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ON A REVERSE TRIGONOMETRIC MASJED-JAMEI INEQUALITY

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Abstract. The Masjed-Jamei inequality is a fascinating mathematical result involving the logarithmic, power, and inverse tangent functions. In this article, we propose an original 'reverse trigonometric' Masjed-Jamei inequality. It involves a sine function to prove a lower bound for the square of the inverse tangent function. A graphical study shows the sharpness of the inequality obtained. Some open problems are formulated for future work in this direction.

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1. Introduction and result

The Masjed-Jamei inequality was established by [2], proposing an interesting upper bound for the square of the inverse tangent function. More precisely, it is formulated as follows: for $x \in (-1, 1)$, we have

$$(\arctan(x))^2 \leq \frac{x \ln(x + \sqrt{1 + x^2})}{\sqrt{1 + x^2}}. \tag{1.1}$$

One can remark that the right term involves the inverse hyperbolic sine function defined by $\operatorname{arcsinh}(x) = \ln(x + \sqrt{1 + x^2})$. Besides certain originality in the functions involved, the Masjed-Jamei inequality is known to be particularly sharp, attracting the researchers for various purposes. In 2021, it is referenced in over 15 articles presenting diverse inequalities of interest. Among the recent developments, it is proved that the Masjed-Jamei inequality holds

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for wider values for x; the whole real line can be considered instead of $(-1, 1)$. The rigorous proof is given in [4]. This work was the inspiration for sharp inequalities centered around the hyperbolic inverse tangent function and its generalized version. See [1] and [3], respectively.

In this paper, we present a result that can be viewed as a 'reverse trigonometric' Masjed-Jamei inequality. Precisely, 'reverse' because it provides a lower bound for the square of the inverse tangent function similar to the upper bound in (1.1) and 'trigonometric' because it is achieved through the use of an additional sine function. This result is specified in the following theorem.

Theorem 1.1. *For $x \in (-\pi, \pi)$, we have*

$$\frac{\sin(x)\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} \leq (\arctan(x))^2.$$

The next section is devoted to the proof of this theorem.

2. Proof of Theorem 1.1

Several techniques are possible to conduct this proof; here, we opt for classic function analysis. Let us define

$$F(x) = (\arctan(x))^2 - \frac{\sin(x)\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}. \tag{2.1}$$

We thus aim to prove that $F(x) \geq 0$ for $x \in (-\pi, \pi)$. Since $F(-x) = F(x)$, it is enough to prove that $F(x) > 0$ for $x \in (0, \pi)$, the desired inequality being obvious for $x = 0$. Let us study the behaviour of $F(x)$ through standard derivation techniques and well-chosen intermediary functions. After some calculus and simplifications, we have

$$\begin{aligned} F'(x) &= \frac{2\arctan(x)}{1+x^2} - \frac{(x/\sqrt{1+x^2}+1)\sin(x)}{(x+\sqrt{1+x^2})\sqrt{1+x^2}} \\ &\quad + \frac{x\,\mathrm{arcsinh}(x)\sin(x)}{(1+x^2)^{3/2}} - \frac{\mathrm{arcsinh}(x)\cos(x)}{\sqrt{1+x^2}} \\ &= \frac{1}{(1+x^2)^{3/2}}g(x), \end{aligned} \tag{2.2}$$

where

$$g(x) = (2\arctan(x) - \sin(x))\sqrt{1+x^2} + \mathrm{arcsinh}(x)(x\sin(x) - (1+x^2)\cos(x)).$$

The sign of $g(x)$ is thus informative on the eventual monotonicity of $F(x)$. Let us now study this aspect. We have

$$
\begin{aligned}
g'(x) &= \frac{2}{\sqrt{1+x^2}} + \frac{2x\arctan(x)}{\sqrt{1+x^2}} - \sqrt{1+x^2}\cos(x) - x\cos(x)\operatorname{arcsinh}(x) \\
&\quad + (2+x^2)\sin(x)\operatorname{arcsinh}(x) - \sqrt{1+x^2}\cos(x) \\
&= \frac{1}{\sqrt{1+x^2}}[\phi(x)+\psi(x)], \qquad (2.3)
\end{aligned}
$$

where

$$\phi(x) = 2 + 2x\arctan(x) - 2(1+x^2)\cos(x)$$

and

$$\psi(x) = (2+x^2)\sqrt{1+x^2}\sin(x)\operatorname{arcsinh}(x) - x\sqrt{1+x^2}\cos(x)\operatorname{arcsinh}(x).$$

Since $\sin(x)/x > \cos(x)$ for $x \in (0,\pi)$, it is immediate that

$$\psi(x) = \operatorname{arcsinh}(x)\sqrt{1+x^2}\left[(1+x^2)\sin(x) + \sin(x) - x\cos(x)\right] > 0. \qquad (2.4)$$

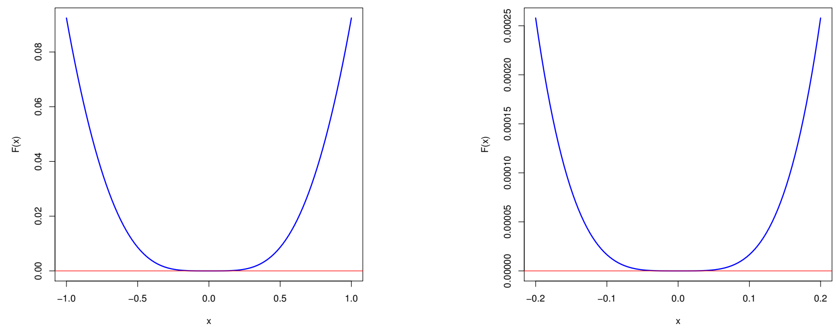
Now, by virtue of $\sin(x)/x > \cos(x)$ for $x \in (0,\pi)$ and $\cos(x) < 1$, we get

$$
\begin{aligned}
\phi'(x) &= \frac{2x}{1+x^2} + 2\arctan(x) - 4x\cos(x) + 2(1+x^2)\sin(x) \\
&= \frac{2}{1+x^2}\Big[(1+2x^2)(\sin(x) - x\cos(x)) \\
&\quad + x(1-\cos(x)) + (1+x^2)\arctan(x) + x^4\sin(x)\Big] > 0.
\end{aligned}
$$

Therefore, $\phi(x)$ is strictly increasing over $(0,\pi)$, implying that $\phi(x) > \phi(0) = 0$. This inequality combined with (2.3) and (2.4) ensures that $g'(x) > 0$ so $g(x)$ is strictly increasing over $(0,\pi)$, implying that $g(x) > g(0) = 0$. Hence, by (2.2), $F'(x) > 0$ which implies that $F(x) > F(0) = 0$. The desired result is obtained, ending the proof of Theorem 1.1. □

3. Graphical analysis

In order to illustrate the validity and sharpness of the inequality in Theorem 1.1, a graphical analysis is performed in Figure 1 by varying the range of values for x.



(a) (b)

FIGURE 1. Plots for $F(x)$ given in (2.1) for (a) $x \in (-1, 1)$ and (b) $x \in (-0.2, 0.2)$.

We see in Figure 1 that the lower bound in Theorem 1.1 is very sharp. This is particularly true for values of x in the neighborhood of 0, as illustrated in Figure 1 (b).

4. Open problem

Based on the result of [4], we conjecture that Theorem 1.1 is still valid for $x \in \mathbb{R}$ instead of $x \in (-\pi, \pi)$, as sketched in Figure 2 for $x \in (-20, 20)$.

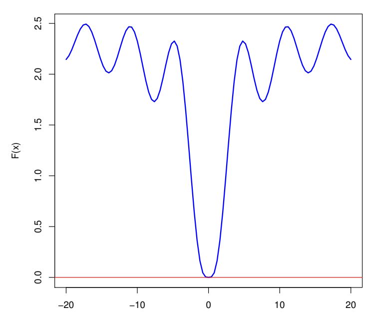


FIGURE 2. Plot for $F(x)$ for $x \in (-20, 20)$ (arbitrary large interval).

The rigorous proof however needs more developments; it remains an open problem.

References

[1] X.-D. Chen, L. Nie, W. Huang, New inequalities between the inverse hyperbolic tangent and the analogue for corresponding functions, J. Inequal. Appl. 2020 (2020), 131.

[2] M. Masjed-Jamei, A main inequality for several special functions. Comput. Math. Appl. 60 (2010), 1280-1289.

[3] K. Nantomah, O. Kasimu, On some properties of generalized inverse hyperbolic functions, Asia Pac. J. Math. 7 (2020), 15.

[4] L. Zhu, B. Malesevic, (2019). Inequalities between the inverse hyperbolic tangent and the inverse sine and the analogue for corresponding functions. J. Inequal. Appl. 2019 (2019), 93.