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ON COUNTABLE MCCOYNESS OF THE IDEALIZATION

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Abstract. It is known that the idealization $R \ltimes M$ of a reduced ring R over a flat R-module M is a McCoy ring if and only if R is a McCoy ring [20, Proposition 3.5]. The main purpose of this paper is to extend Lucas's result to the countably McCoyness of the idealization. Effectively, we drop the reduceness hypotheses and prove that, given an arbitrary commutative ring R and any submodule M of a flat R-module F, $R \ltimes M$ is a countably McCoy-ring if and only if R is a countably McCoy-ring.

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1. Introduction

Throughout this paper, all rings are supposed to be commutative with unit element and all R-modules are unital. Let R be a commutative ring and M an R-module. We denote by $\mathrm{Z}_R(M) = \{r \in R : rm = 0$ for some nonzero element $m \in M\}$ the set of zero divisors of R on M and by $\mathrm{Z}(R) := \mathrm{Z}_R(R)$ the set of zero divisors of the ring R. In [4], the notions of $\mathcal{A}$-module and $\mathcal{SA}$-module are extensively studied. In fact, an R-module M satisfies Property $\mathcal{A}$, or M is an $\mathcal{A}$-module over R (or $\mathcal{A}$-module if no confusion is likely), if for every finitely generated ideal I of R with $I \subseteq \mathrm{Z}_R(M)$, there exists a nonzero $m \in M$ with $Im = 0$, or equivalently, $\mathrm{ann}_M(I) \neq 0$. M is said to satisfy strong Property $\mathcal{A}$, or is an $\mathcal{SA}$-module over R (or an $\mathcal{SA}$-module if no confusion is likely), if for any $r_1, \cdots, r_n \in \mathrm{Z}_R(M)$, there exists a nonzero $m \in M$ such that $r_1 m = \cdots = r_n m = 0$. The ring R is said to satisfy Property $\mathcal{A}$,

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or an $\mathcal{A}$-ring, (respectively, $\mathcal{SA}$-ring) if R is an $\mathcal{A}$-module (resp., an $\mathcal{SA}$-module). One may easily check that M is an $\mathcal{SA}$-module if and only if M is an $\mathcal{A}$-module and $\mathrm{Z}_R(M)$ is an ideal of R. It is worthwhile reminding the reader that the Property $\mathcal{A}$ for commutative rings was introduced by Quentel in [25] who called it Property C and Huckaba used the term Property $\mathcal{A}$ in [17, 18]. In [12], Faith called rings satisfying Property $\mathcal{A}$ McCoy rings. The Property $\mathcal{A}$ for modules was introduced by Darani [10] who called such modules F-McCoy modules (for Faith McCoy terminology). He also introduced the strong Property $\mathcal{A}$ under the name super coprimal and called a module M coprimal if $\mathrm{Z}_R(M)$ is an ideal. In [21], the strong Property $\mathcal{A}$ for commutative rings was independently introduced by Mahdou and Hassani and further studied by Dobbs and Shapiro in [11]. Note that a finitely generated module over a Noetherian ring is an $\mathcal{A}$-module (for example, see [19, Theorem 82]) and thus a Noetherian ring is an $\mathcal{A}$-ring. Also, it is well knwon that a zero-dimensional ring R is an $\mathcal{A}$-ring as well as any ring R whose total quotient ring $\mathrm{Q}(R)$ is zero-dimensional. In fact, it is easy to see that R is an $\mathcal{A}$-ring if and only if so is $\mathrm{Q}(R)$ [9, Corollary 2.6]. Any polynomial ring $R[X]$ is an $\mathcal{A}$-ring [17] as well as any reduced ring with a finite number of minimal prime ideals [17]. For recent achievements related to the Property $\mathcal{A}$ and $\mathcal{SA}$, we refer the reader to [2–5, 16, 20] and for details on the idealization we refer to [6].

In [4], Anderson and Chun considered rings and modules which satisfy the $\mathcal{A}$-property or $\mathcal{SA}$-property. They proved that any module over a zero-dimensional commutative ring is an $\mathcal{A}$-module as well as any Noetherian or Artinian module is an $\mathcal{A}$-module. Moreover, given an additive submonoid Γ of $\mathbb{R}$, a Γ-graded ring R and a Γ-graded R-module M, they proved that if R has a homogeneous element $x \in \mathrm{Z}_R(M)$ of nonzero degree, then M is an $\mathcal{A}$-module. In particular, if M is an R-module, then $M[X]$ is an $\mathcal{A}$-module as an $R[X]$-module. Also, Anderson and Chun gave plenty of examples among which figure many pathological examples for modules satisfying the $\mathcal{A}$-property. For instance, they exhibited an example of two $\mathcal{A}$-modules A_1 and A_2 such that $A_1 \oplus A_2$ is not an $\mathcal{A}$-module as well as an example of two R-modules A_1 and A_2 which are not $\mathcal{A}$-modules while $A_1 \oplus A_2$ is an $\mathcal{A}$-module. This ensures that the $\mathcal{A}$-property is not preserved by submodules, homomorphic images, direct sums or direct summands. Furthermore, Anderson and Chun were interested in the $\mathcal{A}$-property of the idealization $R \ltimes M$ of a ring R over an R-module M. They seeked in [4] necessary and sufficient conditions for the idealization $R \ltimes M$ to be an $\mathcal{A}$-ring (resp., an $\mathcal{SA}$-ring) in terms of module-theoretic properties of R and M. In this context, they proved that if R is an integral domain, then $R \ltimes M$ is an $\mathcal{A}$-ring (resp., $\mathcal{SA}$-ring) if and only if M is an $\mathcal{A}$-module (resp., an $\mathcal{SA}$-module).

In [20], Lucas proves that the idealization $R \ltimes M$ of a reduced ring R over a flat R-module M is a McCoy ring if and only if R is a McCoy ring [20, Proposition 3.5]. The main purpose of this paper is to extend Lucas's result to the countably McCoyness of the idealization. In effect, we drop the reduceness hypotheses and prove that, given an arbitrary commutative ring R and any submodule M of a flat R-module F, $R \ltimes M$ is a countably $\mathcal{A}$-ring (resp., countably $\mathcal{SA}$-ring) if and only if R is a countably $\mathcal{A}$-ring (resp., countably $\mathcal{SA}$-ring). To this goal, we introduce in Section 2 a new property that we term the countable annihilator vanishing property, countable AV-property for short. A module M over a ring R is said to be a countable AV-module over a module N if, given a nonzero countably generated ideal I of R, $\operatorname{ann}_M(I) = 0$ implies that $\operatorname{ann}_N(I) = 0$. M is said to be a countable AV-module, if M is a countable AV-module over any R-module N and R is said to be a countable AV-ring if R is a countable AV-module. We prove that any semi-regular ring R is a countable AV-ring (see Corollary 3.6). Also, a pair (M, N) is said to be a countable AV-pair of R-modules if M and N are countable AV-module over each other. For instance, the pair (R, L) is a countable AV-pair for any free R-module L. It turns out that the countable AV-module notion enjoys nice properties. The countable AV-modules are stable under direct sums and direct products. Concerning direct systems of modules, we introduce the new notion of direct systems $(A_i)_i$ over a countably directed set Δ. We prove that, given a direct system $(A_i)_i$ over a directed set Λ, if Δ denotes the set of all countable subsets α of Λ and if $B_\alpha := \bigoplus_{i \in \alpha} A_i$ for each $\alpha \in \Delta$, then Δ is a countable directed set, B_α is a direct system over Δ and

$$\varinjlim_{i \in \Lambda} A_i = \varinjlim_{\alpha \in \Delta} B_\alpha.$$

In this context, one of our main theorems proves that, given a module M and a direct system $(N_i)_{i \in \Delta}$ of modules over a countably directed set Δ, if M is a countable AV-module over each N_i, then M is a countable AV-module over $\varinjlim_{i \in \Delta} N_i$. Furthermore, one of the key results allowing to generalize Lucas's proposition is our finding that R is a countable AV-module over any submodule of a flat R-module M. In section 3, we examine the impact of the countable AV-property on studying the $\mathcal{CA}$-property. In fact, given a countable AV-pair (M, N), we prove that M is a $\mathcal{CA}$-module (resp., $\mathcal{SCA}$-module) if and only if so is N. As a consequence of this, we show that if M is a countable AV-module over N, then $M \oplus N$ is a $\mathcal{CA}$-module (resp.,

$\mathcal{SCA}$-module) if and only if so is M. These results allow us to give the following general analog of Lucas's proposition for countably McCoy rings: given a ring R and a submodule M of a flat module, then the following assertions are equivalent:

1) $R \ltimes M$ is a $\mathcal{CA}$-ring (resp., $\mathcal{SCA}$-ring);
2) $R \oplus M$ is a $\mathcal{CA}$-module (resp., $\mathcal{SCA}$-module);
3) R is a $\mathcal{CA}$-ring (resp., $\mathcal{SCA}$-module).

2. The countable annihilator vanishing property for rings and modules

This section introduces and studies the countable annihilator vanishing property. In particular, we examine the stability of this property under direct sums and direct products.

We begin by recalling the definitions of the countably McCoy rings introduced by Lucas.

Definition 2.1. Let R be a ring and M an R-module.

(1) R is said to be a countably McCoy ring or a countably $\mathcal{A}$-ring ($\mathcal{CA}$-ring for short), if for any ideal $J \subseteq \mathrm{Z}(R)$ such that J is countably generated, $\mathrm{ann}_R(J) \neq 0$.

(2) M is said to be a countably McCoy module or a countably $\mathcal{A}$-module ($\mathcal{CA}$-module for short), if for any ideal $I \subseteq \mathrm{Z}_R(M)$ such that I is countably generated, $\mathrm{ann}_M(I) \neq 0$.

(3) R is said to be a strongly countably McCoy ring or a strongly countably $\mathcal{A}$-ring ($\mathcal{SCA}$-ring for short), if for any countably generated ideal $J = (a_1, a_2, \cdots, a_n, \cdots)$ such that $a_n \in \mathrm{Z}(R)$ for each integer $n \geq 1$, we have $\mathrm{ann}_R(J) \neq 0$.

(4) M is said to be a strongly countably McCoy module or a strongly countably $\mathcal{A}$-module ($\mathcal{SCA}$-module for short), if for any countably generated ideal $J = (a_1, a_2, \cdots, a_n, \cdots)$ such that $a_n \in \mathrm{Z}_R(M)$ for each integer $n \geq 1$, we have $\mathrm{ann}_M(J) \neq 0$.

Next, we introduce the annihilator vanishing property.

Definition 2.2. Let R be a ring.

(1) We say that an R-module M has the *countable annihilator vanishing* property over an R-module N, or is a countable AV-module over N for short, if, given a nonzero countably generated ideal I of R, ann$_M(I) = 0$ implies that ann$_N(I) = 0$.

(2) We say that an R-module M is a countable AV-module if M is a countable AV-module over any R-module N.

(3) R is said to be a countbale AV-ring, if R, as an R-module, is a countable AV-module.

(4) We say that a pair (M, N) of R-modules is a countable AV-pair, if M and N has the countable AV-property over each other.

The following is a simple result characterizing the countable AV-modules.

Proposition 2.3. *Let R be a commutative ring and M an R-module. Then the following assertions are equivalent:*

(1) *M is a countable AV-module;*

(2) *$\mathrm{ann}_M(I) \neq (0)$ for any nonzero countably generated ideal I of R.*

Proof. 1) $\Rightarrow$ 2) Assume that M is a countable AV-module. Let I be a nonzero countably generated ideal of R. Then $\mathrm{ann}_{\frac{R}{I}}(I) \neq (\overline{0})$ as $I\overline{1} = (\overline{0})$. Now, since M is a countable AV-module over $\dfrac{R}{I}$, we get $\mathrm{ann}_M(I) \neq (0)$, as desired.
2) $\Rightarrow$ 1) It follows easily from the definition. □

We present next a bunch of examples of countable AV-modules. We denote by $\mathrm{Spec}(R)$ the set of prime ideals of R and by $\max(R)$ the set of maximal ideals of R.

Corollary 2.4. *Let R be a commutative ring. Then the R-modules $\bigoplus\limits_{m\in\max(R)} \dfrac{R}{m}$, $\bigoplus\limits_{p\in\mathrm{Spec}(R)} \dfrac{R}{p}$, $\bigoplus\limits_{\Gamma} \dfrac{R}{I_\gamma}$ and $\bigoplus\limits_{\Lambda} \dfrac{R}{I_\lambda}$ are countable AV-modules, where Γ denotes the set of ideals of R and Λ its subset of countably generated ideals.*

Proof. It is easy to check that these modules satisfies (2) of Proposition 2.3.

□

Corollary 2.5. *Let R be a commutative ring. Then any countable AV-module is a $\mathcal{CA}$-module.*

Proof. It is direct from Proposition 2.3. □

The next proposition presents characteristics of modules possessing the countable AV-property.

Proposition 2.6. *Let R be a commutative ring.*

(1) *If $N \subseteq M$ are R-modules, then M is a countable AV-module over N.*

(2) *Let M be a faithful R-module. Then M is a countable AV-module over R.*

(3) *Let M and N be R-modules. Then $M \oplus N$ is a countable AV-module over M and N. Moreover, if M is a countable AV-module over N, then $(M, M \oplus N)$ is a countable AV-pair.*

Proof. 1) It is clear from the definition since, if $N \subseteq M$ are R-modules, then $\text{ann}_N(I) \subseteq \text{ann}_M(I)$ for each ideal I of R.
2) Assume that M is a faithful R-module. Then $\text{ann}_R(M) = (0)$. Let I be a countably generated R-module such that $\text{ann}_R(I) \neq (0)$. Then there exists a nonzero element $r \in R$ such that $Ir = (0)$. As $r \neq 0$, $r \notin \text{ann}_R(M)$ and thus there exists $m \in M$ such that $rm \neq 0$. Now, it is easy to verify that $I(rm) = (0)$ and $rm \neq 0$. Therefore $\text{ann}_M(I) \neq (0)$. Hence M is a countable AV-module over R, as desired.
3) Let M and N be R-modules. Then, by (1), $M \oplus N$ is a countable AV-module over M and N. Assume that M is a countable AV-module over N. Let I be a nonzero countably generated ideal of R such that $\text{ann}_M(I) = (0)$. Then, as M is a countable AV-module over N, we get $\text{ann}_N(I) = (0)$. Therefore $\text{ann}_{M \oplus N}(I) = \text{ann}_M(I) \oplus \text{ann}_N(I) = (0)$. Hence M is a countable AV-module over $M \oplus N$. It follows that $(M, M \oplus N)$ is a countable AV-pair completing the proof.

□

In the following result, we record the simple fact that the countable AV-property is transitive.

Proposition 2.7. *Let R be a ring. Then*

(1) *Let M, N, K be R-modules. If M is a countable* AV*-module over N and N is a countable* AV*-module over K, then M is a countable* AV*-module over K.*
(2) *Let $M' \subseteq M$ and N be R-modules. If M' is a countable* AV*-module over N, then so is M.*
(3) *Let M be an R-module and $N' \subseteq N$ be R-modules. If M is a countable* AV*-module over N, then so is M over N'.*

Proof. 1) Assume that M is a countable AV-module over N and N is a countable AV-module over K. Let I be a nonzero countably generated ideal of R such that $\text{ann}_M(I) = (0)$. Then $\text{ann}_N(I) = (0)$ as M is a countable AV-module over N. Now, since N is a countable AV-module over K, we obtain, $\text{ann}_K(I) = (0)$. It follows that M is a countable AV-module over K, as desired.
2) It follows from (1) and Proposition 2.6 (1).
3) Note that N is a countable AV-module over N', by Proposition 2.6. Using the transitivity of the countable AV-property yields the desired result. □

The next proposition establishes the stability of the countable AV-property under direct sums and direct products.

Proposition 2.8. *Let R be a ring. Then*

1) Let $(M_i)_i$ and $(N_i)_i$ be families of R-modules. If, for each i, M_i is a countable AV-module over N_i, then so is the direct sum $\bigoplus_i M_i$ over $\bigoplus_i N_i$ and the direct product $\prod_i M_i$ over $\prod_i N_i$.

2) Let $(M_i)_i$ be a family of R-modules and N an R-module. If each M_i is a countable AV-module over N, then so is their direct sum $\bigoplus_i M_i$ and their direct product $\prod_i M_i$ over N.

3) Let M be an R-module. Then $(M, \bigoplus_i M)$ and $(M, \prod_i M)$ are countable AV-pairs.

4) Let M be an R-modules and $(N_i)_i$ be a family of R-modules. If M is a countable AV-module over each N_i, then M is a countable AV-module over $\bigoplus N_i$ and $\prod_i N_i$.

Proof. 1) First, observe that

$$\text{ann}_{\oplus_i M_i}(I) = \bigoplus_i \text{ann}_{M_i}(I) \text{ and } \text{ann}_{\prod_i M_i}(I) = \prod_i \text{ann}_{M_i}(I)$$

for any ideal I of R. Similar equalities hold for the N_i. Then the result easily follows.

2) Applying (1), $\bigoplus_i M_i$ is a countable AV-module over $\bigoplus_i N$. Also, note that, as $N \subseteq \bigoplus_i N$, $\bigoplus_i N$ is a countable AV-module over N. Then, by transitivity of the countable AV-property, we get $\bigoplus_i M_i$ is a countable AV-module over N. A similar argument applies for the direct product.

3) It follows from the above two equalities of $\text{ann}_{\oplus_i M_i}(I)$ and $\text{ann}_{\prod_i M_i}(I)$ for any ideal I of R.

4) Assume that M is a countable AV-module over each N_i. Then, using (1) we get $\bigoplus_i M$ is a countable AV-module over $\bigoplus_i N_i$ and $\prod_i M$ is a countable AV-module over $\prod_i N_i$. As, by (3), M is a countable AV-module over $\bigoplus_i M$ and $\prod_i M$, we get by transitivity of the countable AV-property, that M is a countable AV-module over $\bigoplus_i N_i$ and over $\prod_i N_i$ completing the proof. □

Corollary 2.9. *Let R be a commutative ring. Then*

1) For any free R-module L, (R, L) is a countable AV-pair.

2) R is a countable AV-module over any submodule of a free R-module. In particular, R is a countable AV-module over any projective R-module.

3) $(R, R[X])$ and $(R, R[[X]])$ are countable AV-pairs.

Proof. 1) Apply Proposition 2.8 (3), as $L \cong \bigoplus_J R$ for some set J.

2) Use (1), Proposition 2.6(1) and Proposition 2.7 (1).

3) Note that $R[X] \cong \bigoplus R$ and $R[[X]] \cong \prod R$ as R-modules. Proposition 2.8 (3) completes the proof. □

Corollary 2.10. *Any commutative semisimple ring R is a countable* AV*-ring. In particular, any field k is a countable* AV*-ring.*

Proof. Let R be a semisimple ring. Then any module M is projective over R and thus, by Corollary 2.9, R is a countable AV-module over M. Hence R is a countable AV-ring. □

3. The countable AV-property and direct limits

This section examines the behavior of the countable annihilator vanishing property under the inverse limits and direct limits. This stands as starting block in order to give the promised analog of Lucas result for the countable McCoyness.

Our first result discusses the stability of the countable AV-property under inverse limits.

Proposition 3.1. *Let R be a ring. Let $(N_i)_i$ be an inverse system of modules and M be an R-module. If M is a countable* AV*-module over each N_i, then M is a countable* AV*-module over the inverse limit $\varprojlim N_i$ of the N_i.*

Proof. Assume that each M is a countable AV-module over each N_i. Then, by Proposition 2.8, M is a countable AV-module over $\prod_i N_i$. Now, the inverse limit of the N_i is isomorphic to a submodule of the direct product $\prod_i N_i$. Hence, by Proposition 2.7 (1), M is a countable AV-module over $\varprojlim N_i$, as desired. □

The next theorem proves the stability of the countable AV-property under direct limits over countably directed sets. This permits us to generalize Lucas proposition [20, Proposition 3.5]. First, we set the following definition.

Definition 3.2. A directed set Λ is said to be countably directed, or a countably directed set, if for any countable subset $\{i_1, \cdots, i_n, \cdots\}$ of Λ, there exists $j \in \Lambda$ such that $i_k \leq j$ for each integer $k \geq 1$.

Theorem 3.3. *Let R be a ring. Let $(N_i, \varphi_j^i)_{i\in\Lambda}$ be a direct system of R-modules over a countably directed set Λ and M an R-module. Assume that M is a countable* AV*-module over each N_i. Then M is a countable* AV*-module over $\varinjlim N_i$.*

Proof. Put $N := \varinjlim N_i$ and $\lambda_i : N_i \longrightarrow \bigoplus_i N_i$ be the natural injection. Recall that $M = \dfrac{\bigoplus_i M_i}{S}$, where S is the submodule of $\bigoplus_i N_i$ generated by all elements $\lambda_j \varphi_j^i(a_i) - \lambda_i(a_i)$ with $a_i \in N_i$ and $i \le j$. Let $I = (a_1, a_2, \cdots, a_n, \cdots)$ be a countably generated ideal of R such that $\text{ann}_M(I) = (0)$. Let $m \in \text{ann}_N(I)$. Then $Im = 0$. By [26, Theorem 2.17(i)], there exists an index i and $x_i \in N_i$ such that $m = \overline{\lambda_i(x_i)}$. Hence $\overline{\lambda(a_k x_i)} = \overline{0}$ for each $k \in \mathbb{N}^*$. Then, by [26, Theorem 2.17(ii)], for each $k \in \mathbb{N}^*$, there exists $j_k \ge i$ such that $\varphi^i_{j_k}(a_k x_i) = 0$. Note that, as Λ is a countably directed set, there exists $j \in \Lambda$ such that $j \ge j_k$ for each $k \in \mathbb{N}^*$. Therefore $\varphi_j^i(a_k x_i) = a_k \varphi_j^i(x_i) = 0$ for each $k \in \mathbb{N}^*$. It follows that $I\varphi_j^i(x_i) = 0$, that is, $\varphi_j^i(x_i) \in \text{ann}_{N_j}(I)$. As M is a countable AV-module over N_j, we get $\text{ann}_{N_j}(I) = 0$ and thus $\varphi_j^i(x_i) = 0$. A second application of [26, Theorem 2.17(ii)] yields $\overline{\lambda_i(x_i)} = \overline{0}$ (as $j \ge i$), so that, $m = 0$. Therefore $\text{ann}_N(I) = (0)$. Consequently, M is a countable AV-module over N, as desired. □

We proved via Corollary 2.9 that, given a ring R, R is a countable AV-module over any projective R-module. The following theorem extends this result to flat R-modules and allows to give an analog of Lucas's result for the countable McCoy property.

Theorem 3.4. *Let R be a ring. Then R is a countable* AV*-module over any submodule of a flat R-module.*

We need the following lemma. First, let us adopt the following notation. Let $(A_i, f_{ij})_{i,j \in \Lambda}$ be a direct system over a directed set Λ. Let Δ denote the set of all countable subsets of Λ. For each $\alpha \in \Delta$, put $B_\alpha = \bigoplus_{i \in \alpha} A_i$. Given $\alpha, \beta \in \Delta$, we say that $\alpha \le \beta$, if for each $i \in \alpha$, there exists $j \in \beta$ such that $i \le j$. Let $\alpha \le \beta$ with $\alpha = \{i_1, \cdots, i_n, \cdots\}$ and $\beta = \{j_1, \cdots, j_n, \cdots\}$ and $i_k \le j_k$ for integer $k \ge 1$. We put $g_{\alpha\beta} = (f_{i_k j_k})_{k \ge 1} : B_\alpha \longrightarrow B_\beta$ such that, if $t = (a_{i_1}, a_{i_2}, \cdots, a_{i_n}) \in B_\alpha$, $g_{\alpha\beta}(t) = (f_{i_1 j_1}(a_{i_1}), f_{i_2 j_2}(a_{i_2}), \cdots, f_{i_n j_n}(a_{i_n}))$.

Lemma 3.5. *Let $(A_i, f_{ij})_{i,j \in \Lambda}$ be a direct system over a directed set Λ and let Δ be the set of all countable subsets of Λ. For each $\alpha \in \Delta$, put $B_\alpha = \bigoplus_{i \in \alpha} A_i$. Then Δ is a countable directed set, $(B_\alpha, g_{\alpha\beta})_{\alpha,\beta \in \Delta}$ is a direct system over Δ and*

$$\varinjlim_{i \in \Lambda} A_i = \varinjlim_{\alpha \in \Delta} B_\alpha.$$

Proof. Let $\alpha, \beta, \gamma \in \Delta$ such that $\alpha \leq \beta \leq \gamma$. Consider the following diagram

$$\begin{array}{ccc} B_\alpha & \xrightarrow{g_{\alpha\gamma}} & B_\gamma \\ g_{\alpha\beta} \searrow & & \nearrow g_{\beta\gamma} \\ & B_\beta & \end{array}$$

It is easy to check that this diagram is commutative. Then $(B_\alpha, g_{\alpha\beta})_{\alpha,\beta\in\Delta}$ is a direct system over Δ. Moreover, since with $A_i = B_{\{i\}}$, $A_j = B_{\{j\}}$ and $g_{\{i\}\{j\}} = f_{ij}$, it is easy to see that if $i \leq j$, then the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{f_{ij}} & A_j \\ g_i \searrow & & \swarrow g_j \\ & \varinjlim_{\alpha\in\Delta} B_\alpha & \end{array}$$

commutes with $g_\beta : B_\beta \longrightarrow \varinjlim_{\alpha\in\Delta} B_\alpha$ is the canonical homomorphism for each $\beta \in \Delta$. Hence, by the universal mapping property for direct limits, there exists a unique homomorphism $\varphi : \varinjlim_{i\in\Lambda} A_i \longrightarrow \varinjlim_{\alpha\in\Delta} B_\alpha$. Moreover, consider the following commutative diagram

$$\begin{array}{ccc} \beta_\alpha & \xrightarrow{\psi_{\alpha i}} & A_i \\ g_{\alpha\beta} \downarrow & & \downarrow f_{ij} \\ B_\beta & \xrightarrow{\psi_{\beta j}} & A_j \end{array}$$

where:

-) $\alpha = \{i_k\}_{k\geq1}$, $\beta = \{j_k\}_{k\geq1}$ such that $i_k \leq j_k$ for each k and thus $\alpha \leq \beta$.

-) $i \in \alpha$, $j \in \beta$ such that $i \leq j$ and $g_{\alpha\beta} = (f_{i_k j_k})_k$ with $f_{ij} = f_{i_k j_k}$ for some integer $k \geq 1$.

-) $\psi_{\alpha i}$, $\psi_{\beta j}$ are the canonical surjections.

Therefore, by the universel mapping property for direct limits, there exists a unique homomorphism $\psi : \varinjlim_{\alpha\in\Delta} B_\alpha \longrightarrow \varinjlim_{i\in\Lambda} A_i$. It follows that $\varphi \circ \psi = \mathrm{id}_{\varinjlim_{\alpha\in\Delta} B_\alpha}$ and $\psi \circ \varphi = \mathrm{id}_{\varinjlim_{i\in\Lambda} A_i}$ and thus $\varinjlim_{i\in\Lambda} A_i = \varinjlim_{\alpha\in\Delta} B_\alpha$ completing the proof. □

Proof of Theorem 3.4. Let M be a flat module. It is known that there exists a directed set Λ such that $M = \varinjlim_{i\in\Lambda} L_i$ with the L_i are finitely generated free R-modules. Then, by Lemma 3.5, there exists a countable directed set Δ such that $M = \varinjlim_{\alpha\in\Delta} B_\alpha$ with the B_α are countably generated free R-modules. By Corollary 2.9(1), R is a countable AV-module over each B_α. Hence, by Theorem 3.3, R is a countable AV-module over M. Now, by Proposition 2.7(3), R is a countable AV-module over any submodule of M, as desired.

□

Recall that, in 1982, Matlis proved that a ring R is coherent if and only if $\mathrm{Hom}_R(M,N)$ is flat for any injective R-modules M and N [24]. Also, in 1985, he introduced the notion of semi-coherent commutative ring. In effect, he defined a ring R to be semi-coherent if it is commutative and $\mathrm{Hom}_R(M,N)$ is a submodule of a flat R-module for any injective R-modules M and N. Then, inspired by this definition and by von Neumann regularity, he defined a ring to be semi-regular if it is commutative and if any module can be embedded in a flat module. He then provided a connection between this notion and coherence; namely, a commutative ring R is semi-regular if and only if R is coherent and R_M is semi-regular for every maximal ideal M of R. He also proved that a ring R is a Prüfer domain if and only if $\dfrac{R}{I}$ is a semi-regular ring for each nonzero finitely generated ideal I of R. A semi-regular ring is also termed an IF-ring (a ring in which any injective module is flat). The class of semi-regular rings then includes von Neumann regular rings, Quasi-Frobenius rings and quotients of Prüfer domains by nonzero finitely generated ideals.

Our next result records that semi-regular rings are countable AV-rings. This generalizes Corollary 2.10 as any semisimple ring is semi-regular.

Corollary 3.6. *Any semi-regular ring R is a countable* AV*-ring.*

Proof. Assume that R is semi-regular. Then any module M is a submodule of a flat module F. Now, Theorem 3.4 completes the proof.

□

4. $\mathcal{CA}$-property and the countable AV-property

In [20], Lucas proved that if R is a reduced ring and M is a flat R-module, then $R \ltimes M$ is an $\mathcal{A}$-ring if and only if R is an $\mathcal{A}$-ring [20, Proposition 3.5]. The aim of this section is to give the analog of Lucas's result for the countable McCoy property. Also, we give a version of Lucas result for the strongly countable McCoy property. First, we examine the impact of the countable AV-property on studying the $\mathcal{CA}$-property and $\mathcal{SCA}$-property of the direct sum of two modules over a ring R as well as of the idealization $R \ltimes M$ of R on a module M.

We begin by studying the transfer of the $\mathcal{CA}$-property and $\mathcal{SCA}$-property between two modules possessing the countable AV-property.

Theorem 4.1. *Let R be a ring. Let M and N be R-modules.*
1) If M is a countable AV-module over N, then $\mathrm{Z}_R(N) \subseteq \mathrm{Z}_R(M)$.
2) If M is a $\mathcal{CA}$-module, then the following assertions are equivalent:
a) M is a countable AV-module over N;
b) $(M, M \oplus N)$ is a countable AV-pair;
c) $\mathrm{Z}_R(N) \subseteq \mathrm{Z}_R(M)$;
d) $\mathrm{Z}_R(M \oplus N) = \mathrm{Z}_R(M)$.
3) Assume that M is a countable AV-module over N and that $\mathrm{Z}_R(M) \subseteq \mathrm{Z}_R(N)$. If N is a $\mathcal{CA}$-module (resp., an $\mathcal{SCA}$-module), then so is M.
4) Assume that (M, N) is a countable AV-pair. Then the following assertions are equivalent:
i) M is a $\mathcal{CA}$-module (resp., an $\mathcal{SCA}$-module);
ii) N is a $\mathcal{CA}$-module (resp., an $\mathcal{SCA}$-module).

Proof. 1) Assume that M is a countable AV-module over N. Let $x \in \mathrm{Z}_R(N)$ and take $I = Rx$. Then $\mathrm{ann}_N(I) \neq (0)$, so that by applying the countable AV-property of M over N, we get $\mathrm{ann}_M(I) \neq (0)$. Thus $\mathrm{ann}_M(x) \neq (0)$ which means that $x \in \mathrm{Z}_R(M)$, as desired.
2) Assume that M is a $\mathcal{CA}$-module.
a) $\Leftrightarrow$ b) It holds by Proposition 2.6(3).
a) $\Rightarrow$ c) Use (1).
c) $\Leftrightarrow$ d) It is direct.
c) $\Rightarrow$ a) Assume that $\mathrm{Z}_R(N) \subseteq \mathrm{Z}_R(M)$. Let I be a countably generated ideal of R such that $\mathrm{ann}_N(I) \neq (0)$. Then $I \subseteq \mathrm{Z}_R(N)$ and thus $I \subseteq \mathrm{Z}_R(M)$. Therefore, as M is a $\mathcal{CA}$-module, we get $\mathrm{ann}_M(I) \neq (0)$. It follows that M is a countable AV-module over N, as desired.
3) Assume that M is a countable AV-module over N and that $\mathrm{Z}_R(M) \subseteq \mathrm{Z}_R(N)$. Then, by (1), $\mathrm{Z}_R(M) = \mathrm{Z}_R(N)$. Suppose that N is a $\mathcal{CA}$-module and let I be a countably generated ideal of R such that $I \subseteq \mathrm{Z}_R(M)$. Then $I \subseteq \mathrm{Z}_R(N)$. Therefore, as N is a $\mathcal{CA}$-module, we get $\mathrm{ann}_N(I) \neq (0)$. Now, since M is a countable AV-module over N, we obtain $\mathrm{ann}_M(I) \neq (0)$. Hence M is a $\mathcal{CA}$-module, as desired. The proof is similar for the $\mathcal{SCA}$-property.
4) It follows from the combination of (1) and (3) completing the proof. □

Corollary 4.2. *Let R be a commutative ring and L a free R-module. Then R is a $\mathcal{CA}$-ring (resp., $\mathcal{SCA}$-ring) if and only if L is a $\mathcal{CA}$-module (resp., $\mathcal{SCA}$-module).*

Proof. It is direct from Theorem 4.1 since (R, L) is a countable AV-pair, by Corollary 2.9. □

In the next corollaries, we discuss the $\mathcal{CA}$-property and $\mathcal{SCA}$-property of the direct sum and idealization of modules sharing the countable AV-property. Lucas's proposition turns out to be a particular case of this result.

Corollary 4.3. *Let R be a ring. Let M and N be R-modules. Assume that M is a countable* AV*-module over N. Then the following assertions are equivalent:*

1) $M \oplus N$ is a $\mathcal{CA}$-module (resp, an $\mathcal{SCA}$-module);

2) M is a $\mathcal{CA}$-module (resp., an $\mathcal{SCA}$-module).

Proof. As M is a countable AV-module over N, then, by Proposition 2.6, the pair $(M, M \oplus N)$ is a countable AV-pair. Hence Theorem 4.1 completes the proof. □

We state next one of the main theorems of this section. It is a further step towards establishing the analog of Lucas's proposition for the $\mathcal{CA}$-property.

Theorem 4.4. *Let R be a ring and M an R-module. Then the following assertions are equivalent:*

1) $R \ltimes M$ is a $\mathcal{CA}$-ring (resp., $\mathcal{SCA}$-ring);

2) $R \oplus M$ is a $\mathcal{CA}$-module (resp., $\mathcal{SCA}$-module).

Proof. **I) $\mathcal{SCA}$-property.**

1) ⇒ 2) Suppose that $T := R \ltimes M$ is an $\mathcal{SCA}$-ring. Let $I = (a_1, a_2, \cdots, a_n, \cdots)$ be a countably generated ideal of R such that $a_n \in \mathrm{Z}_R(R \oplus M) = \mathrm{Z}(R) \cup \mathrm{Z}_R(M)$ for each integer $n \geq 1$. Then $(a_n, 0) \in \mathrm{Z}(T)$ for each integer $n \geq 1$. Consider the ideal $J := ((a_1, 0), (a_2, 0), \cdots, (a_n, 0), \cdots)T$ with $(a_n, 0) \in \mathrm{Z}(T)$ for each integer $n \geq 1$. As T is an $\mathcal{SCA}$-ring, there exists $(a, m) \in T$ with $(a, m) \neq (0, 0)$ such that $J(a, m) = (0, 0)$. Then $a_n a = 0$ and $a_n m = 0$ for each integer $n \geq 1$. Hence $a_n(a, m) = 0$ for each integer $n \geq 1$. It follows that $I(a, m) = (0, 0)$ and $(a, m) \neq (0, 0)$, that is, $\mathrm{ann}_{R\oplus M}(I) \neq (0, 0)$. Therefore $R \oplus M$ is an $\mathcal{SCA}$-module over R proving (2).

2) ⇒ 1) Assume $R \oplus M$ is an $\mathcal{SCA}$-module over R.

Let $J = ((a_1, m_1), (a_2, m_2), \cdots, (a_n, m_n), \cdots)T$ be a countably generated ideal of T such that $(a_n, m_n) \in \mathrm{Z}(T)$ for each integer $n \geq 1$. Then $a_n \in \mathrm{Z}(R) \cup \mathrm{Z}_R(M) = \mathrm{Z}_R(R \oplus M)$ for each integer $n \geq 1$. Let $I = (a_1, a_2, \cdots, a_n, \cdots)$ be the countably generated ideal of R generated by the a_n. As $R \oplus M$ is an $\mathcal{SCA}$-module over R, there exists $(a, m) \in R \oplus M$ with $(a, m) \neq (0, 0)$ and $I(a, m) = (0, 0)$. Then $a_n a = 0$ and $a_n m = 0$ for each integer $n \geq 1$. Two cases arise.

Case 1: $m \neq 0$. Then $(a_n, m_n)(0, m) = (0, 0)$ for each integer $n \geq 1$ and thus $J(0, m) = (0, 0)$.

Case 2: $m = 0$. Then $a \neq 0$ and $a_n a = 0$ for each integer $n \geq 1$. Thus $(a_n, m_n)(a, 0) = (a_n a, am_i) = (0, am_n)$ for each integer $n \geq 1$. If $am_n = 0$ for each integer $n \geq 1$, then $(a_n, m_n)(a, 0) = (0, 0)$ for each integer $n \geq 1$, so that $J(a, 0) = (0, 0)$ and $(a, 0) \neq (0, 0)$. Now, suppose that there exists $j \in \mathbb{N} \setminus \{1\}$ such that $am_j \neq 0$. Then it is easy to verify that $(a_n, m_n)(0, am_j) = (0, 0)$ for each integer $n \geq 1$ as $aa_n = 0$ for each integer $n \geq 1$. Therefore $J(0, am_j) = (0, 0)$ and $(0, am_j) \neq (0, 0)$. It follows that T is an $\mathcal{SCA}$-ring, as desired.

II) $\mathcal{CA}$-property.

1) $\Rightarrow$ 2) Put $T := R \ltimes M$ and assume that T is a $\mathcal{CA}$-ring. Let $I = (a_1, a_2, \cdots, a_n, \cdots)$ be a countably generated ideal of R such that $I \subseteq \mathrm{Z}_R(R \oplus M) = \mathrm{Z}(R) \cup \mathrm{Z}_R(M)$. Consider the ideal $J := ((a_1, 0), (a_2, 0), \cdots, (a_n, 0), \cdots)T$. Let us prove that $J \subseteq \mathrm{Z}(T)$. In fact, let $t = (a_1, 0)(r_1, m_1) + (a_2, 0)(r_2, m_2) + \cdots + (a_n, 0)(r_n, m_n) \in J$. Then $t = (a_1 r_1 + \cdots + a_n r_n, a_1 m_1 + \cdots + a_n m_n) =: (a', m')$. Note that $a_1 r_1 + \cdots + a_n r_n \in I \subseteq \mathrm{Z}(R) \cup \mathrm{Z}_R(M)$. Then $t \in \mathrm{Z}(T)$. It follows that $J \subseteq \mathrm{Z}(T)$, as contended. As T is a $\mathcal{CA}$-ring, there exists $(a, m) \in T$ such that $(a, m) \neq (0, 0)$ and $J(a, m) = (0, 0)$. Let $x = a_1 r_1 + \cdots + a_n r_n \in I$. Then

$$
\begin{aligned}
x(a, m) &= (xa, xm) \\
&= (\sum_{i=1}^{n} a_i r_i a, \sum_{i=1}^{n} a_i r_i m) \\
&= \sum_{i=1}^{n} (a_i r_i, 0)(a, m) \\
&= \Big(\sum_{i=1}^{n} (a_i, 0)(r_i, 0) \Big)(a, m) \\
&= (0, 0) \text{ as } \sum_{i=1}^{n} (a_i, 0)(r_i, 0) \in J.
\end{aligned}
$$

Hence $I(a, m) = (0, 0)$ and $(a, m) \neq (0, 0)$. It follows that $\mathrm{ann}_{R \oplus M}(I) \neq (0, 0)$. Hence $R \oplus M$ is a $\mathcal{CA}$-module over R.

2) $\Rightarrow$ 1) Assume that $R \oplus M$ is a $\mathcal{CA}$-module. Let $J = ((a_1, m_1), \cdots, (a_n, m_n), \cdots)T$ be a countably generated ideal of T such that $J \subseteq \mathrm{Z}(T)$. Let $I := (a_1, \cdots, a_n, \cdots)R$. Next, we prove that $I \subseteq \mathrm{Z}_R(R \oplus M)$. In fact, let $x = \sum_{i=1}^{n} a_i r_i \in I$. Then, as J is an ideal of T contained in $\mathrm{Z}(T)$,

$$
\sum_{i=1}^{n} (a_i, m_i)(r_i, 0) = \sum_{i=1}^{n} (a_i r_i, m_i r_i) = (x, \sum_{i=1}^{n} m_i r_i) \in \mathrm{Z}(T).
$$

Hence $x \in \mathrm{Z}(R) \cup \mathrm{Z}_R(M) = \mathrm{Z}_R(R \oplus M)$. It follows that $I \subseteq \mathrm{Z}_R(R \oplus M)$. As $R \oplus M$ is a $\mathcal{CA}$-module over R, there exists $(a, m) \in R \oplus M$ such that $(a, m) \neq (0, 0)$ and $I(a, m) = (0, 0)$.

Thus $a_n a = 0$ and $a_n m = 0$ for each integer $n \geq 1$. Note that, for each integer $n \geq 1$,

$$(a_n, m_n)(a, m) = (a_n a, a_n m + a m_n) = (0, a m_n).$$

If $a m_n = 0$ for each integer $n \geq 1$, then $(a_n, m_n)(a, m) = (0, 0)$ for each integer $n \geq 1$ and thus $J(a, m) = (0, 0)$ which means that $\text{ann}_T(J) \neq (0, 0)$. Assume that there exists an integer $j \geq 1$ such that $a m_j \neq 0$. Then

$$(a_n, m_n)(0, a m_j) = (0, a a_n m_j) = (0, 0)$$

for each integer $n \geq 1$. It follows that $\text{ann}_T(J) \neq (0, 0)$. Consequently, T is a $\mathcal{CA}$-ring completing the proof.

□

As an immediate consequence, we deduce the following result on the $\mathcal{CA}$-Property and $\mathcal{SCA}$-property of the idealization of a ring R on a module M such that R is a countable AV-module over M.

Corollary 4.5. *Let R be a ring and M an R-module. Assume that R is a countable* AV*-module over M. Then the following assertions are equivalent:*

1) $R \ltimes M$ is a $\mathcal{CA}$-ring (resp, an $\mathcal{SCA}$-ring);

2) $R \oplus M$ is a $\mathcal{CA}$-module (resp, an $\mathcal{SCA}$-module);

3) R is a $\mathcal{CA}$-ring (resp., an $\mathcal{SCA}$-ring).

Proof. 1) ⇔ 2) It holds by Theorem 4.4.
2) ⇔ 3) It holds by Corollary 4.3. □

Here next we provide the promised analog version of Lucas proposition [20, Proposition 3.5] for the $\mathcal{CA}$-property. Notice that we drop the reduceness hypotheses of R and we consider only an R-module M that is a submodule of a flat R-module.

Corollary 4.6. *Let R be a ring and M a submodule of a flat R-module. Then the following assertions are equivalent:*

1) $R \ltimes M$ is a $\mathcal{CA}$-ring (resp, an $\mathcal{SCA}$-ring);

2) $R \oplus M$ is a $\mathcal{CA}$-module (resp, an $\mathcal{SCA}$-module);

3) R is a $\mathcal{CA}$-ring (resp., an $\mathcal{SCA}$-ring).

Proof. Let M be a submodule of a flat R-module F. Then, by Theorem 3.4, R is a countable AV-module over M. Now, apply Corollary 4.5 to get the desired result. □

Corollary 4.7. *Let R be a countable AV-ring (in particular, a semi-regular ring) and M an R-module. Then the following assertions are equivalent:*

1) *$R \ltimes M$ is a $\mathcal{CA}$-ring (resp, an $\mathcal{SCA}$-ring);*
2) *$R \oplus M$ is a $\mathcal{CA}$-module (resp, an $\mathcal{SCA}$-module);*
3) *R is a $\mathcal{CA}$-ring (resp., an $\mathcal{SCA}$-ring).*

The remaining results of this section deal with the case of an idealization $R \ltimes M$, where M is a faithful R-module.

Corollary 4.8. *Let R be a ring and M a faithful R-module. Then the following assertions are equivalent:*

1) *$R \ltimes M$ is a $\mathcal{CA}$-ring (resp., an $\mathcal{SCA}$-ring);*
2) *$R \oplus M$ is a $\mathcal{CA}$-module (resp., an $\mathcal{SCA}$-module);*
3) *M is a $\mathcal{CA}$-module (resp., an $\mathcal{SCA}$-module).*

Proof. It follows directly from Theorem 4.4 and Corollary 4.3 as, by Proposition 2.6, M is a countable AV-module over R. □

Corollary 4.9. *Let R be a ring and M an R-module. Then the following assertions are equivalent:*

1) *$\dfrac{R}{\mathrm{ann}_R(M)} \ltimes M$ is a $\mathcal{CA}$-ring (resp., an $\mathcal{SCA}$-ring);*
2) *$\dfrac{R}{\mathrm{ann}_R(M)} \oplus M$ is a $\mathcal{CA}$-module (resp., an $\mathcal{SCA}$-module);*
3) *M is a $\mathcal{CA}$-module (resp., an $\mathcal{SCA}$-module).*

Proof. First, note that, by [4, Theorem 2.1], M is a $\mathcal{CA}$-module (resp., an $\mathcal{SCA}$-module) over R if and only it is so over $\dfrac{R}{\mathrm{ann}_R(M)}$. Also, note that $\mathrm{ann}_R\left(\dfrac{R}{\mathrm{ann}_R(M)} \oplus M\right) = \mathrm{ann}_R(M)$ and M is faithful over $\dfrac{R}{\mathrm{ann}_R(M)}$. Then the result easily follows from Corollary 4.8.

□

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