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THE BSE PROPERTY FOR VECTOR-VALUED FRECHET LIPSCHITZ ALGEBRAS

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Abstract. Let (X,d) be a metric space with at least two elements and (A,p_l) be a commutative semisimple Frechet algebra over the scalar field $\mathbb{C}$. The correlation between the BSE-property of the Frechet algebra (A,p_l) and $\mathrm{Lip}_d(X,A)$ is assessed. It is found and approved that if $\mathrm{Lip}_d(X,A)$ is a BSE- Frechet algebra, then so is A. The opposite correlation will hold if (A,p_l) is unital.

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1. Introduction

The class of Frechet algebras which is an important class of locally convex algebras has been widely studied by many authors. For a full study of Frechet algebras, one may see ([3], [5], [8]). A Frechet space is a metrizable complete locally convex vector space. The topology of Frechet algebra A can be given by a sequence (p_n) of increasing sub- multiplicative seminorms. Algebra A is called without order if $aA = 0$ concludes that $a = 0$, $(a \in A)$. Let A be commutative and without order Frechet algebra and $\Delta(A)$, be the character space of A with the Gelfand topology. In this study, $\Delta(A)$ represents the set of all non-zero multiplicative linear functionals over A. Assume that $C_b(\Delta(A))$ is the space consisting of all complex-valued continuous and bounded functions on $\Delta(A)$. A linear operator T on A is named a multiplier if $T(xy) = xT(y)$, for all $x, y \in A$. The set of all multipliers on A will be expressed as $M(A)$. The strong operator topology (briefly SOT- topology) on $M(A)$ is generated by the family of seminorms $\{p_{x,l}\}$ defined as

$$p_{x,l}(T) := p_l(T(x))$$

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for all $x \in A$, $l \in \mathbb{N}$ and $T \in M(A)$. If the Frechet algebra A is semisimple, then the Gelfand map $\Gamma : A \to \widehat{A}$, $f \mapsto \hat{f}$, is injective, or equivalently, and the following equation holds:

$$\bigcap_{\varphi \in \Delta(A)} ker(\varphi) = \{0\}$$

Note that every semisimple commutative Frechet algebra is without order. As observed in [2], if the Frechet algebra (A, p_l) is semisimple, then

$$(M(A), SOT) \cong (\widehat{M(A)}, \mathcal{T}_p)$$

Where $\mathcal{T}_p$ is pointwise topology on $\widehat{M(A)}$.

The Bochner-Schoenberg-Eberlein (BSE) is derived from the famous theorem proved in 1980 by Bochner and Schoenberg for the group of real numbers; [11] and [10]. The researcher in [4], revealed that if G is any locally compact abelian group, then the group algebra $L_1(G)$ is a BSE algebra. The researcher in [10], [13], [14] assessed the commutative Banach algebras that meet the Bochner-Schoenberg-Eberlein- type theorem and explained their properties. They are introduced and assessed in [12] the first and second types of BSE algebras. This concept is expanded in [6] and [7].

The researchers are introduced and assessed in [2], the concept of BSE- Frechet algebra.

The big and little Frechet α- Lipschitz vector-valued algebra of order α, where $\alpha \in \mathbb{R}$ with $\alpha > 0$ was introduced in [9]. The researchers are provided a survey of the similarities and differences between Banach and Frechet algebras include some known results and examples. (See [3]).

That the Lipschitz algebra $\mathrm{Lip}_\alpha(K, A)$ is a BSE-algebra if and only if A is a BSE-algebra, where K is a compact metric space, A is a commutative unital semisimple Banach algebra, and $0 < \alpha \leq 1$ is proved in [1]. In this article, this result is generalized, for any metric space (X, d) and any commutative semisimple Frechet algebra (A, p_l). That the $C_{BSE}(\Delta(\mathrm{Lip}_d(X, A)))$ can be embedded in $\mathrm{Lip}_d(X, C_{BSE}(\Delta(A))$ will be proved in the article first, followed by proving that $\mathrm{Lip}_d(X, M(A)) \underset{\to}{\subseteq} M(\mathrm{Lip}_d(X, A))$. By proving that if $\mathrm{Lip}_d(X, A)$ is a BSE- Frechet algebra, so is A. If (A, p_l) is unital Frechet algbra and BSE- Frechet algebra, then $\mathrm{Lip}_d(X, A)$ is so, is assessed in this article.

2. Some basic properties of BSE- Frechet algebra

The basic terminologies and the related information on BSE-Frechet algebras are extracted from [2] and prove some primary, basic results, and properties related to them.

A bounded complex-valued continuous function σ on $\Delta(A)$, is named BSE-Frechet function, if there exists a bounded set M in A and a positive real number β_M in a sense that for every nite complex-number $c_1, \cdots, c_n$ and the same many $\varphi_1, \cdots, \varphi_n$ in $\Delta(A)$ the following inequality

$$| \sum_{j=1}^{n} c_j \sigma(\varphi_j) | \leq \beta_M P_M(\sum_{j=1}^{n} c_j \varphi_j)$$

holds; where P_M is defined as

$$P_M(f) := sup\{| f(x) |: x \in M\} \quad (f \in A^*)$$

The set of all BSE- functions is expressed by $C_{BSE}(\Delta(A))$. The BSE- seminorm of $\sigma \in C_{BSE}(\Delta(A))$, $q_l(\sigma)$, is expressed as:

$$q_l(\sigma) = sup\{| \sum_{i=1}^{n} c_i \sigma(\varphi_i) |: P_{M_l}(\sum_{i=1}^{n} c_i \varphi_i) \leq 1,\ \varphi_i \in \Delta(A), c_i \in \mathbb{C}, n \in \mathbb{N}\}$$

where

$$M_l := \{a \in A : p_l(a) \leq 1\}$$

It was shown that $(C_{BSE}(\Delta(A)), q_l)$ is a semisimple commutative Frechet subalgebra of $C_b(\Delta(A))$. It is easy to prove that

$$q_l(\sigma) = inf\{\beta_M \,|| \sum_{j=1}^{n} c_j \sigma(\varphi_j) | \leq \beta_M P_M(\sum_{j=1}^{n} c_j \varphi_j), c_j \in \mathbb{C}, \varphi_j \in \Delta(A)\}$$

It is obvious that if $x \in A$ then $\hat{x} \in C_{BSE}(\Delta(A))$ and $q_l(\hat{x}) \leq p_l(x)$, where $\hat{x}(\varphi) = \varphi(x)$ for all $\varphi \in \Delta(A)$. The set $M(A)$ with the strong operator topology, is an unital commutative locally convex algebra. It was shown that for each $T \in M(A)$ there exists a unique bounded continuous function $\widehat{T}$ on $\Delta(A)$ expressed as:

$$\varphi(Tx) = \widehat{T}(\varphi)\varphi(x),$$

for all $x \in A$ and $\varphi \in \Delta(A)$. By setting $\{\widehat{T} : T \in M(A)\}$, the $\widehat{M(A)}$ is yield. A commutative Frechet algebra A is called BSE- Frechet- algebra if it meets the following condition:

$$\widehat{M(A)} = C_{BSE}(\Delta(A)).$$

A bounded net $\{e_\beta\}$ in A is named a bounded Δ- weak approximate identity for A if $\varphi(ae_\beta) \to \varphi(a)$ for all $\varphi \in \Delta(A)$ and $a \in A$, equivalently $\varphi(e_\beta) \to 1$.

Proposition 1. *Let (A, p_l) be a commutative semisimple Frechet algebra. Then $\sigma \in C_{BSE}(\Delta(A))$ if and only if there exists a bounded net $\{x_\lambda\}$ in A with*

$$lim \hat{x_\lambda}(\varphi) = \sigma(\varphi)$$

for all $\varphi \in \Delta(A)$.

Theorem 2. *Let (A, p_l) be a commutative semisimple Frechet algebra. Then A has a bounded Δ- weak approximate identity if and only if*

$$\widehat{M(A)} \subseteq C_{BSE}(\Delta(A))$$

3. Some basic properties of vector- valued Frechet- Lipschitz algebra

The basic terminologies and the related information on vector-valued Frechet- Lipchitz algebras are reviewed. In the sequel, some primary, basic results, and properties related to them are proved.

Throughout this section, (X, d) is a metric space with at least two elements and (A, p_l) is a commutative semisimple Frechet algebra over the scaler field $\mathbb{C}$. Let $f : X \to A$ be a function. Set

$$q_{l,A}(f) = \sup_{x \in X} p_l(f(x))$$

and

$$p_{l,A}(f) = \sup_{x \neq y} \frac{p_l(f(x) - f(y))}{d(x, y)}$$

The set of all functions such $f : X \to A$ satisfies in the following conditions:
i) $q_{l,A}(f) < \infty$, for each $l \in \mathbb{N}$;
ii)$p_{l,A}(f) < \infty$, for each $l \in \mathbb{N}$.
is named the vector-valued Frechet Lipschitz algebra and is expressed by $\mathrm{Lip}_d(X, A)$.

Put

$$r_{l,A}(f) = q_{l,A}(f) + p_{l,A}(f) \qquad (f \in Lip_d(X, A))$$

$C_b(X, A)$ is the set of all bounded continuous functions from X into A. Let $f \in C_b(X, A)$. If $f, g \in C_b(X, A)$ and $\lambda \in \mathbb{C}$, define

$$(f + g)(x) = f(x) + g(x) \quad and \quad (\lambda f)(x) = \lambda f(x) \quad (x \in X)$$

It is obvious that $(C_b(X,A), q_{l,A})$ is a Frechet space over $\mathbb{C}$. That $(Lip_d(X,A), r_{l,A})$ is a Frechet subalgebra of $C_b(X,A)$ is in Lemma 3.1 proved in [9].

In this article, $f_a : X \to A \quad (a \in A)$ is the constant function on X, where $f_a(x) = a \quad (x \in X)$. It is obvious that these functions belong to $\mathrm{Lip}(X,A)$ and

$$r_{l,A}(f_a) = q_{l,A}(f_a) = p_l(a),$$

for each $a \in A$ and $f \in \mathrm{Lip}_d(X,A)$.

Let (A, p_l) and (B, q_l) be Frechet algebras. The function $\Phi : (A, p_l) \to (B, q_l)$ is called isometric, if

$$q_l(\Phi(a)) = p_l(a) \quad (a \in A, l \in \mathbb{N})$$

Let $K_A : A \to \mathrm{Lip}_d(X,A)$ such that $a \mapsto f_a$. Then K_A is a continuous, linear and injective function. Furthermore, $\widehat{K_A} : \Delta(\mathrm{Lip}_\alpha(d)) \to \Delta(A) \cup \{0\}$ is homomorphism and A can be considered as a closed subalgebra of $\mathrm{Lip}_d(X,A)$.

Proposition 3. *Let (X,d) be a metric space, (A,p_l) be a commutative Frechet algebra over the scaler field $\mathbb{C}$. Then $\mathrm{Lip}_d(X,A)$ is a semisimple Frechet algebra if and only if A is so.*

Proof. First, Assume that A is semisimple and take $f, g \in \mathrm{Lip}_d(X,A)$ such that $f \neq g$. So there exists $x_0 \in X$ such that $f(x_0) \neq g(x_0)$. Because A is a semisimple algebra, there exists $\varphi \in \Delta(A)$ where

$$\varphi(f(x_0)) \neq \varphi(g(x_0)).$$

$x_0 \otimes \varphi$ defined by $x_0 \otimes \varphi(f) = \varphi(f(x_0))$; for $f \in \mathrm{Lip}_d(X,A)$. It is obvious that $x_0 \otimes \varphi \in \Delta(\mathrm{Lip}_d(X,A))$ and

$$x_0 \otimes \varphi(f) \neq x_0 \otimes \varphi(g)$$

This implies that $\mathrm{Lip}_d(X,A)$ is semisimple.

Now assume that $\mathrm{Lip}_d(X,A)$ is a semisimple Frechet algebra. Let $a, b \in A$ where $a \neq b$, so $f_a \neq f_b$. Because $\mathrm{Lip}_d(X,A)$ is a semisimple, there exists $\psi \in \Delta(\mathrm{Lip}_d(X,A))$ where $\psi(f_a) \neq \psi(f_b)$. Which yield:

$$\widehat{K_A}(\psi)(a) = \psi(K_A(a)) = \psi(f_a) \neq \psi(f_b) = \psi(K_A(b)) = \widehat{K_A}(\psi)(b).$$

Consequently $\widehat{K_A}(\psi)(a) \neq \widehat{K_A}(\psi)(b)$ and $\widehat{K_A}(\psi) \in \Delta(A)$. Then $\Delta(A)$ separates the points of A, this implies that A is a semisimple Frechet algebra. □

Proposition 4. *Let (X, d) be a metric space, (A, p_l) be a commutative Frechet algebra over the scaler field $\mathbb{C}$. Then $\mathrm{Lip}_d(X, A)$ is without order if and only if A is so.*

Proof. Let A be without order Frechet algebra and Assume that $f \in \mathrm{Lip}_d(X, A)$ be non- zero. So there exists $x_0 \in X$ where $f(x_0) \neq 0$. Because A is without order, there exists $b \in A$ where

$$f(x_0)b \neq 0.$$

Which yield:

$$(ff_b)(x_0) = f(x_0)f_b(x_0) = f(x_0)b \neq 0.$$

So $ff_b \neq 0.$, therefore $\mathrm{Lip}_d(X, A)$ is without order.
Conversely, assume that $\mathrm{Lip}_d(X, A)$ be without order and take $a \in A$ where $a \neq 0$, so $f_a \neq 0$. Because $\mathrm{Lip}_d(X, A)$ is without order, there exists $g \in \mathrm{Lip}_d(X, A)$ where $f_a g \neq 0$. This follows that there exists $x_0 \in X$ where $(f_a g)(x_0) \neq 0$, thus $ag(x_0) \neq 0$ and consequently A is without order. □

Lemma 5. *Let (A, p_l) be a commutative semisimple Frechet algebra and (X, d) be a metric space. If $\mathrm{Lip}_d(X, A)$ has a bounded $\Delta-$weak approximate identity, then A has a bounded $\Delta-$weak approximate identity.*

Proof. Assume that $\mathrm{Lip}_d(X, A)$ has a bounded $\Delta-$ weak approximate identity and (f_β) is a bounded $\Delta-$ weak approximate identity for $\mathrm{Lip}_d(X, A)$. By allowing $\varphi \in \Delta(A)$ the following is yield:

$$\lim_{\beta} \varphi(f_\beta(x)) = \lim_{\beta} (x \otimes \varphi)(f_\beta) = 1.$$

because $x \otimes \varphi \in \Delta(\mathrm{Lip}_d(X, A))$, for each $x \in X$ and $\varphi \in \Delta(A)$, thus, the net $(f_\beta(x))$ is a bounded $\Delta-$ weak approximate identity for A. This completes the proof. □

Lemma 6. *Let (A, p_l) and (B, q_l) be a commutative Frechet algebra and (X, d) be a metric space. If $A \cong B$, as two Frechet algebras, then $\mathrm{Lip}_d(X, A) \cong \mathrm{Lip}_d(X, B)$, These two as Frechet algebras are isometric.*

Proof. Assume that $\Theta : \ A \longrightarrow B$ is an isomorphism map. Define

$$\tilde{\Theta} : \ \mathrm{Lip}_d(X, A) \longrightarrow \mathrm{Lip}_d(X, B)$$

Where $\tilde{\Theta}(f)(x) = \Theta(f(x))$, for all $x \in X$ and $f \in \mathrm{Lip}_d(X, A)$. If $f_1, f_2 \in \mathrm{Lip}_d(X, A)$ and $f_1 = f_2$, so $f_1(x) = f_2(x)$ for each $x \in X$. Thus $\Theta(f_1(x)) = \Theta(f_2(x))$, then $\tilde{\Theta}(f_1) = \tilde{\Theta}(f_2)$. This implies that $\tilde{\Theta}$ is well- defined. At this stage, $\tilde{\Theta}(f) \in \mathrm{Lip}_d(X, B)$, for each $f \in \mathrm{Lip}_d(X, A)$ is assesed. For all $x, y \in X$ with $x \neq y$ the following is yield:

$$\begin{aligned} \frac{q_l(\tilde{\Theta}(f)(x) - \tilde{\Theta}(f)(y))}{d(x,y)} &= \frac{q_l(\Theta(f(x)) - \Theta(f(y)))}{d(x,y)} \\ &\leq K\frac{p_l(f(x) - f(y))}{d(x,y)} \end{aligned}$$

This follows that

$$p_{l,B}(\tilde{\Theta}(f)) \leq K p_{l,A}(f)$$

Moreover, for all $x \in X$ Which yield:

$$q_l(\tilde{\Theta}(f)(x)) = q_l(\Theta(f(x))) \leq K p_l(f(x))$$

This implies that $q_{l,B}(\tilde{\Theta}(f)) \leq K q_{l,A}(f)$, which K is an upper bound for Θ. Therefore $\tilde{\Theta}(f) \in \mathrm{Lip}_d(X, B)$.

In the sequel, it will be concluded that $\tilde{\Theta}$ is injective. To that end, take $f, g \in \mathrm{Lip}_d(X, A)$, such that $\tilde{\Theta}(f) = \tilde{\Theta}(g)$. So $\Theta(f(x)) = \Theta(g(x))$, for all $x \in X$, thus $f(x) = g(x)$ for all $x \in X$, because Θ is injective. Then $\tilde{\Theta}$ is injective. It remains to prove that $\tilde{\Theta}$ is surjective. Assume that $g \in \mathrm{Lip}_d(X, B)$ and define $f(x) = \Theta^{-1}(g(x))$, for all $x \in X$. Which yield:

$$\begin{aligned} \frac{p_l(f(x) - f(y))}{d(x,y)} &= \frac{p_l(\Theta^{-1}(g(x)) - \Theta^{-1}(g(y)))}{d(x,y)} \\ &\leq M\frac{q_l(g(x) - g(y))}{d(x,y)} \end{aligned}$$

This follows that

$$p_{l,A}(f) \leq M p_{l,B}(g)$$

In the same way, It will be concluded that $q_{l,A}(f) \leq M q_{l,B}(g)$, for some $M > 0$. At the result $f \in \mathrm{Lip}_d(X, A)$ and $\tilde{\Theta}(f)(x) = \Theta(f(x)) = \Theta(\Theta^{-1}(g(x))) = g(x)$ and thus $\tilde{\Theta}(f) = g$. This completes the proof. □

Lemma 7. *Let (X, d) be a metric space, (A, p_l) be a commutative Frechet algebra over the scaler field $\mathbb{C}$. Assume that M is a bounded set in A, $x \in X$, $\varphi \in \Delta(A)$, $c_1, \cdots, c_n \in \mathbb{C}$ and the same number $\varphi_1, \cdots, \varphi_n \in \Delta(A)$, then the following is yield:*

$$P_{M'}(\sum_{i=1}^{n} c_i(x \otimes \varphi_i)) = P_M(\sum_{i=1}^{n} c_i\varphi_i).$$

where $M' = \{K_A(a) \mid a \in M\}$.

Proof. By allowing $c_1, \cdots, c_n \in \mathbb{C}$ and the same number $\varphi_1, \cdots, \varphi_n \in \Delta(A)$, the following is yield:

$$\begin{aligned}
P_{M'}(\sum_{i=1}^{n} c_i(x\otimes\varphi_i)) &= \sup\left\{ |\sum_{i=1}^{n} c_i(x\otimes\varphi_i)(f)| : f\in M^{'} \right\} \\
&= \sup\left\{ |\sum_{i=1}^{n} c_i\varphi_i(f(x))| : f(x)\in M \right\} \\
&\leq \sup\left\{ |\sum_{i=1}^{n} c_i\varphi_i(a)| : a\in M \right\} \\
&= P_M(\sum_{i=1}^{n} c_i\varphi_i).
\end{aligned}$$

For the reverse inclusion, which yield:

$$\begin{aligned}
P_{M}(\sum_{i=1}^{n} c_i\varphi_i) &= \sup\left\{ |\sum_{i=1}^{n} c_i\varphi_i(a)| : a\in M \right\} \\
&= \sup\left\{ |\sum_{i=1}^{n} c_i\varphi_i(f_a(x))| : a\in M \right\} \\
&= \sup\left\{ |\sum_{i=1}^{n} c_i(x\otimes\varphi_i)(f_a)| : f_a\in M^{'} \right\} \\
&\leq \sup\left\{ |\sum_{i=1}^{n} c_i(x\otimes\varphi_i)(f)| : f\in M^{'} \right\} \\
&= P_{M'}(\sum_{i=1}^{n} c_i(x\otimes\varphi_i))
\end{aligned}$$

Consequently,

$$P_{M'}(\sum_{i=1}^{n} c_i(x\otimes\varphi_i)) = P_M(\sum_{i=1}^{n} c_i\varphi_i).$$

□

3. Main results

The structure of the BSE functions on $\Delta(\mathrm{Lip}_d(X,A))$ is characterization and the correlations between the BSE property of A and $\mathrm{Lip}_d(X,A)$ are assessed.

Let $f\in \mathrm{Lip}_d(X,A)$, define $r^{'}_{l,A}(f) = max\{p_{l,A}(f), q_{l,A}(f)\}$. It is obvious that $r^{'}_{d,A}$ is a semi-norm on $\mathrm{Lip}_d(X,A)$. Clearly

$$(\mathrm{Lip}_d(X,A), r_{l,A}) \cong (\mathrm{Lip}_d(X,A), r^{'}_{l,A})$$

Proposition 8. *Let (X, d) be a metric space and (A, p_l) be a commutative semisimple Frechet algebra. Assume that $\text{Lip}_d(X, A)$ is a BSE- Frechet- algebra. Then A is so.*

Proof. Because $\text{Lip}_d(X, A)$ is a BSE- algebra, by referring to Theorem 2, $\text{Lip}_d(X, A)$ has a bounded $\Delta-$ weak approximate identity. Lemma 5 and Theorem 2, implies that

$$\widehat{M(A)} \subseteq C_{BSE}(\Delta(A)).$$

For the reverse inclusion, take $\sigma \in C_{BSE}(\Delta(A))$. There exist a bounded set M in A and a positive real number β_M where by allowing $\psi_1, \cdots, \psi_n$ of $\Delta(\text{Lip}_d(X, A))$ and the same number of complex numbers $c_1, \cdots, c_n$, the following is yield:

$$\begin{aligned} \mid \sum_{i=1}^{n} c_i \sigma o \widehat{K_A}(\psi_i) \mid &= \mid \sum_{i=1}^{n} c_i \sigma(\psi_i o K_A) \mid \\ &\leq \beta_M P_M(\sum_{i=1}^{n} c_i(\psi_i o K_A)) \\ &\leq \beta_M K P_{M'}(\sum_{i=1}^{n} c_i \psi_i) \end{aligned}$$

for some $K > 0$, where $M^{'} = \{K_A(a) \mid a \in M\}$. It follows that $\sigma o \widehat{K_A} \in C_{BSE}(\Delta(\text{Lip}_d(X, A)))$. By applying the BSE- property of $\text{Lip}_d(X, A)$, there exists $T \in M(\text{Lip}_d(X, A))$ where $\hat{T} = \sigma o \widehat{K_A}$. Now define $T^{'} \in M(A)$ as follows:

$$T^{'}(a) = T(K_A(a))(x_0), \quad (a \in A)$$

where $x_0 \in X$ is an arbitrary member of X. If $a_1, a_2 \in A$;

$$\begin{aligned} T^{'}(a_1 a_2) = T(K_A(a_1 a_2))(x_0) &= T(K_A(a_1) K_A(a_2))(x_0) \\ &= (T(K_A(a_1)) K_A(a_2))(x_0) \\ &= T(K_A(a_1))(x_0) K_A(a_2)(x_0) = T^{'}(a_1).a_2 \end{aligned}$$

Hence $T^{'} \in M(A)$. Let $\varphi \in \Delta(A)$; It is easy to prove that $\hat{K_A}(x_0 \otimes \varphi) = \varphi$ and the following is yield:

$$
\begin{aligned}
\hat{T}'(\varphi) = \frac{\varphi(T'(a))}{\varphi(a)} &= \frac{\varphi(T(K_A(a))(x_0))}{\varphi(a)} \\
&= \frac{(x_0 \otimes \varphi)(T(K_A(a)))}{\varphi(f_a(x_0))} \\
&= \frac{(x_0 \otimes \varphi)(T(K_A(a)))}{(x_0 \otimes \varphi)(K_A(a))} \\
&= \hat{T}(x_0 \otimes \varphi) \\
&= \sigma o \widehat{K_A}(x_0 \otimes \varphi) \\
&= \sigma(\widehat{K_A}(x_0 \otimes \varphi)) \\
&= \sigma(\varphi)
\end{aligned}
$$

Therefore $\hat{T}' = \sigma$ and consequently $C_{BSE}(\Delta(A)) \subseteq \widehat{M((A)}$. Thus A is a Frechet- BSE- algebra. □

The correlation between the $C_{\text{BSE}}(\Delta(\text{Lip}_d(X, A)))$ and $\text{Lip}_d(X, C_{\text{BSE}}(\Delta(A)))$ is assessed as follows:

Theorem 9. *Let (X, d) be a metric space, (A, p_l) be a commutative semisimple Frechet algebra. Then $C_{BSE}(\Delta(\text{Lip}_d(X, A)))$ can be embedded in $\text{Lip}_d(X, C_{BSE}(\Delta(A))$, These two as Frechet algebras are isometric;*

Proof. Let

$$\phi : \; C_{BSE}(\Delta(\text{Lip}_d(X, A))) \to \text{Lip}_d(X, C_{BSE}(\Delta(A))),$$

defined by

$$\phi(\Sigma) = \phi_\Sigma \quad (\Sigma \in C_{BSE}(\Delta(\text{Lip}_d(X, A))),$$

Where

$$\phi_\Sigma(x)(\varphi) = \Sigma(x \otimes \varphi), \quad (x \in X, \; \varphi \in \Delta(A))$$

Assume that $\Sigma_1, \Sigma_2 \in C_{BSE}(\Delta(\text{Lip}_d(X, A)))$, where $\Sigma_1 = \Sigma_2$.
So $\Sigma_1(x \otimes \varphi) = \Sigma_2(x \otimes \varphi)$, at the result $\phi_{\Sigma_1}(x)(\varphi) = \phi_{\Sigma_2}(x)(\varphi)$, for all $x \in X$ and $\varphi \in \Delta(A)$. Then $\phi_{\Sigma_1} = \phi_{\Sigma_2}$ and, therefore, ϕ is well defined. It is obvious that ϕ is linear. Let $\Sigma_1, \Sigma_2 \in$

$C_{BSE}(\Delta(\mathrm{Lip}_d(X,A)))$, so $\phi(\Sigma_1.\Sigma_2)=\phi_{\Sigma_1.\Sigma_2}$, By allowing $x\in X$ and $\varphi\in\Delta(A)$, the following is yield:

$$\begin{aligned}\phi_{\Sigma_1.\Sigma_2}(x)(\varphi) &= \Sigma_1.\Sigma_2(x\otimes\varphi)\\ &= \Sigma_1(x\otimes\varphi).\Sigma_2(x\otimes\varphi)\\ &= \phi_{\Sigma_1}(x)(\varphi).\phi_{\Sigma_2}(x)(\varphi)\end{aligned}$$

thus $\phi_{\Sigma_1.\Sigma_2}(x)=\phi_{\Sigma_1}(x).\phi_{\Sigma_2}(x)$, so $\phi_{\Sigma_1.\Sigma_2}=\phi_{\Sigma_1}.\phi_{\Sigma_2}$. Then ϕ is homomorphism. First of all, $\phi_\Sigma(x)\in C_{BSE}(\Delta(A))$, for each $x\in X$ and $\Sigma\in C_{BSE}(\Delta(\mathrm{Lip}_d(X,A))$ is assessed. In fact, Since $\Sigma\in C_{BSE}(\Delta(\mathrm{Lip}_d(X,A))$, so there exists a bounded set M in $\mathrm{Lip}_d(X,A)$ such that for every complex number $c_1,\cdots,c_n$ and the same number $\varphi_1,\cdots,\varphi_n\in\Delta(A)$, we have

$$\begin{aligned}P_M(\sum_{i=1}^n c_i(x\otimes\varphi_i)) &= sup\{\mid\sum_{i=1}^n c_i(x\otimes\varphi_i)(f)\mid:\ f\in M\}\\ &= sup\{\mid\sum_{i=1}^n c_i\varphi_i(f(x))\mid:\ f\in M\}\\ &\leq sup\{\mid\sum_{i=1}^n c_i\varphi_i(a)\mid:\ a\in M^{'}\}\\ &= P_{M'}(\sum_{i=1}^n c_i\varphi_i).\end{aligned}$$

Where $M^{'}:=\hat{x}(M)$. This implies that

$$\begin{aligned}\mid\sum_{i=1}^n c_i\phi_\Sigma(x)(\varphi_i)\mid &= \mid\sum_{i=1}^n c_i\Sigma(x\otimes\varphi_i)\mid\\ &\leq q_l(\Sigma)P_M(\sum_{i=1}^n c_i x\otimes\varphi_i)\\ &= q_l(\Sigma)P_{M'}(\sum_{i=1}^n c_i\varphi_i)\end{aligned}$$

Hence $\phi_\Sigma(x)\in C_{BSE}(\Delta(A))$ and $q_l(\Sigma)\leq q_l(\phi_\Sigma(x))$, for each $x\in X$, since $q_l(\Sigma)=inf\{\beta_M\mid\mid\sum_{j=1}^n c_j\Sigma(\varphi_j)\mid\leq\beta_M P_M(\sum_{j=1}^n c_j\varphi_j)\}$ In the other hand

$$\begin{aligned}q_l(\phi_\Sigma(x)) &= sup\left\{\mid\sum_{i=1}^n c_i(\phi_\Sigma(x))(\varphi_i)\mid:\ P_{M_l}(\sum_{i=1}^n c_i\varphi_i)\leq 1,\ \varphi_i\in\Delta(A)\right\}\\ &= sup\left\{\mid\sum_{i=1}^n c_i\Sigma(x\otimes\varphi_i)\mid:\ P_{M_l}(\sum_{i=1}^n c_i(x\otimes\varphi_i))\leq 1,\ \varphi_i\in\Delta(A)\right\}\\ &\leq q_l(\Sigma)\end{aligned}$$

Therefore

$$q_l(\phi_\Sigma(x)) \leq q_l(\Sigma).$$

Consequently, for all $x \in X, \Sigma \in C_{BSE}(\Delta(\mathrm{Lip}_d(X, A)))$ and $l \in \mathbb{N}$, we have

$$q_l(\phi_\Sigma(x)) = q_l(\Sigma). \tag{1}$$

Note that

$$\begin{aligned} q_l(\phi_\Sigma(x) - \phi_\Sigma(y)) = & \ sup\{|\sum_{i=1}^{n} c_i(\phi_\Sigma(x) - \phi_\Sigma(y))(\varphi_i)|: \ P_{M_l}(\sum_{i=1}^{n} c_i\varphi_i) \leq 1, \ \varphi_i \in \Delta(A)\} \\ = & \ sup\{|\sum_{i=1}^{n} c_i(\Sigma(x\otimes\varphi_i) - \Sigma(y\otimes\varphi_i))|: \ P_{M_l}(\sum_{i=1}^{n} c_i\varphi_i) \leq 1, \ \varphi_i \in \Delta(A)\} \\ \leq & \ q_l(\Sigma) sup\{P_{M_l}(\sum_{i=1}^{n} c_i(x\otimes\varphi_i - y\otimes\varphi_i)) : \ P_{M_l}(\sum_{i=1}^{n} c_i\varphi_i) \leq 1, \ \varphi_i \in \Delta(A)\}. \end{aligned}$$

This follows that

$$\begin{aligned} p_{l,C_{BSE}(\Delta(A))}(\phi_\Sigma) = & \ sup\{\frac{q_l(\phi_\Sigma(x) - \phi_\Sigma(y))}{d(x,y)} : x \neq y\} \\ & \leq q_l(\Sigma) sup \ sup\{\{\frac{P_{M_l}(\sum_{i=1}^{n} c_i(x\otimes\varphi_i - y\otimes\varphi_i))}{d(x,y)} : \\ & P_{M_l}(\sum_{i=1}^{n} c_i\varphi_i) \leq 1, \ \varphi_i \in \Delta(A)\} : x \neq y\} \\ & \leq q_l(\Sigma) \end{aligned}$$

Therefore $p_{l,C_{BSE}(\Delta(A))}(\phi_\Sigma) \leq q_l(\Sigma)$. Also

$$q_{l,C_{BSE}(\Delta(A))}(\phi_\Sigma) = sup\{q_l(\phi_\Sigma)(x)) : x \in X\}$$

and by using relation (1), $q_l(\phi_\Sigma(x)) = q_l(\Sigma)$ and so $q_{l,C_{BSE}(\Delta(A))}(\phi_\Sigma) = q_l(\Sigma)$. This show that

$$r'_{l,C_{BSE}(\Delta(A))}(\phi_\Sigma) = max\{p_{l,C_{BSE}(\Delta(A))}(\phi_\Sigma), q_{l,C_{BSE}(\Delta(A))}(\phi_\Sigma)\} = q_l(\Sigma)$$

Which implies that ϕ is isometry. This completes the proof. □

Let $T \in M(A)$. Define

$$q'_l(T) = sup\{p_l(T(a)) : a \in A, \ p_l(a) \leq 1\}.$$

It is obvious that q'_l is a seminorm on $M(A)$. In the following theorem, it will be shown that $Lip_d(X, A)$ can be embedded in $M(\mathrm{Lip}_d(X, A))$, isometrically as two locally convex algebras which are isometric.

Theorem 10. *Let (X, d) be a metric space and (A, p_l) be a commutative semisimple Frechet algebra. Then*

$$\mathrm{Lip}_d(X, M(A)) \underset{\rightarrow}{\subseteq} M(\mathrm{Lip}_d(X, A)),$$

As two locally convex algebras which are isometric.

Proof. Let

$$\phi \ : \mathrm{Lip}_d(X, M(A)) \rightarrow M(\mathrm{Lip}_d(X, A))$$

Where

$$\phi(F) = \phi_F \quad (F \in \mathrm{Lip}_d(X, M(A))).$$

Defined by

$$\phi_F(g) = F \odot g \qquad (g \in \mathrm{Lip}_d(X, A))$$
$$F \odot g(x) = F(x)(g(x)) \qquad (x \in X).$$

It will be concluded that ϕ is an isomorphism map. Assume that $F_1, F_2 \in \mathrm{Lip}_d(X, M(A))$ where $F_1 = F_2$, so $F_1(x) = F_2(x)$, for each $x \in X$. Thus $F_1(x)(g(x)) = F_2(x)(g(x))$, for all $g \in \mathrm{Lip}_d(X, A)$, then $F_1 \odot g(x) = F_2 \odot g(x)$, for all $x \in X$ and $g \in \mathrm{Lip}_d(X, A)$. Therefore $\phi(F_1) = \phi(F_2)$ and so ϕ is well-defined.

1) ϕ_F is a continuous linear multiplier on $Lip_\alpha(X, A)$ is assessed in the following:

Assume that $g_1, g_2 \in \mathrm{Lip}_d(X, M(A))$, so

$$\phi_F(g_1 g_2) = F \odot g_1 g_2.$$

for any $x \in X$, the following is yield:

$$\begin{aligned} F \odot g_1 g_2(x) &= F(x)(g_1 g_2(x)) \\ &= F(x)(g_1(x) g_2(x)) \\ &= g_1(x) F(x)(g_2(x)) \\ &= g_1(x) F \odot g_2(x) \end{aligned}$$

This implies that

$$F \odot g_1 g_2 = g_1.(F \odot g_2).$$

Then $\phi_F(g_1 g_2) = g_1.\phi_F(g_2)$, for all $g_1, g_2 \in \mathrm{Lip}_d(X, M(A))$, at the result ϕ_F is a multiplier. It is obvious that for each $F \in \mathrm{Lip}_d(X, M(A))$, the map ϕ_F is linear. In the sequel, that

ϕ_F is a continuous map will be proved. Let (g_n) be a sequence in $\text{Lip}_d(X, A)$ converges to $g \in \text{Lip}_d(X, A)$. Then $r_{l,A}(g_n) \to r_{l,A}(g)$, thus $q_{l,A}(g_n) \to q_{l,A}(g)$ and so $p_{l,A}(g_n) \to p_{l,A}(g)$. Which yield:

$$\begin{aligned} q_{l,A}(F \odot g_n - F \odot g) &= sup\left\{p_l(F(x)(g_n(x) - g(x))) \ : \ x \in X\right\} \\ &\leq K \ sup\left\{p_l(g_n(x) - g(x)) \ : \ x \in X\right\} \\ &= K q_{l,A}(g_n - g) \to 0, \end{aligned}$$

Which

$$K := q_{l,M(A)}(F) = sup\left\{q_l^{'}(F(x)) \mid x \in X\right\}$$

Note that

$$q_l^{'}(F(x)) = sup\left\{p_l(F(x))(a) \mid a \in A, p_l(a) \leq 1\right\}.$$

This shows that

$$q_{l,A}(F \odot g_n - F \odot g) \to 0.$$

Also:

$$\begin{aligned} p_{l,A}(F \odot g_n - F \odot g) &= sup\left\{\frac{p_l(F(x)(g_n(x) - g(x)) - F(y)(g_n(y) - g(y)))}{d(x,y)} \ : \ x \neq y\right\} \\ &\leq sup\left\{\frac{p_l\left(F(x)(g_n(x) - g(x)) - (g_n(y) - g(y))\right)}{d(x,y)} \ : \ x \neq y\right\} \\ &+ sup\left\{\frac{p_l(F(x) - F(y))}{d(x,y)})(g_n(y) - g(y) \ : \ x \neq y\right\} \\ &\leq q_{l,M(A)}(F) p_{l,A}(g_n - g) + p_{l,M(A)}(F) q_{l,A}(g_n - g) \to 0 \end{aligned}$$

Which yield:

$$p_{l,A}(F \odot g_n - F \odot g) \to 0.$$

Therefore $r_{l,A}(F \odot g_n - F \odot g) \to 0$. Hence ϕ_F is a continuous map. This follows that $\phi_F \in M(\text{Lip}_d(X, A))$.

2) In following, it will be concluded that ϕ is an isomorphism map.

It is obvious that ϕ is a linear map. Assume that (F_n) is a sequence in $\text{Lip}_d(X, M(A))$ which converges to some $F \in \text{Lip}_d(X, M(A))$, so $p_{l,M(A)}(F_n - F) \to 0$ and $q_{l,M(A)}(F_n - F) \to 0$.

yielding the following:

$$q_l^{'}(\phi(F_n)-\phi(F)) = sup\left\{r_{\alpha,A}(F_n\odot g - F\odot g) : g\in \mathrm{Lip}_d(X,A)\right\}$$

Hence

$$\begin{aligned} q_{l,A}(F_n\odot g - F\odot g) &= sup\left\{p_l((F_n(x)-F(x))g(x)) : x\in X\right\} \\ &\leq supsup\left\{p_l((F_n-F)(x)(a)) : a\in A, x\in X\right\} \\ &= sup\left\{q_l^{'}((F_n-F)(x)) : x\in X\right\} = q_{l,M(A)}(F_n-F)\to 0 \end{aligned}$$

Thus

$$q_{l,A}(F_n\odot g - F\odot g)\to 0.$$

Moreover

$$\begin{aligned} p_{l,A}(F_n\odot g - F\odot g) &= sup\left\{\frac{p_l((F_n-F)(x)(g(x))) - ((F_n-F)(y)(g(y)))}{d(x,y)} : x\neq y\right\} \\ &\leq sup\left\{\frac{p_l(((F_n-F)(x)-(F_n-F)(y))(g(x))}{d(x,y)} : x\neq y\right\} \\ &+ sup\left\{\frac{(p_l(F_n-F)(y)(g(x)-g(y))}{d(x,y)} : x\neq y\right\} \\ &\leq p_{l,M(A)}(F_n-F).q_{l,A}(g) + q_{l,M(A)}(F_n-F).p_{l,A}(g)\to 0, \end{aligned}$$

and so

$$p_{l,A}(F_n\odot g - F\odot g)\to 0.$$

It follows that $r_{l,A}(\phi_{F_n}(g)-\phi_F(g))\to 0$, for all $g\in \mathrm{Lip}_d(X,A)$. Consequently $q_l^{''}(\phi(F_n)-\phi(F))\to 0$. Therefore ϕ is a continuous map.

In this stage, that ϕ is injective is assessed. Assume that $F\in \mathrm{Lip}_d(X,M(A))$ where $\phi_F = \phi(F) = 0$. If $F\neq 0$, then there exists $x_0\in X$ where $F(x_0)\neq 0$, so there exists $a_0\in A$ where $(F(x_0))(a_0)\neq 0$. Put $g = f_{a_0}$, thus $g\in \mathrm{Lip}_d(X,A)$ and

$$F\odot g(x_0) = F(x_0)(g(x_0)) = (F(x_0))(a_0))\neq 0$$

i.e.

$$\phi_F(g)(x_0) = F\odot g(x_0)\neq 0$$

This is a contradiction. Therefore ϕ is injective.

□

At this stage, based on the established prerequisite the primary Theorem, is expressed as follows:

Theorem 11. *Let (X, d) be a metric space and (A, p_l) be a commutative semisimple Frechet algebra. Then $\mathrm{Lip}_d(X, A)$ is a Frechet- BSE-algebra if and only if A is a Frechet- BSE algebra. Then*
1) If $\mathrm{Lip}_d(X, A)$ is a Frechet- BSE-algebra, then A is a Frechet- BSE-algebra.
2) If A is unital and Frechet- BSE-algebra, then $\mathrm{Lip}_d(X, A)$ is a Frechet- BSE-algebra.

Proof. 1) If $\mathrm{Lip}_d(X, A)$ is a Frechet- BSE-algebra, then by Proposition 8, A is a Frechet- BSE-algebra.
2) Assume that A is a BSE-algebra. Since A is semisimple, then by applying Proposition 3, $\mathrm{Lip}_d(X, A)$ is semisimple. By applying Proposition 8 imply that

$$(M(\mathrm{Lip}_d(X, A))\widehat{)} \subseteq C_{\mathrm{BSE}}(\Delta(\mathrm{Lip}_d(X, A)).$$

For the reverse inclusion, according to Theorem 9 and Theorem 10, the following is yield:

$$\begin{aligned} C_{\mathrm{BSE}}(\Delta(\mathrm{Lip}_d(X, A)) &\underset{\rightarrow}{\subseteq} \mathrm{Lip}_d(X, C_{\mathrm{BSE}}(\Delta(A))) \\ &\cong \mathrm{Lip}_d(X, \widehat{M(A)}) \\ &= \mathrm{Lip}_d(X, M(A)) \\ &\underset{\rightarrow}{\subseteq} M(\mathrm{Lip}_d(X, A) \\ &\cong (M(\mathrm{Lip}_d(X, A))\widehat{)}. \end{aligned}$$

Thus

$$C_{\mathrm{BSE}}(\Delta(\mathrm{Lip}_d(X, A)) \cong (M(\mathrm{Lip}_d(X, A)))\widehat{)}.$$

□

Because every commutative C^*- Banach algebra is BSE algebra, [13], so by using Theorem 11, the following example is immediate:

Example 1. *Let (X, d) be a metric space and A be a commutative C^*- Banach algebra. Then $\mathrm{Lip}_d(X, A)$ is a BSE-Frechet algebra.*

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