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NEUMANN PROBLEM INVOLVING THE $p(x)$-KIRCHHOFF-LAPLACIAN-LIKE OPERATOR IN VARIABLE EXPONENT SOBOLEV SPACE

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ABSTRACT. This paper is concerned with a class of $p(x)$-Kirchhoff type problem for the $p(x)$-Laplacian-like operator under Neumann boundary condition. We apply the topological degree theory for a class of demicontinuous operator of generalized (S_+) type and the theory of variable exponent Sobolev space to etablish the existence of weak solution of this problem. Our result extend and generalize several corresponding results from the existing literature.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N (N > 1)$ with smooth boundary denoted by $\partial\Omega$, and let δ, μ and λ be three real parameters and $p(x), a(x) \in C_+(\overline{\Omega})$.

In this paper, we establish the existence of weak solution for a class of $p(x)$-Kirchhoff type problem for the $p(x)$-Laplacian-like operator of the following form:

$$(1.1) \quad \begin{cases} -\mathcal{M}\Big(\mathcal{K}(u)\Big)\Big(\Delta^l_{p(x)}u - |u|^{p(x)-2}u\Big) + \delta|u|^{a(x)-2}u = \mu\, g(x,u) + \lambda\, f(x,u,\nabla u) & \text{in } \Omega, \\ \\ \frac{\partial u}{\partial \eta} = \frac{\partial}{\partial \eta}(\Delta^l_{p(x)}u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\mathcal{K}(u) := \int_\Omega \frac{1}{p(x)}\Big(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}} + |u|^{p(x)}\Big)dx,$$

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and

$$\Delta^{l}_{p(x)} u := \operatorname{div}\Big(|\nabla u|^{p(x)-2} \nabla u + \frac{|\nabla u|^{2p(x)-2} \nabla u}{\sqrt{1 + |\nabla u|^{2p(x)}}} \Big)$$

is the $p(x)$-Laplacian-like operator, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions that satisfy the assumption of growth and $\mathcal{M} : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function.

The motivation for this research originated from the application of similar models in physics to represent the behavior of elasticity [24] and electrorheological fluids (see [1, 4, 5, 22]), which have the ability to modify their mechanical properties when exposed to an electric field (see [18, 19]), specifically the phenomenon of capillarity, which depends on solid-liquid interfacial characteristics as surface tension, contact angle, and solid surface geometry.

Problems related to (1.1) have been studied by many scholars, for example, in the case when $\mathcal{M}\Big(\mathcal{K}(u)\Big) \equiv 1$, $\mu = \delta = 0$, $\lambda > 0$, f independent of ∇u and without the term $|u|^{p(x)-2}u$ with Dirichlet boundary condition, we know that the problem (1.1) has a nontrivial solutions from [21] (see also [6–8]).

Note that, in the case when $\mathcal{K}(u) = \displaystyle\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)})dx$, $\mu = \delta = 0$, $\lambda = 1$, f independent of ∇u and without the term $|u|^{p(x)-2}u$ with Dirichlet boundary condition, then we obtain the following problem

$$(1.2) \qquad \begin{cases} -\mathcal{M}\Big(\displaystyle\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx \Big) \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = f(x, u) & \text{in } \Omega, \\[2ex] u = 0 & \text{on } \partial\Omega, \end{cases}$$

which is called the $p(x)$-Kirchhoff type problem. In this case, Dai et al. [3], by a direct variational approach, established conditions ensuring the existence and multiplicity of solutions to (1.2). Furthermore, the problem (1.2) is a generalization of the stationary problem of a model introduced by Kirchhoff [11] of the following form:

$$\rho \frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \Big| \frac{\partial u}{\partial x} \Big|^2 dx \Big) \frac{\partial^2 u}{\partial x^2} = 0,$$

where ρ, ρ_0, h, E, L are all constants, which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration.

The remainder of the paper is organized as follows. In Section 2, we review some fundamental preliminaries about the functional framework where we will treat our problem and we introduce some classes of operators of generalized (S_+) type, as well as the Berkovits topological degrees. Finaly, in Section 3, we give our basic assumptions, some technical lemmas, and we will state and prove the main result of the paper.

2. Preliminaries

2.1. **Variable exponent Sobolev space $W^{1,p(x)}(\Omega)$.** In the analysis of problem (1.1), we will use the theory of variable exponent Sobolev space $W^{1,p(x)}(\Omega)$. We refer to [9, 12, 14–17] for more details.

Let Ω be a smooth bounded domain in $\mathbb{R}^N (N > 2)$, with a Lipschitz boundary denoted by $\partial\Omega$. Set

$$C_+(\overline{\Omega}) = \Big\{p : p \in C(\overline{\Omega}) \text{ such that } p(x) > 1 \text{ for any } x \in \overline{\Omega}\Big\}.$$

For each $p \in C_+(\overline{\Omega})$, we define

$$p^+ := \max\Big\{p(x),\ x \in \overline{\Omega}\Big\} \text{ and } p^- := \min\Big\{p(x),\ x \in \overline{\Omega}\Big\}.$$

For every $p \in C_+(\overline{\Omega})$, we define

$$L^{p(x)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ is measurable such that } \int_\Omega |u(x)|^{p(x)}dx < +\infty\Big\},$$

equipped with the Luxemburg norm

$$|u|_{p(x)} = \inf\Big\{\lambda > 0 : \rho_{p(x)}\Big(\frac{u}{\lambda}\Big) \le 1\Big\},$$

where

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}dx,\ \forall\, u \in L^{p(x)}(\Omega).$$

Proposition 2.1. [9, *Theorem 1.3 and Theorem 1.4*] *Let (u_n) and $u \in L^{p(x)}(\Omega)$, then*

$$|u|_{p(x)} < 1\big(resp. = 1; > 1\big) \Leftrightarrow \rho_{p(x)}(u) < 1\big(resp. = 1; > 1\big), \tag{2.1}$$

$$|u|_{p(x)} > 1 \ \Rightarrow\ |u|^{p^-}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^+}_{p(x)}, \tag{2.2}$$

$$|u|_{p(x)} < 1 \ \Rightarrow\ |u|^{p^+}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^-}_{p(x)}, \tag{2.3}$$

$$\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \ \Leftrightarrow\ \lim_{n\to\infty} \rho_{p(x)}\big(u_n - u\big) = 0. \tag{2.4}$$

Remark 2.2. According to (2.2) and (2.3), we have

$$|u|_{p(x)} \leq \rho_{p(x)}(u) + 1, \tag{2.5}$$

$$\rho_{p(x)}(u) \leq |u|^{p^-}_{p(x)} + |u|^{p^+}_{p(x)}. \tag{2.6}$$

Proposition 2.3. *[12, Theorem 2.5 and Corollary 2.7] The space $\left(L^{p(x)}(\Omega), |\cdot|_{p(x)}\right)$ is a separable and reflexive Banach space.*

Proposition 2.4. *[12, Theorem 2.1] The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \Omega$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the following Hölder-type inequality*

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(x)}|v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)}. \tag{2.7}$$

Remark 2.5. [9, Theorem 1.11] If $p_1, p_2 \in C_+(\overline{\Omega})$ with $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then we have $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

Now, we define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ as

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) \text{ such that } |\nabla u| \in L^{p(x)}(\Omega)\right\},$$

with the norm

$$|u|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Furthermore, we have the compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$(see [12]).

Remark 2.6. Note that for all $u \in W^{1,p(x)}(\Omega)$, we have

$$|u|_{p(x)} \leq |u|_{1,p(x)} \text{ and } |\nabla u|_{p(x)} \leq |u|_{1,p(x)}.$$

Next, for all $u \in W^{1,p(x)}(\Omega)$, we introduce the following notation

$$\rho_{1,p(x)}(u) = \rho_{p(x)}(u) + \rho_{p(x)}(\nabla u).$$

Then, from [9, Theorem 1.3], we have the following result.

Proposition 2.7. *If $u \in W^{1,p(x)}(\Omega)$, then the following properties hold true*

$$|u|_{1,p(x)} < 1(resp. = 1; > 1) \;\Leftrightarrow\; \rho_{1,p(x)}(u) < 1(resp. = 1; > 1), \tag{2.8}$$

$$|u|_{1,p(x)} > 1 \;\Rightarrow\; |u|^{p^-}_{1,p(x)} \leq \rho_{1,p(x)}(u) \leq |u|^{p^+}_{1,p(x)}, \tag{2.9}$$

(2.10) $$|u|_{1,p(x)} < 1 \ \Rightarrow \ |u|^{p^+}_{1,p(x)} \leq \rho_{1,p(x)}(u) \leq |u|^{p^-}_{1,p(x)}.$$

Proposition 2.8. [9, 12] *The space* $\left(W^{1,p(x)}(\Omega), |\cdot|_{1,p(x)}\right)$ *is a separable and reflexive Banach space.*

Remark 2.9. The dual space of $W^{1,p(x)}(\Omega)$ denoted $W^{-1,p'(x)}(\Omega)$, is equipped with the norm

$$|u|_{-1,p'(x)} = \inf\left\{|u_0|_{p'(x)} + \sum_{i=1}^{N} |u_i|_{p'(x)}\right\},$$

where the infinimum is taken on all possible decompositions $u = u_0 - \text{div}F$ with $u_0 \in L^{p'(x)}(\Omega)$ and $F = (u_1, \dots, u_N) \in (L^{p'(x)}(\Omega))^N$.

2.2. **Topological degree theory.** Now, we give some results and properties from the theory of topological degree. The readers can find more information about the history of this theory in [2, 10].

In what follows, let X be a real separable reflexive Banach space and X^* be its dual space with dual pairing $\langle \cdot, \cdot \rangle$ and given a nonempty subset Ω of X. Strong (weak) convergence is represented by the symbol $\to$ ($\rightharpoonup$).

Definition 2.10. Let Y be a real Banach space. A operator $F : \Omega \subset X \to Y$ is said to be :

(1) bounded, if it takes any bounded set into a bounded set.
(2) demicontinuous, if for any sequence $(u_n) \subset \Omega,\ u_n \to u$ implies that $F(u_n) \rightharpoonup F(u)$.
(3) compact, if it is continuous and the image of any bounded set is relatively compact.

Definition 2.11. A mapping $F : \Omega \subset X \to X^*$ is said to be :

(1) of class (S_+), if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle Fu_n, u_n - u\rangle \leq 0$, we have $u_n \to u$.
(2) quasimonotone, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, we have $\limsup_{n\to\infty}\langle Fu_n,\ u_n - u\rangle \geq 0$.

Definition 2.12. Let $T : \Omega_1 \subset X \to X^*$ be a bounded operator such that $\Omega \subset \Omega_1$. For any operator $F : \Omega \subset X \to X$, we say that

(1) F of class $(S_+)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup_{n\to\infty}\langle Fu_n,\ y_n - y\rangle \leq 0$, we have $u_n \to u$.
(2) F has the property $(QM)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$, we have $\limsup_{n\to\infty}\langle Fu_n, y - y_n\rangle \geq 0$.

In the rest of this paper, we consider the following classes of operators:

$$\mathcal{F}_1(\Omega) := \Big\{F : \Omega \to X^* \mid F \text{ is bounded, demicontinuous and of class } (S_+)\Big\},$$
$$\mathcal{F}_T(\Omega) := \Big\{F : \Omega \to X \mid F \text{ is demicontinuous and of class } (S_+)_T\Big\},$$

for any $\Omega \subset D(F)$, where $D(F)$ denotes the domain of F, and any $T \in \mathcal{F}_1(\Omega)$.
Now, let $\mathcal{O}$ be the collection of all bounded open sets in X and we define

$$\mathcal{F}(X) := \Big\{F \in \mathcal{F}_T(\overline{E}) \mid E \in \mathcal{O},\ \mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})\Big\},$$

where, $\mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})$ is called an essential inner map to F.

Lemma 2.13. [10, *Lemma 2.3*] *Let $T \in \mathcal{F}_1(\overline{E})$ be continuous and $S : D(S) \subset X^* \to X$ be demicontinuous such that $T(\overline{E}) \subset D(S)$, where E is a bounded open set in a real reflexive Banach space X. Then the following statements are true :*

(1) *If S is quasimonotone, then $I + S \circ T \in \mathcal{F}_T(\overline{E})$, where I denotes the identity operator.*
(2) *If S is of class (S_+), then $S \circ T \in \mathcal{F}_T(\overline{E})$.*

Definition 2.14. Suppose that E is bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\overline{E})$ is continuous and $F, S \in \mathcal{F}_T(\overline{E})$. The affine homotopy
$\mathcal{H} : [0,1] \times \overline{E} \to X$ defined by

$$\mathcal{H}(t,u) := (1-t)Fu + tSu, \quad \text{for all} \quad (t,u) \in [0,1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 2.15. [10, Lemma 2.5] *The above affine homotopy is of class $(S_+)_T$.*

Next, as in [10] we give the topological degree for the class $\mathcal{F}(X)$.

Theorem 2.16. *Let*

$$M = \Big\{(F, E, h) \mid F \in \mathcal{F}(X),\ E \in \mathcal{O},\ h \notin F(\partial E)\Big\}.$$

Then, there exists a unique degree function $d : M \longrightarrow \mathbb{Z}$ that satisfies the following properties:

(1) (Normalization) *For any $h \in E$, we have*

$$d(I, E, h) = 1.$$

(2) (Homotopy invariance) *If* $\mathcal{H} : [0,1] \times \overline{E} \to X$ *is a bounded admissible affine homotopy with a common continuous essential inner map and* h*:* $[0,1] \to X$ *is a continuous path in* X *such that* $h(t) \notin \mathcal{H}(t, \partial E)$ *for all* $t \in [0,1]$*, then*

$$d(\mathcal{H}(t,\cdot), E, h(t)) = C \text{ for all } t \in [0,1].$$

(3) (Existence) *If* $d(F, E, h) \neq 0$*, then the equation* $Fu = h$ *has a solution in* E*.*

Definition 2.17. [10, Definition 3.3] The above degree is defined as follows:

$$d(F, E, h) \ := d_B(F|_{\overline{E}_0}, E_0, h),$$

where d_B is the Berkovits degree [2] *and* E_0 *is any open subset of* E *with* $F^{-1}(h) \subset E_0$ *and* F *is bounded on* $\overline{E}_0$*.*

3. Main result

In this section, we will discuss the existence of weak solution of (1.1).

We assume that $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, p, $a \in C_+(\overline{\Omega})$ with $1 < a^- \leq a(x) \leq a^+ < p^-$, $\mathcal{M} : \mathbb{R}^+ \to \mathbb{R}^+$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are functions such that:

(A_1)**:** f is a Carathéodory function.

(A_2)**:** There exists $C_1 > 0$ and $l \in L^{p'(x)}(\Omega)$ such that

$$|f(x, y, z)| \leq C_1(l(x) + |y|^{q(x)-1} + |z|^{q(x)-1}).$$

(A_3)**:** g is a Carathéodory function.

(A_4)**:** There are $C_2 > 0$ and $k \in L^{p'(x)}(\Omega)$ such that

$$|g(x, y)| \leq C_2(k(x) + |y|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and all $(y, z) \in \mathbb{R} \times \mathbb{R}^N$, where $q, s \in C_+(\overline{\Omega})$ with $1 < q^- \leq q(x) \leq q^+ < p^-$ and $1 < s^- \leq s(x) \leq s^+ < p^-$.

(M_0)**:** $\mathcal{M} : [0, +\infty) \to (c_0, +\infty)$ is a continuous and increasing function with $c_0 > 0$.

Remark 3.1. • Note that, for all $u, \varphi \in W^{1,p(x)}(\Omega)$

$$\mathcal{M}\Big(\mathcal{K}(u)\Big) \int_\Omega \Big(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1 + |\nabla u|^{2p(x)}}}\Big)\nabla\varphi + |u|^{p(x)-2}u\ \varphi\Big)dx$$

is well defined (see [13]).

- We have $\delta|u|^{a(x)-2}u \in L^{p'(x)}(\Omega)$, $\mu\, g(x,u) \in L^{p'(x)}(\Omega)$ and $\lambda\, f(x,u,\nabla u) \in L^{p'(x)}(\Omega)$ under $u \in W^{1,p(x)}(\Omega)$, the assumptions (A_2) and (A_4) and the given hypotheses about the exponents p, a, q and s because: $l \in L^{p'(x)}(\Omega)$, $k \in L^{p'(x)}(\Omega)$, $r(x) = (q(x)-1)p'(x) \in C_+(\overline{\Omega})$ with
  $r(x) < p(x)$, $\beta(x) = (a(x)-1)p'(x) \in C_+(\overline{\Omega})$ with $\beta(x) < p(x)$ and
  $\kappa(x) = (s(x)-1)p'(x) \in C_+(\overline{\Omega})$ with $\kappa(x) < p(x)$.
  Then, by Remark 2.5 we can conclude that $L^{p(x)} \hookrightarrow L^{r(x)}$, $L^{p(x)} \hookrightarrow L^{\beta(x)}$ and $L^{p(x)} \hookrightarrow L^{\kappa(x)}$.
  Hence, since $\varphi \in L^{p(x)}(\Omega)$, we have
  $$\Big(-\delta|u|^{a(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u)\Big)\varphi \in L^1(\Omega).$$
  This implies that, the integral
  $$\int_\Omega \Big(-\delta|u|^{a(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u)\Big)\varphi dx$$
  exists.

Then, we shall use the definition of weak solution for (1.1) in the following sense:

Definition 3.2. We say that a function $u \in W^{1,p(x)}(\Omega)$ is a weak solution of (1.1), if for any $\varphi \in W^{1,p(x)}(\Omega)$, it satisfies the following:

$$\mathcal{M}\Big(\mathcal{K}(u)\Big)\int_\Omega \bigg(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla\varphi + |u|^{p(x)-2}u\,\varphi\bigg)dx \\ = \int_\Omega \Big(-\delta|u|^{a(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u)\Big)\varphi dx.$$

Before giving our main result we first give two lemmas that will be used later.

Lemma 3.3. *If (M_0) holds, then the operator $\mathcal{I} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by*

$$\langle \mathcal{I}u, \vartheta\rangle = \mathcal{M}\Big(\mathcal{K}(u)\Big)\int_\Omega \bigg(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla\vartheta + |u|^{p(x)-2}u\,\vartheta\bigg)dx,$$

is continuous, bounded, strictly monotone and is of type (S_+).

Proof. Let us consider the following functional:

$$\mathcal{J}(u) := \widehat{\mathcal{M}}\Big(\mathcal{K}(u)\Big), \quad \text{where} \quad \widehat{\mathcal{M}}(s) = \int_0^s \mathcal{M}(\tau)\mathrm{d}\tau,$$

such that $\mathcal{M}(\tau)$ satisfies the assumption (M_0).

From [13], it is obvious that $\mathcal{J}$ is a continuously Gâteaux differentiable function whose Gâteaux derivative at the point $u \in W_0^{1,p(x)}(\Omega)$ is the functional $\mathcal{I}(u) := \mathcal{J}'(u) \in W^{-1,p'(x)}(\Omega)$ given by

$$\langle \mathcal{I}u, \varphi \rangle = \mathcal{M}\Big(\Theta(u)\Big) \int_\Omega \Big(\Big(|\nabla u|^{p(x)-2} \nabla u + \frac{|\nabla u|^{2p(x)-2} \nabla u}{\sqrt{1 + |\nabla u|^{2p(x)}}} \Big) \nabla \varphi + |u|^{p(x)-2} u \, \varphi \Big) dx,$$

for all $u, \varphi \in W_0^{1,p(x)}(\Omega)$ where $\langle \cdot, \cdot \rangle$ means the duality pairing between $W^{-1,p'(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$.

Hence, by using the similar argument as in [13, Theorem 3.1.] and in [21, Proposition 3.1.], we conclude that $\mathcal{I}$ is continuous, bounded, strictly monotone and is of type (S_+). □

Lemma 3.4. *Assume that the assumptions $(A_1) - (A_4)$ hold, then the operator $\mathcal{G}$ defined by*

$$\mathcal{G} : W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$$
$$\langle \mathcal{G}u, \varphi \rangle = -\int_\Omega \Big(-\delta |u|^{a(x)-2} u + \mu \, g(x,u) + \lambda \, f(x,u,\nabla u) \Big) \varphi dx,$$

for all $u, \varphi \in W^{1,p(x)}(\Omega)$, is compact.

Proof. In order to prove this lemma, we proceed in four steps.

Step 1 : Let $\Upsilon : W^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be an operator defined by

$$\Upsilon u(x) := -\mu \, g(x,u).$$

In this step, we prove that the operator Υ is bounded and continuous.

First, let $u \in W^{1,p(x)}(\Omega)$, bearing (A_4) in mind and using (2.5) and (2.6), we infer

$$\begin{aligned}
|\Upsilon u|_{p'(x)} &\leq \rho_{p'(x)}(\Upsilon u) + 1 \\
&= \int_\Omega |\mu \, g(x,u(x))|^{p'(x)} dx + 1 \\
&= \int_\Omega |\mu|^{p'(x)} |g(x,u(x)|^{p'(x)} dx + 1 \\
&\leq \Big(|\mu|^{p'^-} + |\mu|^{p'^+} \Big) \int_\Omega |C_2 \Big(k(x) + |u|^{s(x)-1} \Big) |^{p'(x)} dx + 1 \\
&\leq const \Big(|\mu|^{p'^-} + |\mu|^{p'^+} \Big) \int_\Omega \Big(|k(x)|^{p'(x)} + |u|^{\kappa(x)} \Big) dx + 1 \\
&\leq const \Big(|\mu|^{p'^-} + |\mu|^{p'^+} \Big) \Big(\rho_{p'(x)}(k) + \rho_{\kappa(x)}(u) \Big) + 1 \\
&\leq const \Big(|k|_{p(x)}^{p'^+} + |u|_{\kappa(x)}^{\kappa^+} + |u|_{\kappa(x)}^{\kappa^-} \Big) + 1.
\end{aligned}$$

Then, we deduce from Remark 2.6 and $L^{p(x)} \hookrightarrow L^{\kappa(x)}$, that

$$|\Upsilon u|_{p'(x)} \leq const\Big(|k|_{p(x)}^{p'^+} + |u|_{1,p(x)}^{\kappa^+} + |u|_{1,p(x)}^{\kappa^-}\Big) + 1,$$

that means Υ is bounded on $W^{1,p(x)}(\Omega)$.

Second, we show that the operator Υ is continuous. To this purpose let $u_n \to u$ in $W^{1,p(x)}(\Omega)$. We need to show that $\Upsilon u_n \to \Upsilon u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.

Note that if $u_n \to u$ in $W^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$. Hence there exist a subsequence (u_k) of (u_n) and ϕ in $L^{p(x)}(\Omega)$ such that

$$u_k(x) \to u(x) \ \text{ and } \ |u_k(x)| \leq \phi(x), \tag{3.1}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.

Hence, from (A_2) and (3.1), we have

$$|g(x, u_k(x))| \leq C_2(k(x) + |\phi(x)|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.

On the other hand, thanks to (A_3) and (3.1), we get, as $k \longrightarrow \infty$

$$g(x, u_k(x)) \to g(x, u(x)) \ \text{ a.e. } \ x \in \Omega.$$

Seeing that

$$k + |\phi|^{s(x)-1} \in L^{p'(x)}(\Omega) \ \text{ and } \ \rho_{p'(x)}(\Upsilon u_k - \Upsilon u) = \int_\Omega |g(x, u_k(x)) - g(x, u(x))|^{p'(x)} dx,$$

then, from the Lebesgue's theorem and the equivalence (2.4), we have

$$\Upsilon u_k \to \Upsilon u \ \text{ in } \ L^{p'(x)}(\Omega),$$

and consequently

$$\Upsilon u_n \to \Upsilon u \ \text{ in } \ L^{p'(x)}(\Omega),$$

that is, Υ is continuous.

Step 2 : We define the operator $\Psi : W^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Psi u(x) := \delta|u(x)|^{a(x)-2}u(x).$$

We will prove that Ψ is bounded and continuous.

It is clear that Ψ is continuous. Next we show that Ψ is bounded.

Let $u \in W^{1,p(x)}(\Omega)$ and using (2.5) and (2.6), we obtain

$$\begin{aligned}
|\Psi u|_{p'(x)} &\leq \rho_{p'(x)}(\Psi u) + 1 \\
&= \int_\Omega |\delta |u|^{a(x)-2} u|^{p'(x)} dx + 1 \\
&= \int_\Omega |\delta|^{p'(x)} |u|^{(a(x)-1)p'(x)} dx + 1 \\
&\leq \left(|\delta|^{p'^-} + |\delta|^{p'^+}\right) \int_\Omega |u|^{\beta(x)} dx + 1 \\
&= \left(|\delta|^{p'^-} + |\delta|^{p'^+}\right) \rho_{\beta(x)}(u) + 1 \\
&\leq \left(|\delta|^{p'^-} + |\delta|^{p'^+}\right)\left(|u|^{\beta-}_{\beta(x)} + |u|^{\beta+}_{\beta(x)}\right) + 1.
\end{aligned}$$

Hence, we deduce from $L^{p(x)} \hookrightarrow L^{\beta(x)}$ and Remark 2.6 that

$$|\Psi u|_{p'(x)} \leq const\left(|u|^{\beta-}_{1,p(x)} + |u|^{\beta+}_{1,p(x)}\right) + 1,$$

and consequently, Ψ is bounded on $W^{1,p(x)}(\Omega)$.

Step 3 : Let us define the operator $\Phi : W^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Phi u(x) := -\lambda\, f(x, u(x), \nabla u(x)).$$

We will show that Φ is bounded and continuous.

Let $u \in W^{1,p(x)}(\Omega)$. According to (A_2) and the inequalities (2.5) and (2.6), we obtain

$$\begin{aligned}
|\Phi u|_{p'(x)} &\leq \rho_{p'(x)}(\Phi u) + 1 \\
&= \int_\Omega |\lambda\, f(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&= \int_\Omega |\lambda|^{p'(x)} |f(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&\leq \left(|\lambda|^{p'^-} + |\lambda|^{p'^+}\right) \int_\Omega |C_1\left(l(x) + |u|^{q(x)-1} + |\nabla u|^{q(x)-1}\right)|^{p'(x)} dx + 1 \\
&\leq const\left(|\lambda|^{p'^-} + |\lambda|^{p'^+}\right) \int_\Omega \left(|l(x)|^{p'(x)} + |u|^{r(x)} + |\nabla u|^{r(x)}\right) dx + 1 \\
&\leq const\left(|\lambda|^{p'^-} + |\lambda|^{p'^+}\right)\left(\rho_{p'(x)}(l) + \rho_{r(x)}(u) + \rho_{r(x)}(\nabla u)\right) + 1 \\
&\leq const\left(|l|^{p'^+}_{p(x)} + |u|^{r^+}_{r(x)} + |u|^{r^-}_{r(x)} + |\nabla u|^{r^+}_{r(x)} + |\nabla u|^{r^-}_{r(x)}\right) + 1.
\end{aligned}$$

Taking into account that $L^{p(x)} \hookrightarrow L^{r(x)}$ and Remark 2.6, we have then

$$|\Phi u|_{p'(x)} \leq const\left(|l|^{p'^+}_{p(x)} + |u|^{r^+}_{1,p(x)} + |u|^{r^-}_{1,p(x)}\right) + 1,$$

and consequently Φ is bounded on $W^{1,p(x)}(\Omega)$.

It remains to show that Φ is continuous. Let $u_n \to u$ in $W^{1,p(x)}(\Omega)$, we need to show that $\Phi u_n \to \Phi u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.

Note that if $u_n \to u$ in $W^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$ and $\nabla u_n \to \nabla u$ in $(L^{p(x)}(\Omega))^N$. Hence, there exist a subsequence (u_k) and ϕ in $L^{p(x)}(\Omega)$ and ψ in $(L^{p(x)}(\Omega))^N$ such that

$$u_k(x) \to u(x) \text{ and } \nabla u_k(x) \to \nabla u(x), \tag{3.2}$$

$$|u_k(x)| \leq \phi(x) \text{ and } |\nabla u_k(x)| \leq |\psi(x)|, \tag{3.3}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.

Hence, thanks to (A_1) and (3.2), we get, as $k \longrightarrow \infty$

$$f(x, u_k(x), \nabla u_k(x)) \to f(x, u(x), \nabla u(x)) \text{ a.e. } x \in \Omega.$$

On the other hand, from (A_2) and (3.3), we can deduce the estimate

$$|f(x, u_k(x), \nabla u_k(x))| \leq C_1(l(x) + |\phi(x)|^{q(x)-1} + |\psi(x)|^{q(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.

Seeing that

$$l + |\phi|^{q(x)-1} + |\psi(x)|^{q(x)-1} \in L^{p'(x)}(\Omega),$$

and taking into account the equality

$$\rho_{p'(x)}(\Phi u_k - \Phi u) = \int_\Omega |f(x, u_k(x), \nabla u_k(x)) - f(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

then, we conclude from the Lebesgue's theorem and (2.4) that

$$\Phi u_k \to \Phi u \text{ in } L^{p'(x)}(\Omega)$$

and consequently

$$\Phi u_n \to \Phi u \text{ in } L^{p'(x)}(\Omega),$$

and then Φ is continuous.

Step 4: Let $I^* : L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be the adjoint operator of the operator $I : W^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$.

We then define

$$I^* \circ \Upsilon : W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

$$I^* \circ \Psi : W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

and

$$I^* \circ \Phi : W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega).$$

On another side, taking into account that I is compact, then I^* is compact. Thus, the composition $\mathcal{G} = I^* \circ \Upsilon + I^* \circ \Psi + I^* \circ \Phi$ is compact. □

We are now in the position to get the existence result of weak solution for (1.1).

Theorem 3.5. *Assume that $(A_1) - (A_4)$ and (M_0) hold, then the problem (1.1) admits a weak solution u in $W^{1,p(x)}(\Omega)$.*

Proof. The basic idea of our proof is to reduce the problem (1.1) to a new one governed by a Hammerstein equation, and apply the theory of topological degree introduced in Subsection 2.2 to show the existence of weak solution to the state problem.

For all $u, \varphi \in W^{1,p(x)}(\Omega)$, we define the operators $\mathcal{I}$ and $\mathcal{G}$, as in Lemmas 3.3 and 3.4, by

$$\langle \mathcal{I}u, \varphi \rangle = \mathcal{M}\Big(\mathcal{K}(u)\Big) \int_\Omega \Big(\Big(|\nabla u|^{p(x)-2} \nabla u + \frac{|\nabla u|^{2p(x)-2} \nabla u}{\sqrt{1 + |\nabla u|^{2p(x)}}} \Big) \nabla \varphi + |u|^{p(x)-2} u \, \varphi \Big) dx,$$

and

$$\langle \mathcal{G}u, \varphi \rangle = -\int_\Omega \Big(-\delta |u|^{a(x)-2} u + \mu \, g(x,u) + \lambda \, f(x,u,\nabla u) \Big) \varphi dx.$$

Consequently, the problem (1.1) is equivalent to the equation

$$\mathcal{I}u + \mathcal{G}u = 0, \quad u \in W^{1,p(x)}(\Omega). \tag{3.4}$$

Taking into account that, by Lemma 3.3, the operator $\mathcal{I}$ is a continuous, bounded, strictly monotone and of class (S_+), then, by [23, Theorem 26 A], the inverse operator

$$\mathcal{A} := \mathcal{I}^{-1} : W^{-1,p'(x)}(\Omega) \to W^{1,p(x)}(\Omega),$$

is also bounded, continuous, strictly monotone and of class (S_+).

On another side, according to Lemma 3.4, we have that the operator $\mathcal{G}$ is bounded, continuous and quasimonotone.

Consequently, following Zeidler's terminology [23], the equation (3.4) is equivalent to the following abstract Hammerstein equation

$$u = \mathcal{A}\varphi \text{ and } \varphi + \mathcal{G} \circ \mathcal{A}\varphi = 0, \quad u \in W^{1,p(x)}(\Omega) \text{ and } \varphi \in W^{-1,p'(x)}(\Omega). \tag{3.5}$$

Seeing that (3.4) is equivalent to (3.5), then to solve (3.4) it is thus enough to solve (3.5). In order to solve (3.5), we will apply the Berkovits topological degree introduced in Section 2.2.

First, let us set

$$\mathcal{R} := \Big\{\varphi \in W^{-1,p'(x)}(\Omega) : \exists\, t \in [0,1] \text{ such that } \varphi + t\mathcal{G}\circ\mathcal{A}\varphi = 0\Big\}.$$

Next, we show that $\mathcal{R}$ is bounded in $\in W^{-1,p'(x)}(\Omega)$.

Let us put $u := \mathcal{A}\varphi$ for all $\varphi \in \mathcal{R}$. Taking into account that $|\mathcal{A}\varphi|_{1,p(x)} = |u|_{1,p(x)}$, then we have the following two cases:

First case : If $|u|_{1,p(x)} \le 1$, then $|\mathcal{A}\varphi|_{1,p(x)} \le 1$, that means $\Big\{\mathcal{A}\varphi : \varphi \in \mathcal{R}\Big\}$ is bounded.

Second case : If $|u|_{1,p(x)} > 1$, then we deduce from (2.9), (A_2) and (A_4), the inequalities (2.7) and (2.6) and the Young's inequality that

$$\begin{aligned}
|\mathcal{A}\varphi|^{p^-}_{1,p(x)} &= |u|^{p-}_{1,p(x)} \\
&\le \rho_{1,p(x)}(u) \\
&= \rho_{p(x)}(u) + \rho_{p(x)}(\nabla u) \\
&\le \langle \varphi,\ \mathcal{A}\varphi\rangle \\
&= -t\langle \mathcal{G}\circ\mathcal{A}\varphi,\ \mathcal{A}\varphi\rangle \\
&= t\int_\Omega \Big(-\delta|u|^{a(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u)\Big) u dx \\
&\le t\max(|\delta|, C_2|\mu|, C_1|\lambda|)\Big(\rho_{a(x)}(u) + \int_\Omega |k(x)u(x)|dx + \int_\Omega |l(x)u(x)|dx \\
&\qquad\qquad + \rho_{s(x)}(u) + \rho_{q(x)}(u) + \int_\Omega |\nabla u|^{q(x)-1}|u|dx\Big) \\
&\le const\Big(|u|^{a^-}_{a(x)} + |u|^{a^+}_{a(x)} + |k|_{p'(x)}|u|_{p(x)} + |l|_{p'(x)}|u|_{p(x)} + |u|^{s^+}_{s(x)} \\
&\qquad + |u|^{s^-}_{s(x)} + |u|^{q^+}_{q(x)} + |u|^{q^-}_{q(x)} + \frac{1}{q'^-}\rho_{q(x)}(\nabla u) + \frac{1}{q-}\rho_{q(x)}(u)\Big) \\
&\le const\Big(|u|^{a^-}_{a(x)} + |u|^{a^+}_{a(x)} + |u|_{p(x)} + |u|^{s^+}_{s(x)} + |u|^{s^-}_{s(x)} + |u|^{q^+}_{q(x)} \\
&\qquad\qquad + |u|^{q^-}_{q(x)} + |\nabla u|^{q^+}_{q(x)}\Big),
\end{aligned}$$

then, according to $L^{p(x)} \hookrightarrow L^{a(x)}$, $L^{p(x)} \hookrightarrow L^{s(x)}$ and $L^{p(x)} \hookrightarrow L^{q(x)}$, we get

$$|\mathcal{A}\varphi|^{p^-}_{1,p(x)} \le const\Big(|\mathcal{A}\varphi|^{a^+}_{1,p(x)} + |\mathcal{A}\varphi|_{1,p(x)} + |\mathcal{A}\varphi|^{s^+}_{1,p(x)} + |\mathcal{A}\varphi|^{q^+}_{1,p(x)}\Big),$$

what implies that $\Big\{\mathcal{A}\varphi : \varphi \in \mathcal{R}\Big\}$ is bounded.

On the other hand, we have that the operator is $\mathcal{G}$ is bounded, then $\mathcal{G}\circ\mathcal{A}\varphi$ is bounded. Thus, thanks to (3.5), we have that $\mathcal{R}$ is bounded in $W^{-1,p'(x)}(\Omega)$.

However, $\exists\, r > 0$ such that

$$|\varphi|_{-1,p'(x)} < r \text{ for all } \varphi \in \mathcal{R},$$

which leads to

$$\varphi + t\mathcal{G} \circ \mathcal{A}\varphi \neq 0, \ \ \varphi \in \partial\mathcal{R}_r(0) \text{ and } t \in [0,1],$$

where $\mathcal{R}_r(0)$ is the ball of center 0 and radius r in $W^{-1,p'(x)}(\Omega)$.
Moreover, by Lemma 2.13, we conclude that

$$I + \mathcal{G} \circ \mathcal{A} \in \mathcal{F}_{\mathcal{A}}(\overline{\mathcal{R}_r(0)}) \text{ and } I = \mathcal{I} \circ \mathcal{A} \in \mathcal{F}_{\mathcal{A}}(\overline{\mathcal{R}_r(0)}).$$

Next, we define the homotopy

$$\mathcal{H} : [0,1] \times \overline{\mathcal{R}_r(0)} \to W^{-1,p'(x)}(\Omega)$$
$$(t,\varphi) \mapsto \mathcal{H}(t,\varphi) := \varphi + t\mathcal{G} \circ \mathcal{A}\varphi.$$

Applying the homotopy invariance property of the degree d seen in Theorem 2.16, we obtain

$$d(I + \mathcal{G} \circ \mathcal{A}, \mathcal{R}_r(0), 0) = d(I, \mathcal{R}_r(0), 0).$$

Then, by the normalization property of the degree d, we have $d(I, \mathcal{R}_r(0), 0) = 1$ and consequently $d(I + \mathcal{G} \circ \mathcal{A}, \mathcal{R}_r(0), 0) = 1$.
Since $d(I + \mathcal{G} \circ \mathcal{A}, \mathcal{R}_r(0), 0) \neq 0$, then by the existence property of the degree d stated in Theorem 2.16, we conclude that there exists $\varphi \in \mathcal{R}_r(0)$ which verifies

$$\big(I + \mathcal{G} \circ \mathcal{A}\big)(\varphi) = 0 \Leftrightarrow \varphi + \mathcal{G} \circ \mathcal{A}\varphi = 0 \Leftrightarrow \mathcal{I} \circ \mathcal{A}\varphi + \mathcal{G} \circ \mathcal{A}\varphi = 0.$$

Finally, we infer that $u = \mathcal{A}\varphi$ is a weak solution of (1.1). The proof is completed. □

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