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DIFFERENTIABLE AND ANALYTIC RESULTS ON $\omega-$ORDER PRESERVING PARTIAL CONTRACTION MAPPING IN SEMIGROUP OF LINEAR OPERATOR

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Abstract. In this paper, we established a differentiable results leading to analytic function thereby laying special emphasis on equicontinuous and uniformly differentiable semigroup on a special class of $C_0 - Semigroup$ called $\omega-$order preserving partial contraction mapping in semigroup of linear operator. We were able to obtained some differentiable and analytic results on ω-OCP_n and their proves were stated accordingly. 2010 Mathematics Subject Classification. 06F15; 06F05; 20M05.

Key words and phrases: ω-OCP_n; equicontinuous; differentiable semigroup; analytic semigroup.

1. Introduction

Differentiability concept of a semigroup of linear operator is considered an important area of semigroup of linear operator because differentiable semigroup on a complex Banach space can be extended as analytic functions acting from a certain sector in the complex plane including the resolvent set of the generator to linear bounded operator. Suppose X is a Banach space, $X_n \subseteq X$ be a finite set, $(T(t))_{t\geq 0}$ the C_0-semigroup, $\omega - OCP_n$ be ω-order-preserving partial contraction mapping (semigroup of linear operator), $\omega - OCP_n \subseteq OCP_n$(Order Preserving Partial Contraction Mapping). let $Mm(\mathbb{N} \cup 0)$ be a matrix, $L(X)$ be a bounded linear operator in X, P_n, a partial transformation semigroup, $\rho(A)$, a resolvent of A, $\sigma(A)$, a spectrum of A, where A is the generator of a semigroup of linear operator. This paper will focus on results of

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differentiabiity, extending to analytic and uniform differentiability on $\omega - OCP_n$ in a semigroup of linear operator. Balakrishnan [1], established fractional powers of closed operators and the semigroup generated by them. Banach [2], established and introduced the concept of Banach spaces. Dunford and Schartz [3], deduced integral of analytic function in linear operator. Engel and Nagel [4], obtained one-parameter semigroup for linear evolution equations. Feller [5], obtained parabolic differential equations and associated semigroups of transformation. Hale [6], established functional differential equations in applied mathematics. Rauf and Akinyele [7], obtained ω-order-preserving partial contraction mapping and established its properties, also in [8], Rauf *et.al.* established some results of stability and spectra properties on semigroup of linear operator. Vrabie [9], characterized new generator of differentiable semigroups and also in [10], Vrabie deduced some results of C_0-semigroup and its applications. Yosida [11], established and proved some results on differentiability and representation of one-parameter semigroup of linear operators.

2. Preliminaries

Definition 2.1 ($C_0 - Semigroup$) [10]
A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

Definition 2.2 (Differentiable Semigroup) [10]
A C_0-Semigroup is a is called:
(i) differentiable at $\tau \geq 0$, if, for each $x \in X$, the function $t \to T(t)x$ is differentiable at τ;
(ii) differentiable, if it differentiable at each $\tau \in (0, +\infty)$; and
(iii) eventually differentiable if there exists $\theta > 0$ such that $t \to T(t)x$ is differentiable at each $\tau \in (\theta, +\infty)$.

Definition 2.3 (Uniformly Differentiable Semigroup) [10]
A C_0-Semigroup is a is called:
(i) Uniformly differentiable at $\tau \geq 0$ if the function $t \to T(t)$ from $(0, +\infty)$ to $L(X)$ is differentiable at τ; and
(ii) uniformly differentiable if it is uniformly differentiable at each point $\tau \in (0, +\infty)$.

Definition 2.4 (ω-OCP_n) [7]
A transformation $\alpha \in P_n$ is called ω-order-preserving partial contraction mapping if $\forall x, y \in$Domα : $x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 2.5 (Analytic Semigroup) [4]

We say that a C_0-semigroup $\{T(t); t \geq 0\}$ is analytic if there exists $0 < \theta \leq \pi$, and a mapping $S : \bar{\mathbb{C}}_\theta \to L(X)$ such that:

(i) $T(t) = S(t)$ for each $t \geq 0$;

(ii) $S(z_1 + z_2) = S(z_1)S(z_2)$ for $z_1, z_2 \in \bar{\mathbb{C}}_\theta$;

(iii) $lim_{z_1 \in \bar{\mathbb{C}}_\theta, z_1 \to 0} S(z_1)x = x$ for $x \in X$; and

(iv) the mapping $z_1 \to S(z_1)$ is analytic from $\bar{\mathbb{C}}_\theta$ to $L(X)$. In addition, for each $0 < \delta < \theta$, the mapping $z_1 \to S(z_1)$ is bounded from $\mathbb{C}_\delta$ to $L(X)$, then the C_0-Semigroup $\{T(t); t \geq 0\}$ is called analytic and uniformly bounded.

Definition 2.6 (Equicontinuous)[10]

A C_0-semigroup $\{T(t); t \geq 0\}$ is equicontinuous if the function $t \to T(t)$ is continuous from $(0, +\infty)$ to $L(X)$ endowed with the uniform operator norm $\|.\|_{L(X)}$.

Definition 2.7 (Convergent Sequence)[10]

We say that the sequence $(\varphi_n)_n$ is convergent in $D(\Omega)$ to φ and write $D(\Omega) : \varphi \to \varphi$ if

(i) there exists a compact subset $K \subset \Omega$ such that, for each $n \in \mathbb{N}$, supp $\varphi_n \subset K$; and

(ii) for each multi - index α, we have $lim_{n \to \infty} D^\alpha \varphi_n = D^\alpha \varphi$ uniformly on Ω, or equivalent on K.

Example 1

3×3 matrix $[M_m(\mathbb{C})]$, we have

for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X.

Suppose we have

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{2t\lambda} & e^{2t\lambda} & e^{3t\lambda} \\ e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Example 2

By the translation semigroup starting from $Af = f^{'}$ on $C_0(\mathbb{R}_+)$ or $L^p(\mathbb{R}_+)$, $1 \leq p < \infty$, the operator

$A^2 f = f^{''}$

generates a bounded analytic semigroup. Let us consider the slightly more involved case of several space dimensions, that is we consider the spaces $C_0(\mathbb{R}_+)$ or $L^p(\mathbb{R}_+)$, $1 \leq p < \infty$. Denote by $(\cup_i(t))_{t\in\mathbb{R}_+}$ the strongly continuous semigroup given by

$(\cup_i(t)f)(x) = f(x_1, ..., x_{i-1}, x_i + t, ..., x_n)$,

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$ and $1 \leq i \leq n$, and let Ai be its generator where $A \in \omega - OCP_n$. Obviously, these semigroups commutes as do the resolvent of Ai and hence of A^2i.

Example 3

Suppose $A : D(A) \subseteq X \to X$ is an unbounded generator of a strongly continuous semigroup and take an isomorphism $S \in L(X)$ such that $D(A) \cap S(D(A)) = \{0\}$. Then $B = SAS^{-1}$ is a generator as well, but $A+B$ is defined only on $D(A+B) = D(A) \cap D(B) = D(A) \cap S(D(A)) = \{0\}$.

A concrete example for this situation is given on $X = C_0(\mathbb{R}_+)$ by $Af = f'$ with its canonical domain $D(A) = C_0'(\mathbb{R}_+)$ and $Sf = q.f$ for some continuous, positive function q such that q and q^{-1} are bounded and nowhere differentiable. Defining the operator B as $Bf = q.(q^1.f)'$ on $D(B) = \{f \in X : q^{-1}.f \in D(A)\}$, we obtain that the sum $A + B$ is defined only on $\{0\}$.

Theorem 2.1(Dunford) [3]

Let $A \in L(X)$, $f, g \in \zeta(A)$ and $\alpha, \beta \in \mathbb{C}$ then :

i. $\alpha f + \beta g \in \zeta(A)$ and $\alpha f(A) + \beta g(A) = (\alpha f + \beta g)A$;
ii. $fg \in \zeta(A)$ and $f(A)g(A) = (fg)A$;
iii. if $f(\lambda) = \sum_{n=0}^{\infty} C_n \lambda^n$ on an open neighborhood D of $\sigma(A)$, then $f(A) = \sum_{n=0}^{\infty} C_n A^n$ in the in the norm of $L(X)$;
iv. if $(f_n)_{n\in\mathbb{N}}$ are analytic on an open neighborhood D of $\sigma(A)$ and $\lim fn = f$ uniformly on D, then $\lim_{n\to\infty} fn(A) = f(A)$ in the norm of $L(X)$;
v. $f \in \zeta(A^*)$ and $f(A^*) = (f(A))^*$; and
vi. $f(\sigma(A)) = \sigma(f(A))$.

3. Main Results

In this section, differentiable and analytic results on ω-OCP_n in semigroup of linear operator (C_0-semigroup) were established:

Theorem 3.1

Suppose $\{T(t); t \geq 0\}$ is a C_0-semigroup differentiable at each $t > \theta$, let $A : D(A) \subseteq X \to X$ be its infinitesimal generator where $A \in \omega$-OCP_n and $n \in \mathbb{N}^*$. Then:

(i). for $t > n\theta$, $T(t)X \subseteq D(A^n)$ and $T^{(n)}(t) = A^n(t)$ is a linear bounded operator; and

(ii). the mapping $t \to T^{n-1}(t)$ is continuous from $(n\theta, +\infty)$ to $L(X)$ in the uniform operator topology.

Proof :

Let $n = 1$, since , for each $x \in X$, $t \to T(t)x$ is differentiable at each $t > 0$, we have $T(t)x \in D(A)$ and $T^{'}(t)x = AT(t)x$, for each $x \in X$, $t > \theta$ and $A \in \omega$-OCP_n. Since A is closed and $T(t)$ is bounded, it follows that $AT(t)$ is closed and thus, by the closed graph theorem, we conclude that it is bounded, and this complete the proof of (i) for $n = 1$.
To prove (ii), let us observe that there exists $M \geq 1$ such that

$$\|T(t)\|_{L(X)} \leq M \tag{3.1}$$

for each $t \in [0, 1]$. Then for each $\theta < t \leq s < \theta + 1$, we have

$$T(s)x - T(t)x = \int_t^s AT(\tau)d\tau = \int_t^s T(\tau - t)AT(t)xd\tau, \tag{3.2}$$

and thus

$$\|T(s) - T(t)\|_{L(X)} \leq |s - t|M\|AT(t)\|_{L(X)}\|x\|. \tag{3.3}$$

so that, $t \to T(t)$ is continuous from $(\theta, +\infty)$ to $L(X)$ in the uniform operator topology, and this proves (ii) for $n = 1$. Next, we proceed by induction on n. Suppose that both (i) and (ii) hold for n and let $t > (n + 1)\theta$. Choose $s > n\theta$ such that $t - s > \theta$. Then, for each $x \in X$ and $A \in \omega$-OCP_n, we have

$$T^n(t)x = T(t - s)A^nT(s)x. \tag{3.4}$$

Clearly the right hand side is differentiable at t, and thus $t \to T(t)x$ is $(n + 1)$ - times differentiable and $T^{(n+1)}(t)x = A^{n+1}T(t)x$. Hence the proof is complete.

Theorem 3.2

Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup of contraction $\{T(t); t \leq 0\}$, where $A \in \omega$-OCP_n. Then $\{T(t); t \geq 0\}$ is differentiable (and thus uniformly differentiable) if and only if for each $\alpha \in (0, 1)$, there exists

$\lim_{n\to\infty} A(I - \frac{t}{n}A)^{-n}$,

uniformly for $t \in [\alpha, 1/\alpha]$ in the norm topology of $L(X)$.

Proof :

Suppose $x \in X$, $A \in \omega$-OCP_n and $\alpha \in (0,1)$. From Hille experimental formula, we have

$$\lim_{n\to\infty}(I-\frac{t}{n}A)^{-n}x = T(t)x \tag{3.5}$$

uniformly for $t \in [\alpha, 1/\alpha]$. As A is closed operator, from this, it follows that

$$\lim_{n\to\infty}(I-\frac{t}{n}A)^{-n}x = AT(t)x \tag{3.6}$$

uniformly for $t \in [\alpha, 1/\alpha]$. But this means that $T(t)x \in D(A)$ for each $x \in X$, $A \in \omega$-OCP_n and each $t > 0$. Again, Using the closedness of A, we get

$$A(I-\frac{t}{n}A)^{-n-1}x - AT(t)x = \frac{n^{n+1}}{n!}\int_0^{+\infty}(ve^{-v})^n[AT(tv)x - AT(t)x]dv \tag{3.7}$$

for each $x \in X$, $A \in \omega$-OCP_n and $t > 0$. Suppose $\alpha \in (0,1)$, and fix $\beta \in (0,\alpha)$ and since $\{T(t) : t \geq 0\}$ is differentiable, it follows that, for each $x \in X$ and $A \in \omega$-OCP_n, the mapping $t \to AT(t)x$ is continuous on $(0,+\infty)$. Then for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\|AT(t)x - AT(s)x\| \leq \epsilon \tag{3.8}$$

for each $t, s \in [\beta, 1/\beta]$ with $|t-s| \leq \delta(\epsilon)$. More so, for the very same $\epsilon > 0$, there exists $a = a(\epsilon)$ with $0 < a < 1 < b < +\infty$ such that, for each $t \in [\alpha, 1/\alpha]$ and $v \in [a,b]$, we have $tv \in [\beta, 1/\beta]$ and $|t - tv| \leq \delta(\epsilon)$ so that

$$\|AT(tv)x - AT(t)x\| \leq \epsilon \tag{3.9}$$

for each $t \in [\alpha, 1/\alpha]$ and $v \in [a,b]$. From (3.7), we deduced

$$\|A(I-\frac{t}{n}A)^{-n-1}x - AT(t)x\| \leq \sum_{k=1}^{5}\|J_k^n(t)\|, \tag{3.10}$$

for each $n \in \mathbb{N}^*$ and each $t \in [\alpha, 1/\alpha]$, where

$J_1^n(t) = \frac{n^{n+1}}{n!}\int_0^a(ve^{-v})^n\frac{1}{t}\frac{d}{dt}(T(tv))xdv,$
$J_2^n(t) = \frac{n^{n+1}}{n!}\int_0^a(ve^{-v})^nAT(t)xdv,$
$J_3^n(t) = \frac{n^{n+1}}{n!}\int_a^b(ve^{-v})^n(AT(tv)x - AT(t)x)dv,$
$J_4^n(t) = \frac{n^{n+1}}{n!}\int_b^{+\infty}(ve^{-v})^n\frac{1}{t}\frac{d}{dv}(T(tv))xdv,$
$J_5^n(t) = \frac{n^{n+1}}{n!}\int_0^{+\infty}(ve^{-v})^nAT(t)xdv,$

for each $n \in \mathbb{N}^*$ and each $t \in [\alpha, 1/\alpha]$. We shall then evaluate each of the five terms on the right hand side of (3.10). To achieve this, let us put $\rho > 0$ and each $\tau \in [\rho, +\infty)$, then we have

$$\|AT(\tau)x\| \leq \|AT(\rho)x\|. \tag{3.11}$$

Since the semigroup is differentiable, then the mapping $\tau \to AT(\tau)x$ is a C'-solution of the equation $U' = AU$ on the interval $(0, +\infty)$ and then since the semigroup generated by A is a contractions where $A \in \omega$-OCP_n, we deduced (3.11). We begin to evaluate $J_1^n(t)$, integrating by parts, we observed that, for each $v \in (0,1)$, $|e^{-v} - ve^{-v}| \leq 1$ and taking into account that the mapping $v \to ve^{-v}$ is non-decreasing on $(0,1)$, then we deduced

$$\begin{aligned} \|J_1^n(t)\| &= \|\frac{n^{n+1}}{n!}\int_0^a (ve^{-v})^n \frac{1}{t}\frac{d}{dt}(T(tv))xdv\| \\ &= \frac{1}{t}\|\frac{n^{n+1}}{n!}(ae^{-a})^n T(ta)x - \frac{n^{n+1}}{n!}\int_0^a n(ve^{-v})^{n-1}(e^{-v} - ve^{-v})T(tv)xdv\| \\ &\leq \frac{\|x\|}{\alpha}[\frac{n^{n+1}}{n!}(ae^{-a})^n + \frac{n^{n+1}}{(n-1)!}(ae^{-a})^{n-1}] \end{aligned} \tag{3.12}$$

for each $n \in \mathbb{N}^*$ and $t \in [\alpha, 1/\alpha]$. We have that
$\frac{n^{n+1}}{n!}(ae^{-a})^n = \frac{n^n}{n!}e^{-n}.n(ae^{1-a})^n$ and
$\frac{n^{n+1}}{(n-1)!}(ae^{-a})^{n-1} = \frac{(n-1)^{n-1}}{(n-1)!}e^{-(n-1)}.n^2(\frac{n}{n-1})^{n-1}(ae^{1-a})^{n-1}$.
By Stirling's formular, we have

$$\lim_{n\to\infty} \frac{n^n}{n!}e^{-n} = 0. \tag{3.13}$$

In addition, since $a \in (0,1)$, we have $ae^{1-a} < 1$. Therefore, we deduced both

$$\lim_{n\to\infty} \frac{n^{n+1}}{n!}(ae^{-a})^n = 0 \textit{ and } \lim_{n\to\infty} \frac{n^{n+1}}{(n-1)!}(ae^{-a})^{n-1} = 0, \tag{3.14}$$

so, we concludes that

$$\lim_{n\to\infty} \|J_1^n(t)\| = 0 \tag{3.15}$$

uniformly with respect to $t \in [\alpha, 1/\alpha]$.
Regarding $J_2^n(t)$, from (3.11), we have

$$\|J_2^n(t)\| \leq \|AT(\alpha)x\| \frac{n^{n+1}}{n!}\int_0^a (ve^{-v})^n dv \tag{3.16}$$

for each $n \in \mathbb{N}^*$, and $t \in [\alpha, 1/\alpha]$. From (3.16) and by definition (2.6), we have,

$$\lim_{n\to\infty} \|J_2^n(t)\| = 0, \tag{3.17}$$

uniformly for $t \in [\alpha, 1/\alpha]$. In order to evaluate $J_3^n(t)$, let us observe that from (3.9) and by equicontinuous of class C^∞, we have

$$\begin{aligned} \|J_3^n(t)\| &\leq \frac{n^{n+1}}{n!}\int_0^b (ve^{-v})^n \|AT(tv)x - AT(t)x\| dv \\ &\leq \epsilon\frac{n^{n+1}}{n!}\int_a^b (ve^{-v})^n dv \leq \epsilon\frac{n^{n+1}}{n!}\int_0^\infty (ve^{-v})^n dv = \epsilon \end{aligned} \tag{3.18}$$

for each $n \in \mathbb{N}^*$ and $t \in [\alpha.1/\alpha]$. Consequently

$$\lim_{n\to\infty} \|J_3^n(t)\| \leq \epsilon. \tag{3.19}$$

In order to evaluate $J_4^n(t)$, let us observe that, in view of (3.11), we have

$\|AT(tv)x\| \leq \|AT(\alpha b)x\|$

for each $t \in]\alpha, 1/\alpha]$ and each $v \in (b, +\infty)$. As consequence

$$\begin{aligned} \|J_4^n(t)\| &= \|\frac{n^{n+1}}{n!}\int_b^\infty (ve^{-v})^n AT(tv)x dv\| \\ &\leq \|AT(\alpha b)x\|\frac{n^{n+1}}{n!}\int_b^\infty (ve^{-v})^n dv, \end{aligned} \tag{3.20}$$

for each $n \in \mathbb{N}^*$ and $t \in [\alpha.1/\alpha]$. From (3.20), (3.14) and by equicontinuousness, we have

$$\lim_{n\to\infty} \|J_4^n(t)\| = 0. \tag{3.21}$$

Finally, from (3.11), we have

$$\|J_5^n(t)\| \leq \|AT(\alpha)x\|\frac{n^{n+1}}{n!}\int_b^\infty (ve^{-v})^n dv \tag{3.22}$$

for each $n \in \mathbb{N}^*$, and each $t \in [\alpha.1/\alpha]$. We have

$$\lim_{n\to\infty} \|J_5^n(t)\| = 0. \tag{3.23}$$

Therefore

$\lim_{n\to\infty} sup\|A(I - \frac{t}{n}A)^{-n-1}x - AT(t)x\| \leq \epsilon$

for each $\epsilon > 0$. Consequently, for each $\alpha \in (0, 1)$, we have

$$\lim_{n\to\infty} A(I - \frac{t}{n}A)^{-n-1}x = AT(t)x \tag{3.24}$$

uniformly for $t \in [\alpha.1/\alpha]$ and $A \in \omega\text{-}OCP_n$. To complete the proof, we need to show that

$$\lim_{n\to\infty} A(I - \frac{t}{n+1}A)^{-n-1}x = AT(t)x \tag{3.25}$$

uniformly for $t \in [\alpha.1/\alpha]$ and $A \in \omega\text{-}OCP_n$. To achieve this, we need to observe that from (3.24), it follows that for each sequence $(a_n)_{n\in\mathbb{N}}$ of functions from $\mathbb{R}_+{}^*$ in $\mathbb{R}_+{}^*$ satisfying

$\lim_{n\to\infty} a_n(t) = t$,

uniformly on every compact subset in $\mathbb{R}_+{}^*$, then we have

$\lim_{n\to\infty} A(I - \frac{a_n(t)}{n}A)^{-n}x = AT(t)x$

uniformly on every compact subset in $\mathbb{R}_+{}^*$. And this follows from the remark that for each $x \in X$, $A \in \omega\text{-}OCP_n$ and $\alpha \in (0,1)$, the family of functions

$\{t \to A(I - \frac{t}{n}A)^{-n-1}x; n \in \mathbb{N}^*\}$ is relatively compact in $C([\alpha, 1/\alpha]; X)$ and thus equicontinuous on $[\alpha, 1/\alpha]$. To complete the proof, it is sufficient to observe that the choice

$a_n(t) = \frac{nt}{n+1}$, for $n \in \mathbb{N}^*$ gives

$\lim_{n\to\infty} A(I - \frac{t}{n+1}A)^{-n-1}x = AT(t)x$,

which complete the proof.

Theorem 3.3

Let $A : D(A) \subseteq X \to X$ be a $\mathbb{C}$ - linear operator which generates a C_0-semigroup of contractions $\{T(t); t \geq 0\}$ such that $A \in \omega\text{-}OCP_n$. Suppose $0 \in \rho(A)$, then the following conditions are equivalent:

(i) the semigroup $\{T(t); t \geq 0\}$ is analytic and uniformly bounded;

(ii) $\{\lambda \in \mathbb{C}; Re\lambda > 0\} \subseteq \rho(A)$, and there exists $C > 0$ such that for each $\lambda \in \mathbb{C}$ and $Im\lambda \neq 0$, we have

$\|R(\lambda; A)\|_{L(X)} \leq \frac{C}{|Im\lambda|}$;

(iii) there exist $\delta \in (0, \pi/2)$ and $M > 0$, such that $\mathbb{C}_{\pi/2+\delta} \subseteq \rho(A)$ and, for each $\lambda \in \mathbb{C}_{\pi/2+\delta}$, we have

$\|R(\lambda; A)\|_{L(X)} \leq \frac{M}{|\lambda|}$; and

(iv) the semigroup $\{T(t); t \geq 0\}$ is uniformly differentiable for $t > 0$ and there exists $C > 0$ so that

$\|T^{'}(t)\|_{L(X)} \leq \frac{C}{t}$;

for each $t > 0$.

Proof :

Let us begin by showing that (i) implies (ii). Suppose $\delta > 0$ for which there exists $C_1 > 0$ such that

$\|S(z_{1)}\|_{L(X)} \leq C_1$

for each $z_1 \in \mathbb{C}$ with $|arg(z_1)| < \delta$. Let us remark that whenever A generates a C_0-semigroup of contractions where $A \in \omega\text{-}OCP_n$, then $\{\lambda \in \mathbb{C}; Re\lambda > 0\} \subseteq \rho(A)$ and for each $\lambda \in \mathbb{C}$ with

$Re\lambda > 0$, we have

$$\|R(\lambda; A)\|_{L(X)} \leq \frac{1}{Re\lambda} \tag{3.26}$$

and for each $x \in X$, $\sigma > 0$, $A \in \omega\text{-}OCP_n$ and $\tau \in \mathbb{R}$, we have

$$R(\sigma + i\tau; A)x = \int_0^\infty e^{(\sigma+i\tau)t} T(t) x dt. \tag{3.27}$$

Suppose $\theta \in (0, \frac{\pi}{2})$, since the semigroup is analytic, for $\tau > 0$, we can shift the path of integration from $(0, +\infty)$ to the ray $\{\rho e^{-i\theta}; 0 < \rho < +\infty\}$ oriented from 0 to $+\infty$. We obtained

$$\begin{aligned} \|R(\sigma + i\tau; A)x\| &\leq \int_0^\infty e^{-\rho(\sigma cos\theta + \tau sin\theta)} C_1 \|x\| d\rho \\ &= \frac{C_1}{\sigma cos\theta + \tau sin\theta} \|x\| \leq \frac{C}{|\tau|} \|x\|. \end{aligned} \tag{3.28}$$

Analogousy, for $\tau < 0$, shifting the path of integration from $(0, +\infty)$ to $\{\rho e^{i\theta}; 0 < \rho < +\infty\}$ oriented from 0 to $+\infty$. We obtained

$$\begin{aligned} \|R(\sigma + i\tau; A)x\| &\leq \int_0^\infty e^{-\rho(\sigma cos\theta - \tau sin\theta)} C_1 \|x\| d\rho \\ &= \frac{C_1}{\sigma cos\theta - \tau sin\theta} \|x\| \leq \frac{C}{|\tau|} \|x\|, \end{aligned} \tag{3.29}$$

which proves (ii).

To prove that (ii) implies (iii), Let us observe that by (3.26), for each $\lambda \in \mathbb{C}$, $A \in \omega\text{-}OCP_n$ with $Re\lambda > 0$, we have

$$\|R(\lambda; A)\|_{L(X)} \leq \frac{1}{Re\lambda} \tag{3.30}$$

On the other hand, by (ii), we know that, for each $\lambda \in \mathbb{C}$ satisfying $Re\lambda > 0$, and $Im\lambda \neq 0$, we have

$$\|R(\lambda; A)\|_{L(X)} \leq \frac{C}{|Im\lambda|}. \tag{3.31}$$

From (3.30) and (3.31), it follows that there exist $M > 0$ such that

$$\|R(\lambda; A)\|_{L(X)} \leq \frac{M}{|\lambda|}. \tag{3.32}$$

for $\lambda \in \mathbb{C}$ with $Re\lambda > 0$. Let $\sigma > 0$ and $\tau \in \mathbb{R}$. Since the Taylor expansion of the resolvent function around $\sigma + i\tau$ is

$$R(\lambda; A) = \sum_{n=0}^{\infty} (-1)^n R(\sigma + i\tau; A)^{n+1} (\sigma + i\tau - \lambda)^n. \tag{3.33}$$

This series is convergent in $L(X)$ for each $\lambda \in \mathbb{C}$ satisfying

$\|R(\sigma+i\tau;A)\|_{L(X)}|\sigma+i\tau-\lambda|\leq K<1.$

Taking $\lambda = Re\lambda + i\tau$ in (3.33) and using the inequality in (ii), we observe that it is convergent in $L(X)$ for $|\sigma - Re\lambda| \leq k\frac{|\tau|}{C}$. Since both $\sigma > 0$ and $k \in (0,1)$ are arbitrary, it follows that $\rho(A)$ includes all complex numbers λ with $Re\lambda \leq 0$ satisfying $\frac{|Re\lambda|}{|Im\lambda|} < \frac{1}{C}$. In Particular, we have

$C_{\frac{\pi}{2}+\delta} = \{\lambda \in \mathbb{C}; |arg\lambda| < \frac{\pi}{2}+\delta\} \subseteq \rho(A),$

where $\delta = k\ arctan(1/C), \in (0,1)$. Moreover, on $C_{\frac{\pi}{2}+\delta}$, we have

$$\|R(\lambda;A)\|_{L(X)} \leq \frac{C}{1-k} \leq \frac{\sqrt{C^2+1}}{(1-k)}\frac{1}{|\lambda|} = \frac{M}{|\lambda|}, \tag{3.34}$$

which follows that A satisfies (iii).

To prove that (iii) implies (iv), first let us remark that if (iii) holds for each $t > 0$ and $A \in \omega$-OCP_n, $\sigma(tA) \subseteq \{\lambda \in \mathbb{C}; Re\lambda < 0\}$. Then, for a fixed $\frac{\pi}{2} < \theta < \frac{\pi}{2}+\delta$, the Dunford integral of analytic function e^z calculated at tA is well-defined, and

$$e^{tA} = \frac{1}{2\pi i}\int_\Gamma e^{\mu t}R(\mu;A)d\mu, \tag{3.35}$$

where Γ is the path consisting of the two rays $\{\rho e^{-i\theta}; 0 < \rho < +\infty\}$, and $\{\rho e^{i\theta}; 0 < \rho < +\infty\}$, oriented in the sense of increase of the imaginary part of λ. We emphasis that from (iii) and from the condition $0 \in \rho(A)$, it follows that Γ is entirely contained in the domain of analyticity of the resolvent function. We shall show next that, for each $t \geq 0$ and $A \in \omega$-OCP_n, we have

$$e^{tA} = T(t) \tag{3.36}$$

where e^{tA} is defined by (3.35). First let us observe that from (i) in theorem 2.1, it follows that $\{e^{tA}; t > 0\} \cup \{I\}$ is a semigroup. Therefore, to check (3.36), it suffices to show that, for each $\lambda > 0$, we have

$R(\lambda;A) = \int_0^{+\infty} e^{-\lambda}e^{tA}dt.$

Let $\lambda > 0$ and let us multiply (3.35) by $e^{-\lambda t}$, and integrate from 0 to b. From Fubini theorem and residues theorem, we deduced

$$\begin{aligned}\int_0^b e^{-\lambda t}e^{tA}dt &= \frac{1}{2\pi i}\int_\Gamma \frac{1}{\mu-\lambda}(e^{(\mu-\lambda)b}-1)R(\mu;A)d\mu \\ &= R(\lambda;A) + \frac{1}{2\pi i}\int_\Gamma \frac{e^{(\mu-\lambda)}b}{\mu-\lambda}R(\mu;A)d\mu.\end{aligned} \tag{3.37}$$

As

$$\lim_{b\to+\infty}\int_\Gamma \frac{e^{(\mu-\lambda)}b}{\mu-\lambda}R(\mu;A)d\mu = 0, \tag{3.38}$$

from (3.38) we get (3.36). At this point, let us observe that the integral on the right-hand side in (3.35) can be differentiated with respect to the parameter $t > 0$, we can interchange the integration with the differentiation, because the integral $\frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t} R(\lambda; A) d\lambda$ is convergent for $t > 0$ in the uniform operator norm of space $L(X)$. This last accertion follows from the simple observation that for each $t > 0$, we have

$$\|\frac{1}{2\pi i}\int_\Gamma \mu e^{\mu t} R(\mu; A) d\mu\| \leq \frac{1}{\pi}\int_0^{+\infty} e^{-\rho cos\theta t} d\rho = \frac{1}{\pi cos\theta}.\frac{1}{t}. \tag{3.39}$$

Differentiating both sides in (3.35) with respect to $t > 0$ and using (3.36) and (3.39), we deduced (iv) with $C = \frac{1}{\pi cos\theta}$. To prove that (iv) implies (i), let us observe that for each $t > 0$ and each $n \in \mathbb{N}^*$, we have

$$T^{(n)}(t) = [T'(\frac{t}{n})]^n. \tag{3.40}$$

This follows by observing that, if semigroup $\{T(t); t \geq 0\}$ is uniformly differentiable, then, for each $x \in X$, $A \in \omega$-OCP_n, each $t > 0$ and each $n \in \mathbb{N}^*$, we have $T(t)x \in D(A^n)$. Because of the fact that the semigroup commutes with its infinitesimal generator, we have

$$T^{(n)}(t) = A^n T(t) = A^n T(\frac{t}{n})^n = [AT(\frac{t}{n})]^n = [T'(\frac{t}{n})]^n, \tag{3.41}$$

which proves (3.40). From (3.40) and the inequality $n^n \leq n! e^n$, we deduced

$$\frac{1}{n!}\|T^{(n)}(t)\|_{L(X)} \leq (\frac{Ce}{t})^n, \tag{3.42}$$

for each $t > 0$ and $n \in \mathbb{N}^*$. Let us consider now the power series

$$S(z_1) = T(t) + \sum_{n=1}^{\infty} \frac{(z_1 - t)^n}{n!} T^{(n)}(t) \tag{3.43}$$

which by virtue of (3.42) is uniformly convergent in the norm $L(X)$ for $|z_1 - t| \leq k(\frac{t}{Ce})$, for each $k \in (0, 1)$. Obviously, the family of the linear operators $\{S(z_1); z_1 \in \mathbb{C}_0\}$, with $\theta = arctan(\frac{1}{Ce})$, extends $\{T(t) : t \geq 0\}$ to $\mathbb{C}_0$ and therefore it satisfies (i) in definition (2.5). by (i) and (ii) in theorem (2.1), it follows that $\{S(z_1); z_1 \in \bar{\mathbb{C}}_0\}$ satisfies (ii) in definition (2.1). By (3.43) we conclude that $\{S(z_1); z_1 \in \bar{\mathbb{C}}_0\}$ satisfies (iii) and (iv) in definition (2.5). Since by virtue of (3.42), the uniform boundedness condition is obviously satisfied and the proof is complete.

4. Conclusion

In this paper, differentiability and analytic results on $\omega - OCP_n$ were obtained thereby given special consideration to equicontinousness of a semigroup of linear operator (C_0-semigroup) on a Banach space.

Competing interests

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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