'(M) = M'$, that is, f' is an isomorphism of M onto M'. □

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