Asia Pac. J. Math. 2022 **9**:20

EXISTENCE AND STABILITY OF SOLUTIONS FOR A COUPLED SYSTEM OF FUZZY FRACTIONAL PANTOGRAPH STOCHASTIC DIFFERENTIAL EQUATIONS

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Received Sep. 19, 2022

Abstract. In the current paper, we investigate a novel class of coupled system of fuzzy fractional pantograph stochastic differential equations (FFPSDEs), whose derivative is based on Caputo fractional derivative. Firstly, we convert the system under consideration into an analogous integral system. Secondly, using Banach fixed point theorem, the existence and uniqueness results of solutions for FFPSDEs are then established. Additionally, we explore the Ulam-Hyers (UH) stability result of solution.

2010 Mathematics Subject Classification. 34A07, 34A08, 35K05.

Key words and phrases. fuzzy fractional stochastic differential equations; coupled system; Banach fixed point; Ulam-Hyers stability.

1. Introduction

The most significant area of practical mathematics that converts all currently available integer-order operators to arbitrary-order ones is fractional calculus. This significance results from the excellent precision of fractional operators in the simulation of numerous real-world occurrences in the context of various fractional boundary value problems.

Pantograph differential equations are functional differential equations with proportional delays. Because these equations have numerous real-world applications in areas like electrodynamics, astrophysics, and cell development, many researchers have explored them numerically and analytically.

DOI: 10.28924/APJM/9-20

On the other hand, one essential qualitative theory for dynamical systems is the notion of stability. As a result, the theory of stability characteristics has attracted significant interest through applications in a number of study domains.

Particularly, many scholars have looked into the Ulam-Hyers stability analysis and its applicability to many kinds of differential equations. We wish to mention that the theory of FFPSDEs have recently been the subject of important studies. As, for the UH stability of this type of system, even less has been done, with only a few works published in this topic as for as we know.

In [2], Jameel et al. studied approximate solution fuzzy pantograph equation by using homotopy perturbation method. Mikaeilvand et al. [3] introduced a numerical method based on the Taylor polynomials for the approximate solution of fuzzy pantograph equation. Hosseinzadet et al. [4] studied the pantograph volterra fuzzy integrodifferential equation. In [5] Agilan et al. proved the existence of solutions of fuzzy fractional panto-graph equations. Recently, Priyadharsini et al. [22] proposed a new type of equation nemely fuzzy fractional stochastic pantograph delay differential system. Then, Arhrrabi et al. [6,24–26] studied the existence and stability of solutions of fuzzy fractional stochastic differential equations with fractional Brownian motions, averaging principle for fuzzy stochastic differential equations, fuzzy fractional boundary value problem and existence and uniqueness results of fuzzy fractional stochastic differential equations with impulsive, respectively. Melliani et al. [7] studied Ulam-Hyers-Rassias stability for fuzzy fractional integrodifferential equations under Caputo gH-differentiability. On the other hand, UH stability is another interesting topic for mathematicians researchers. Then, stability theory research is currently fairly common and many articles have been published. Sajedi et al. [8] investigate the existence, uniqueness and UH stability of solutions of an impulsive coupled system of fractional differential equation with Caputo-Katugampola fuzzy fractional derivative. For more details, references [9–12] are some of the studies on the UH stability. Chalishajar et al. [13] studied existence and stability of solutions for a coupled system of fractional differential equation. Tamer [14] established the existence and Ulam stability of nonlinear coupled system of fractional differential equation including generalized Caputo fractional derivative. For more details, references [15–20] are some of important studies on the coupled system of FDEs.

Motivated by the above mentioned works and its importance in many applied fields, it is interesting to study the coupled system of FFPSDEs. So, in this paper, we will study the existence, uniqueness and UH stability of the following coupled system

$$(1)\quad \begin{cases} {}^{C}\mathcal{D}^{\gamma_1}x(t) = Ax(t) + f_1\big(t, x(t), x(\lambda_1 t), y(t)\big) + \Big\langle \int_0^t g_1\big(s, x(s), x(\lambda_1 s), y(s)\big)dB(s)\Big\rangle, \\ {}^{C}\mathcal{D}^{\gamma_2}y(t) = Ay(t) + f_2\big(t, y(t), y(\lambda_2 t), x(t)\big) + \Big\langle \int_0^t g_2\big(s, y(s), y(\lambda_2 s), x(s)\big)dB(s)\Big\rangle, \\ x(0) = x_0 \in \mathcal{F}_{\mathbb{R}^n} \qquad \text{and} \qquad y(0) = y_0 \in \mathcal{F}_{\mathbb{R}^n}, \end{cases}$$

where ${}^{C}\mathcal{D}^{\gamma_1}$, ${}^{C}\mathcal{D}^{\gamma_2}$ denote the Caputo fractional derivative of order γ_1 and γ_2 respectively, A is $n\times n$ matrix, the functions $f_1, f_2 : \mathcal{J} := [0,T]\times\mathcal{F}^3_{\mathbb{R}^n} \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ and $g_1, g_2 : \mathcal{J}\times\mathcal{F}^3_{\mathbb{R}^n} \longrightarrow \mathbb{R}^{n\times m}$ are continuous on $\mathcal{J}$, $B(t)$ is standard Brownian motion with n-dimensional.
The rest of this paper is formulated as follows. Section 2 is related to some basic definitions, lemma and remarks for FFPSDEs wich will be needed in the later section. In Section 3, some appropriate conditions are derived for the existence and uniqueness of solutions and UH stability of the given problem. The final Section is provide a conclusion.

2. Preliminaries

Let $\mathcal{F}_{\mathbb{R}^n}$ denote the set of fuzzy subsets of the real axis, if $\Lambda : \mathbb{R}^n \longrightarrow [0,1]$, satisfying the following properties:

(i) Λ is normal, that is, there exists $z_0 \in \mathbb{R}^n$ such that $\Lambda(z_0) = 1$,

(ii) Λ is fuzzy convex, that is, for $0 \le \lambda \le 1$

$$\Lambda\big(\lambda z_1 + (1-\lambda)z_2\big) \ge min\left\{\Lambda(z_1), \Lambda(z_2)\right\},\ for\ any\ z_1, z_2 \in \mathbb{R}^n,$$

(iii) Λ is upper semicontinous on $\mathbb{R}^n$,

(iv) $[\Lambda]^0 = cl\{z \in \mathbb{R}^n : \Lambda(z) > 0\}$ is compact, where cl denotes the closure in $(\mathbb{R}^n, |.|)$.

Then $\mathcal{F}_{\mathbb{R}^n}$ is called the space of fuzzy number. For $\gamma \in (0,1]$, we denote $[\Lambda]^\gamma = \{z \in \mathbb{R}^n | \Lambda(z) \ge \gamma\}$ and $[\Lambda]^0 = \{z \in \mathbb{R}^n | \Lambda(z) > 0\}$. From the conditions (i) to (iv), it follows that the $\gamma - level$ set of Λ, $[\Lambda]^\gamma$, is a nonempty compact interval, for all $\gamma \in [0,1]$ and any $\Lambda \in \mathcal{F}_{\mathbb{R}^n}$.
The notation $[\Lambda]^\gamma = [\underline{\Lambda}(\gamma), \overline{\Lambda}(\gamma)]$, denotes explicitly the $\gamma - level$ set of Λ, for $\gamma \in [0,1]$. We refer to $\underline{\Lambda}$ and $\overline{\Lambda}$ as the lower and upper branches of Λ, respectively. For $\Lambda \in \mathcal{F}_{\mathbb{R}^n}$, we define the lengh of the $\gamma - level$ set of Λ as $len([\Lambda]^\gamma) = \overline{\Lambda}(\gamma) - \underline{\Lambda}(\gamma)$. For addition and scalar multiplication in fuzzy set space $\mathcal{F}_{\mathbb{R}^n}$, we have $[\Lambda_1 + \Lambda_2]^\gamma = [\Lambda_1]^\gamma + [\Lambda_2]^\gamma$, $[\lambda\Lambda]^\gamma = \lambda[\Lambda]^\gamma$.
The Hausdorff distance between fuzzy numbers is given by

$$\begin{aligned} \mathcal{D}_\infty\big(\Lambda_1, \Lambda_2\big) &= \sup_{0\le\gamma\le1} \left\{|\underline{\Lambda}_1(\gamma) - \underline{\Lambda}_2(\gamma)|, |\overline{\Lambda}_1(\gamma) - \overline{\Lambda}_2(\gamma)|\right\}, \\ &= \sup_{0\le\gamma\le1} \mathcal{D}_H\big([\Lambda_1]^\gamma, [\Lambda_2]^\gamma\big). \end{aligned}$$

The metric space $(\mathcal{F}_{\mathbb{R}^n}, \mathcal{D}_\infty)$ is complet metric space and the following properties of the metric $\mathcal{D}_\infty$ are valid.

$$\mathcal{D}_\infty\big(\Lambda_1+\Lambda_3, \Lambda_2+\Lambda_3\big) = \mathcal{D}_\infty\big(\Lambda_1, \Lambda_2\big),$$

$$\mathcal{D}_\infty\big(\lambda\Lambda_1, \lambda\Lambda_2\big) = |\lambda|\mathcal{D}_\infty\big(\Lambda_1, \Lambda_2\big),$$

$$\mathcal{D}_\infty\big(\Lambda_1, \Lambda_2\big) \le \mathcal{D}_\infty\big(\Lambda_1, \Lambda_3\big) + \mathcal{D}_\infty\big(\Lambda_3, \Lambda_2\big),$$

for all $\Lambda_1, \Lambda_2, \Lambda_3 \in \mathcal{F}_{\mathbb{R}^n}$ and $\lambda \in \mathbb{R}^n$.

Definition 2.1. [1] The metric D on $C\big(\mathcal{J}, \mathcal{F}_{\mathbb{R}^n}\big)$ is given by

$$D\big(\Lambda_1, \Lambda_2\big) = \sup_{0\le t\le T} \mathcal{D}_\infty\big(\Lambda_1(t), \Lambda_2(t)\big).$$

Remark 2.2. $\big(C(\mathcal{J}, \mathcal{F}_{\mathbb{R}^n}), D\big)$ is a complete metric space.

Definition 2.3. [1] • The derivative $v'(t)$ of a fuzzy processus u is defined by

$$[v'(t)]^\gamma = \left[(\underline{v}^\gamma)'(t), (\overline{v}^\gamma)'(t)\right],$$

provided that the equation define a fuzzy set $u'(t) \in \mathcal{F}_{\mathbb{R}^n}$.

• The fuzzy integral $\int_a^b v(t)dt$, $a, b \in [0, T]$ is defined by

$$\left[\int_a^b v(t)dt\right]^\gamma = \left[\int_a^b \underline{v}^\gamma(t)dt, \int_a^b \overline{v}^\gamma(t)dt\right],$$

provided that the Lebesgue integral on the right hand side exist.

Definition 2.4. [1] A mapping $f : \mathcal{J} \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ is strongly measurable if for all $\gamma \in [0, 1]$ the set-valued function $F_\gamma : \mathcal{J} \longrightarrow \mathbf{K}(\mathbb{R}^n)$ defined by $F_\gamma(t) = [f(t)]^\gamma$ is Lebesgue measurable when $\mathbf{K}(\mathbb{R}^n)$ has the topology induced by the Hausdorff metric.

Definition 2.5. [1] • A mapping $f : \mathcal{J} \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ is called level wise continuous at $t_0 \in \mathcal{J}$ if the multivalued mapping $F_\gamma(t) = [f(t)]^\gamma$ is continuous at $t = t_0$ with respect to the Hausdorff metric for all $\gamma \in [0, 1]$.

• A mapping $f : \mathcal{J} \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ is said to be integrably bounded if there is an integrable function $g(t)$ such that $\|y(t)\| \le g(t)$ for every $y(t) \in F_0(t)$.

• A strongly measurable and integrably bounded mapping $f : \mathcal{J} \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ is said to be integrable over $\mathcal{J}$ if $\int_0^T f(t)dt \in \mathcal{F}_{\mathbb{R}^n}$.

Remark 2.6. If $f : \mathcal{J} \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ is strongly measurable and integrably bounded, then f is integrable.

Let $\langle . \rangle : \mathbb{R}^n \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ denote the embedding of $\mathbb{R}^n$ into $\mathcal{F}_{\mathbb{R}^n}$, i.e. for $r \in \mathbb{R}^n$ we have

$$\langle r \rangle(a) = \begin{cases} 1\,,\ if \quad a = r, \\ 0\,,\ if \quad a \neq r. \end{cases}$$

Remark 2.7. It is easy to see that if $y : \Omega \longrightarrow \mathbb{R}^n$ is an $\mathbb{R}^n$-valued random variable on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, then $\langle y \rangle : \Omega \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ is a fuzzy random variable. For stochastic processes we have a similar property.

Notations: Let $(\Omega, \mathcal{F}_{\mathbb{R}^n})$ be the complete probability space and $B(t)$ be a n-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}_{\mathbb{R}^n})$. Let $L^2(\Omega, \mathcal{F}_{\mathbb{R}^n})$ be the collection of all strongly measurable square integrable $(\Omega, \mathcal{F}_{\mathbb{R}^n})$-valued random variable, which is a complete metric space equipped with the following metric

$$D^2(\Lambda_1, \Lambda_2) = \mathbb{E}\mathcal{D}^2_\infty(\Lambda_1, \Lambda_2).$$

Let $C(\mathcal{J}, L^2(\Omega, \mathcal{F}_{\mathbb{R}^n}))$ be the Banach space of all continuous process from I into $L^2(\Omega, \mathcal{F}_{\mathbb{R}^n})$ such that $\mathbb{E}\mathcal{D}^2_\infty(\Lambda_1, \Lambda_2) < \infty$. Denote by $\mathcal{B}_h := C(\mathcal{J}, L^2(\Omega, \mathcal{F}_{\mathbb{R}^n}))$ the closed bounded subspace of all continuous fuzzy process Λ in $L^2(\Omega, \mathcal{F}_{\mathbb{R}^n})$ consists of $\mathcal{A}_t$-adapted measurable process $\{\Lambda(t), t \in \mathcal{J}\}$ equipped with the norm

$$\mathbb{E}\mathcal{D}^2_\infty(\Lambda_1, \Lambda_2) = \sup_{0 \le t \le T} \mathbb{E}\mathcal{D}^2_\infty(\Lambda_1(t), \Lambda_2(t)).$$

Remark 2.8. Note that $(\mathcal{B}_h, \mathcal{D}_\infty)$ is a complete metric space.

Proposition 2.9. [22] *Let $\psi : \mathcal{J} \longrightarrow \mathbb{R}^n$, then for $t \in \mathcal{J}$;*

$$\sup_{a \in [0,t]} \mathbb{E}\Big\| \int_0^a \psi(s) dB(s) \Big\|^2 \le C_T \int_0^t \Big\| \psi(s) \Big\|^2 ds.$$

Lemma 2.10. [21] *The nonnegative functions M_γ and $M_{\gamma,\gamma}$ given by*

$$M_\gamma(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\gamma k + 1)},$$

$$M_{\gamma,\gamma}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\gamma k + \gamma)},$$

have the following properties:

$1-$ $\forall A > 0$ *and* $t \in \mathcal{J}$

$$M_\gamma(A t^\gamma) \le 1,$$

$$M_{\gamma,\gamma}(At^{\gamma}) \leq \frac{1}{\Gamma(\gamma)},$$

and $M_{\gamma}(0) = 1$, $M_{\gamma,\gamma}(0) = \frac{1}{\Gamma(\gamma)}$.

$2-$ $\forall A > 0$ *and* $t_1, t_2 \in \mathcal{J}$ *such that* $t_1 \leq t_2$*:*

$$M_{\gamma}(At_2^{\gamma}) \leq M_{\gamma}(At_1^{\gamma}),$$

$$M_{\gamma,\gamma}(At_2^{\gamma}) \leq M_{\gamma,\gamma}(At_1^{\gamma}).$$

3. Main results

3.1. **Existence and uniqueness result.** In this subsection, we show the existence and uniqueness of fuzzy solution for a coupled system of FFPSDEs (1).

Definition 3.1. A couple (x, y) is a solution of problem (1) if it satisfies the equations

$$\begin{cases} {}^{C}\mathcal{D}^{\gamma_1}x(t) = Ax(t) + f_1\big(t, x(t), x(\lambda_1 t), y(t)\big) + \big\langle \int_0^t g_1\big(s, x(s), x(\lambda_1 s), y(s)\big)dB(s)\big\rangle, \\ \\ {}^{C}\mathcal{D}^{\gamma_2}y(t) = Ay(t) + f_2\big(t, y(t), y(\lambda_2 t), x(t)\big) + \big\langle \int_0^t g_2\big(s, y(s), y(\lambda_2 s), x(s)\big)dB(s)\big\rangle, \end{cases}$$

and the conditions initial $x(0) = x_0$, $y(0) = y_0$.

It follows from [23], that the solution of (1) can be expressed as follows:

$$\begin{aligned} x(t) = {} & M_{\gamma_1}(At^{\gamma_1})x_0 + \int_0^t (t-s)^{\gamma_1 - 1} M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big) f_1\big(s, x(s), x(\lambda_1 s), y(s)\big)ds \\ & + \int_0^t (t-s)^{\gamma_1 - 1} M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)\Big\langle \int_0^s g_1\big(u, x(u), x(\lambda_1 u), y(u)\big)dB(u)\Big\rangle ds. \end{aligned}$$

$$\begin{aligned} y(t) = {} & M_{\gamma_2}(At^{\gamma_2})y_0 + \int_0^t (t-s)^{\gamma_2 - 1} M_{\gamma_2,\gamma_2}\big(A(t-s)^{\gamma_2}\big) f_2\big(s, y(s), y(\lambda_2 s), x(s)\big)ds \\ & + \int_0^t (t-s)^{\gamma_2 - 1} M_{\gamma_2,\gamma_2}\big(A(t-s)^{\gamma_2}\big)\Big\langle \int_0^s g_2\big(u, y(u), y(\lambda_2 u), x(u)\big)dB(u)\Big\rangle ds. \end{aligned}$$

We will study the proposed system under the following assumptions:

$(\mathcal{A}1)$ The functions f_1, f_2 are continuous and $\exists L_1, L_2 > 0$ such that

$$\mathbb{E}\mathcal{D}^2_{\infty}\Big(f_1(t,x,y,z), f_1(t,u,v,w)\Big) \leq L_1\Big(\mathbb{E}\mathcal{D}^2_{\infty}(x,u) + \mathbb{E}\mathcal{D}^2_{\infty}(y,v) + \mathbb{E}\mathcal{D}^2_{\infty}(z,w)\Big).$$

$$\mathbb{E}\mathcal{D}^2_{\infty}\Big(f_2(t,x,y,z), f_2(t,u,v,w)\Big) \leq L_2\Big(\mathbb{E}\mathcal{D}^2_{\infty}(x,u) + \mathbb{E}\mathcal{D}^2_{\infty}(y,v) + \mathbb{E}\mathcal{D}^2_{\infty}(z,w)\Big).$$

$(\mathcal{A}2)$ The functions g_1, g_2 are continuous and $\exists N_1, N_2 > 0$ such that

$$\mathbb{E}\Big\|g_1(t,x,y,z)-g_1(t,u,v,w)\Big\|^2 \leq N_1\Big(\mathbb{E}\mathcal{D}_\infty^2(x,u)+\mathbb{E}\mathcal{D}_\infty^2(y,v)+\mathbb{E}\mathcal{D}_\infty^2(z,w)\Big).$$

$$\mathbb{E}\Big\|g_2(t,x,y,z)-g_2(t,u,v,w)\Big\|^2 \leq N_2\Big(\mathbb{E}\mathcal{D}_\infty^2(x,u)+\mathbb{E}\mathcal{D}_\infty^2(y,v)+\mathbb{E}\mathcal{D}_\infty^2(z,w)\Big).$$

$(\mathcal{A}3)$ For all $t \in \mathcal{J}$, $\exists Q_1, Q_2 > 0$ such that

$$\mathbb{E}\mathcal{D}_\infty^2\Big(f_1(t,\hat{0},\hat{0},\hat{0}),\hat{0}\Big) \leq Q_1 \qquad \text{and} \qquad \mathbb{E}\mathcal{D}_\infty^2\Big(f_2(t,\hat{0},\hat{0},\hat{0}),\hat{0}\Big) \leq Q_2.$$

$(\mathcal{A}4)$ For all $t \in \mathcal{J}$, $\exists P_1, P_2 > 0$ such that

$$\mathbb{E}\Big\|g_1(t,\hat{0},\hat{0},\hat{0})\Big\|^2 \leq P_1 \qquad \text{and} \qquad \mathbb{E}\Big\|g_2(t,\hat{0},\hat{0},\hat{0})\Big\|^2 \leq P_2.$$

Theorem 3.2. *Assume that the assumptions* $(\mathcal{A}1)-(\mathcal{A}4)$ *holds. Then the system* (1) *has a unique solution provided that*

$$\frac{6(L_1+L_2)T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}+\frac{6(N_1+N_2)T^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}<1.$$

Proof. We define the operator $F : \mathcal{B}_h \times \mathcal{B}_h \longrightarrow \mathcal{B}_h \times \mathcal{B}_h$ by

$$F(x,y)(t) = \Big(T_1(x,y)(t), T_2(x,y)(t)\Big),$$

where

$$T_1(x,y)(t) = M_{\gamma_1}(At^{\gamma_1})x_0 + \int_0^t (t-s)^{\gamma_1-1}M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)f_1\big(s,x(s),x(\lambda_1 s),y(s)\big)ds$$
$$+\int_0^t (t-s)^{\gamma_1-1}M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)\Big\langle \int_0^s g_1\big(u,x(u),x(\lambda_1 u),y(u)\big)dB(u)\Big\rangle ds.$$

$$T_2(x,y)(t) = M_{\gamma_2}(At^{\gamma_2})y_0 + \int_0^t (t-s)^{\gamma_2-1}M_{\gamma_2,\gamma_2}\big(A(t-s)^{\gamma_2}\big)f_2\big(s,y(s),y(\lambda_2 s),x(s)\big)ds$$
$$+\int_0^t (t-s)^{\gamma_2-1}M_{\gamma_2,\gamma_2}\big(A(t-s)^{\gamma_2}\big)\Big\langle \int_0^s g_2\big(u,y(u),y(\lambda_2 u),x(u)\big)dB(u)\Big\rangle ds.$$

Then, the fixed point of the operator F coincides with a solution of coupled system (1). For each positive number r, we define

$$\mathcal{B}_r = \{(x,y) \in \mathcal{B}_h \times \mathcal{B}_h : \mathbb{E}\mathcal{D}_\infty^2\big((x,y),\hat{0}\big) \leq r\}.$$

Step 1: We prove that $F(\mathcal{B}_r) \subseteq \mathcal{B}_r$. We choose

$$r \geq \frac{3\mathbb{E}\mathcal{D}_\infty^2(x_0,\hat{0})+3\mathbb{E}\mathcal{D}_\infty^2(y_0,\hat{0})+J_1+K_1}{1-J_2-K_2}.$$

By using the assumptions $(\mathcal{A}1)-(\mathcal{A}4)$, Cauchy–Schwarz inequality and It$\hat{o}$ isometry, we get

$$\begin{aligned}
\mathbb{E}\mathcal{D}^2_\infty\big(T_1(x,y)(t),\hat{0}\big) &= \mathbb{E}\mathcal{D}^2_\infty\bigg(M_{\gamma_1}(At^{\gamma_1})x_0 + \int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}} f_1\big(s,x(s),x(\lambda_1 s),y(s)\big)ds \\
&+ \int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}} \Big\langle \int_0^s g_1\big(u,x(u),x(\lambda_1 u),y(u)\big)dB(u)\Big\rangle ds, \hat{0}\bigg), \\
&\le 3\mathbb{E}\mathcal{D}^2_\infty\Big(M_{\gamma_1}(At^{\gamma_1})x_0,\hat{0}\Big) + 3\mathbb{E}\mathcal{D}^2_\infty\bigg(\int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}} f_1\big(s,x(s),x(\lambda_1 s),y(s)\big)ds,\hat{0}\bigg) \\
&+ 3\mathbb{E}\mathcal{D}^2_\infty\bigg(\int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}} \Big\langle \int_0^s g_1\big(u,x(u),x(\lambda_1 u),y(u)\big)dB(u)\Big\rangle ds, \hat{0}\bigg), \\
&\le 3\mathbb{E}\mathcal{D}^2_\infty\big(x_0,\hat{0}\big) + \frac{6T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \mathbb{E}\mathcal{D}^2_\infty\Big(f_1\big(s,x(s),x(\lambda_1 s),y(s)\big), f_1\big(s,\hat{0},\hat{0},\hat{0}\big)\Big)ds \\
&+ \frac{6T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \mathbb{E}\mathcal{D}^2_\infty\Big(f_1\big(s,\hat{0},\hat{0},\hat{0}\big),\hat{0}\Big)ds \\
&+ \frac{6T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \bigg(\int_0^s \mathbb{E}\Big\| g_1\big(u,x(u),x(\lambda_1 u),y(u)\big) - g_1\big(u,\hat{0},\hat{0},\hat{0}\big)\Big\|^2 du\bigg)ds \\
&+ \frac{6C_T T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \bigg(\int_0^s \mathbb{E}\Big\| g_1\big(u,\hat{0},\hat{0},\hat{0}\big)\Big\|^2 du\bigg)ds, \\
&\le 3\mathbb{E}\mathcal{D}^2_\infty\big(x_0,\hat{0}\big) + \frac{6L_1T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \Big[\mathbb{E}\mathcal{D}^2_\infty\big(x(s),\hat{0}\big) + \mathbb{E}\mathcal{D}^2_\infty\big(x(\lambda_1 s),\hat{0}\big) + \mathbb{E}\mathcal{D}^2_\infty\big(y(s),\hat{0}\big)\Big]ds \\
&+ \frac{6N_1T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t (t-s)^{\gamma_1-1}\bigg(\int_0^s \Big[\mathbb{E}\mathcal{D}^2_\infty\big(x(u),\hat{0}\big) + \mathbb{E}\mathcal{D}^2_\infty\big(x(\lambda_1 u),\hat{0}\big) + \mathbb{E}\mathcal{D}^2_\infty\big(y(u),\hat{0}\big)\Big]du\bigg)ds \\
&+ \frac{6Q_1T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{3P_1C_TT^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}, \\
&\le 3\mathbb{E}\mathcal{D}^2_\infty\big(x_0,\hat{0}\big) + \frac{6L_1T^{2\gamma_1}r}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6Q_1T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6N_1T^{2\gamma_1+1}r}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{3P_1C_TT^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}, \\
&\le 3\mathbb{E}\mathcal{D}^2_\infty\big(x_0,\hat{0}\big) + \frac{6Q_1T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{3P_1C_TT^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \bigg(\frac{6L_1T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6N_1T^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\bigg)r, \\
&\le 3\mathbb{E}\mathcal{D}^2_\infty\big(x_0,\hat{0}\big) + J_1 + J_2 r,
\end{aligned}$$

where

$$J_1 = \frac{6Q_1T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{3P_1C_TT^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} \quad \text{and} \quad J_2 = \frac{6L_1T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6N_1T^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}.$$

In the same way, we can obtain that:

$$\mathbb{E}\mathcal{D}^2_\infty\big(T_2(x,y)(t),\hat{0}\big) \le 3\mathbb{E}\mathcal{D}^2_\infty\big(y_0,\hat{0}\big) + K_1 + K_2 r,$$

where

$$K_1 = \frac{6Q_2T^{2\gamma_2}}{(2\gamma_2-1)(\Gamma(\gamma_2))^2} + \frac{3P_2C_TT^{2\gamma_2+1}}{(2\gamma_2-1)(\Gamma(\gamma_2))^2} \quad \text{and} \quad K_2 = \frac{6L_2T^{2\gamma_2}}{(2\gamma_2-1)(\Gamma(\gamma_2))^2} + \frac{6N_2T^{2\gamma_2+1}}{(2\gamma_2-1)(\Gamma(\gamma_2))^2}.$$

Finally, we have

$$\mathbb{E}\mathcal{D}^2_\infty\big(F(x,y)(t),\hat{0}\big) \le \mathbb{E}\mathcal{D}^2_\infty\big(T_1(x,y)(t),\hat{0}\big) + \mathbb{E}\mathcal{D}^2_\infty\big(T_2(x,y)(t),\hat{0}\big) \le r,$$

which implies that $F(\mathcal{B}_r) \subseteq \mathcal{B}_r$.

Step 2: We show that F is a contraction operator. For $(x,y),(x',y') \in \mathcal{B}_h \times \mathcal{B}_h$ and $t \in \mathcal{J}$, using the assumptions $(\mathcal{A}1)-(\mathcal{A}4)$, Cauchy–Schwarz inequality and It$\widehat{o}$ isometry, we have

$$\mathbb{E}\mathcal{D}^2_\infty\big(T_1(x,y)(t),T_1(x',y')(t)\big) = \mathbb{E}\mathcal{D}^2_\infty\bigg(M_{\gamma_1}(At^{\gamma_1})x_0 + \int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}} f_1\big(s,x(s),$$
$$x(\lambda_1 s),y(s)\big)ds + \int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}}\Big\langle \int_0^s g_1\big(u,x(u),x(\lambda_1 u),y(u)\big)dB(u)\Big\rangle ds, M_{\gamma_1}(At^{\gamma_1})x_0$$
$$+\int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}} f_1\big(s,x'(s),x'(\lambda_1 s),y'(s)\big)ds$$
$$+\int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}}\Big\langle \int_0^s g_1\big(u,x'(u),x'(\lambda_1 u),y'(u)\big)dB(u)\Big\rangle ds\bigg),$$
$$\leq 2\mathbb{E}\mathcal{D}^2_\infty\bigg(\int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}} f_1\big(s,x(s),x(\lambda_1 s),y(s)\big)ds, \int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}}$$
$$f_1\big(s,x'(s),x'(\lambda_1 s),y'(s)\big)ds\bigg) + 2\mathbb{E}\mathcal{D}^2_\infty\bigg(\int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}}\Big\langle \int_0^s g_1\big(u,x(u),$$
$$x(\lambda_1 u),y(u)\big)dB(u)\Big\rangle ds, \int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}}\Big\langle \int_0^s g_1\big(u,x'(u),x'(\lambda_1 u),y'(u)\big)dB(u)\Big\rangle ds\bigg),$$
$$\leq \frac{2T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \mathbb{E}\mathcal{D}^2_\infty\Big(f_1\big(s,x(s),x(\lambda_1(s)),y(s)\big), f_1\big(s,x'(s),x'(\lambda_1(s)),y'(s)\big)\Big)ds$$
$$+\frac{2T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \bigg(\int_0^s \mathbb{E}\Big\|g_1\big(u,x(u),x(\lambda_1 u),y(u)\big) - g_1\big(u,x'(u),x'(\lambda_1 u),y'(u)\big)\Big\|^2 du\bigg)ds,$$
$$\leq \frac{2L_1T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \Big(\mathbb{E}\mathcal{D}^2_\infty\big(x(s),x'(s)\big) + \mathbb{E}\mathcal{D}^2_\infty\big(x(\lambda_1 s),x'(\lambda_1 s)\big) + \mathbb{E}\mathcal{D}^2_\infty\big(y(s),y'(s)\big)\Big)ds,$$
$$+\frac{2N_1T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \bigg(\int_0^s \Big[\mathbb{E}\mathcal{D}^2_\infty\big(x(s),x'(s)\big) + \mathbb{E}\mathcal{D}^2_\infty\big(x(\lambda_1 s),x'(\lambda_1 s)\big) + \mathbb{E}\mathcal{D}^2_\infty\big(y(s),y'(s)\big)\Big]du\bigg)ds,$$
$$\leq \frac{6L_1T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\Big(\mathbb{E}\mathcal{D}^2_\infty(x,x') + \mathbb{E}\mathcal{D}^2_\infty(y,y')\Big) + \frac{6N_1T^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\Big(\mathbb{E}\mathcal{D}^2_\infty(x,x') + \mathbb{E}\mathcal{D}^2_\infty(y,y')\Big),$$
$$\leq \bigg(\frac{6L_1T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6N_1T^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\bigg)\Big(\mathbb{E}\mathcal{D}^2_\infty(x,x') + \mathbb{E}\mathcal{D}^2_\infty(y,y')\Big).$$

With a similar method, we also get:

$$\mathbb{E}\mathcal{D}^2_\infty\big(T_2(x,y)(t),T_2(x',y')(t)\big) \leq \bigg(\frac{6L_2T^{2\gamma_2}}{(2\gamma_2-1)(\Gamma(\gamma_2))^2} + \frac{6N_2T^{2\gamma_2+1}}{(2\gamma_2-1)(\Gamma(\gamma_2))^2}\bigg)\Big(\mathbb{E}\mathcal{D}^2_\infty(x,x') + \mathbb{E}\mathcal{D}^2_\infty(y,y')\Big).$$

Finally, we can get:

$$\mathbb{E}\mathcal{D}^2_\infty\big(F(x,y)(t),F(x',y')(t)\big) \leq \bigg(\frac{6(L_1+L_2)T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6(N_1+N_2)T^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\bigg)\Big(\mathbb{E}\mathcal{D}^2_\infty(x,x') + \mathbb{E}\mathcal{D}^2_\infty(y,y')\Big).$$

So, since $\frac{6(L_1+L_2)T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6(N_1+N_2)T^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} < 1$, then F is a contraction operator. Therefore, by using Banach's contraction mapping principale, we conclude that F has a fixed point which is the unique solution of (1). □

3.2. **Stability analysis.** In this subsection, we study Ulam's type stability for the coupled system (1). First, we recall the definition of those types of Ulam stability.

Definition 3.3. [13] The coupled system (1) is said to be UH stable if there exists a constant $\omega = (\omega_1, \omega_2) > 0$ such that for each $\epsilon = (\epsilon_1, \epsilon_2) > 0$ and solution $(x, y) \in \mathcal{B}_h \times \mathcal{B}_h$ of the following inequality

$$(2)\quad \begin{cases} \mathbb{E}\mathcal{D}^2_\infty\Big(\ {}^C\mathcal{D}^{\gamma_1}x(t), Ax(t) + f_1\big(t, x(t), x(\lambda_1 t), y(t)\big) + \Big\langle \int_0^t g_1\big(s, x(s), x(\lambda_1 s), y(s)\big)dB(s)\Big\rangle \ \Big) \leq \epsilon_1, \\ \\ \mathbb{E}\mathcal{D}^2_\infty\Big(\ {}^C\mathcal{D}^{\gamma_2}y(t), Ay(t) + f_2\big(t, y(t), y(\lambda_2 t), x(t)\big) + \Big\langle \int_0^t g_2\big(s, y(s), y(\lambda_2 s), x(s)\big)dB(s)\Big\rangle \ \Big) \leq \epsilon_2, \end{cases}$$

there exists a solution $(v, k) \in \mathcal{B}_h \times \mathcal{B}_h$ of system (1), such that

$$\mathbb{E}\mathcal{D}^2_\infty\Big((x, y)(t), (v, k)(t)\Big) \leq \omega\epsilon, \qquad t \in \mathcal{J}.$$

Definition 3.4. [13] The system (1) is said to be generalized UH (GUH) stable if there exists $\varphi \in C^1(\mathcal{J}, \mathcal{F}_{\mathbb{R}^n})$, $\varphi(0) = 0$ such that for each solution $(x, y) \in \mathcal{B}_h \times \mathcal{B}_h$ of (2), there exists a solution $(v, k) \in \mathcal{B}_h \times \mathcal{B}_h$ of system (1) such that

$$\mathbb{E}\mathcal{D}^2_\infty\Big((x, y)(t), (v, k)(t)\Big) \leq \varphi(\epsilon), \qquad t \in \mathcal{J}.$$

Remark 3.5. A couple $(x, y) \in \mathcal{B}_h \times \mathcal{B}_h$ is a solution of (2) if and only if $\exists(\phi, \psi) \in \mathcal{B}_h \times \mathcal{B}_h$ such that

(i)- $\mathbb{E}\mathcal{D}^2_\infty\big(\phi(t), \hat{0}\big) \leq \epsilon_1$, and $\mathbb{E}\mathcal{D}^2_\infty\big(\psi(t), \hat{0}\big) \leq \epsilon_2$, $t \in \mathcal{J}$.

(ii)- For $\in \mathcal{J}$,

$$ {}^C\mathcal{D}^{\gamma_1}x(t) = Ax(t) + f_1\big(t, x(t), x(\lambda_1 t), y(t)\big) + \Big\langle \int_0^t g_1\big(s, x(s), x(\lambda_1 s), y(s)\big)dB(s)\Big\rangle + \phi(t),$$

$$ {}^C\mathcal{D}^{\gamma_2}y(t) = Ay(t) + f_2\big(t, y(t), y(\lambda_2 t), x(t)\big) + \Big\langle \int_0^t g_2\big(s, y(s), y(\lambda_2 s), x(s)\big)dB(s)\Big\rangle + \psi(t).$$

Lemma 3.6. *Suppose that $(x, y) \in \mathcal{B}_h \times \mathcal{B}_h$ is a solution of the inequality system (2). So, we have*

$$\mathbb{E}\mathcal{D}^2_\infty\big[x(t), L(t)\big] \leq \frac{T^{2\gamma_1 - 1}}{(2\gamma_1 - 1)(\Gamma(\gamma_1))^2}\epsilon_1 \qquad \textit{and} \qquad \mathbb{E}\mathcal{D}^2_\infty\big[y(t), P(t)\big] \leq \frac{T^{2\gamma_2 - 1}}{(2\gamma_2 - 1)(\Gamma(\gamma_2))^2}\epsilon_2$$

Proof. By remark 3.5, we have

$$(3)\quad \begin{cases} {}^C\mathcal{D}^{\gamma_1}x(t) = Ax(t) + f_1\big(t, x(t), x(\lambda_1 t), y(t)\big) + \big\langle \int_0^t g_1\big(s, x(s), x(\lambda_1 s), y(s)\big)dB(s)\big\rangle + \phi(t), \\ \\ {}^C\mathcal{D}^{\gamma_2}y(t) = Ay(t) + f_2\big(t, y(t), y(\lambda_2 t), x(t)\big) + \big\langle \int_0^t g_2\big(s, y(s), y(\lambda_2 s), x(s)\big)dB(s)\big\rangle + \psi(t). \end{cases}$$

Thanks to Definition 3.1, the solution of system (3) can be reformulated immediately as

$$\begin{aligned}x(t) &= M_{\gamma_1}(At^{\gamma_1})x_0 + \int_0^t (t-s)^{\gamma_1-1} M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big) f_1\big(s, x(s), x(\lambda_1 s), y(s)\big)ds \\ &+ \int_0^t (t-s)^{\gamma_1-1} M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)\Big\langle \int_0^s g_1\big(u, x(u), x(\lambda_1 u), y(u)\big)dB(u)\Big\rangle ds \\ &+ \int_0^t (t-s)^{\gamma_1-1} M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)\phi(s)ds, \\ &:= L(t) + \int_0^t (t-s)^{\gamma_1-1} M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)\phi(s)ds.\end{aligned}$$

$$\begin{aligned}y(t) &= M_{\gamma_2}(At^{\gamma_2})y_0 + \int_0^t (t-s)^{\gamma_2-1} M_{\gamma_2,\gamma_2}\big(A(t-s)^{\gamma_2}\big) f_2\big(s, y(s), y(\lambda_2 s), x(s)\big)ds \\ &+ \int_0^t (t-s)^{\gamma_2-1} M_{\gamma_2,\gamma_2}\big(A(t-s)^{\gamma_2}\big)\Big\langle \int_0^s g_2\big(u, y(u), y(\lambda_2 u), x(u)\big)dB(u)\Big\rangle ds \\ &+ \int_0^t (t-s)^{\gamma_2-1} M_{\gamma_2,\gamma_2}\big(A(t-s)^{\gamma_2}\big)\psi(s)ds, \\ &:= P(t) + \int_0^t (t-s)^{\gamma_2-1} M_{\gamma_2,\gamma_2}\big(A(t-s)^{\gamma_2}\big)\psi(s)ds.\end{aligned}$$

Then, using Cauchy–Schwarz inequality and It$\widehat{o}$ isometry, we get

$$\begin{aligned}\mathbb{E}\mathcal{D}^2_\infty\big[x(t), L(t)\big] &= \mathbb{E}\mathcal{D}^2_\infty\Big[\int_0^t (t-s)^{\gamma_1-1} M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)\phi(s)ds, \hat{0}\Big], \\ &\le \frac{T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\mathbb{E}\mathcal{D}^2_\infty[\phi,\hat{0}] \le \frac{T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\epsilon_1.\end{aligned}$$

Similarly, we find

$$\mathbb{E}\mathcal{D}^2_\infty\big[y(t), P(t)\big] \le \frac{T^{2\gamma_2-1}}{(2\gamma_2-1)(\Gamma(\gamma_2))^2}\epsilon_2.$$

□

Now, we prove the UH stability for the system (1).

Theorem 3.7. *Assume that the assumptions* $(\mathcal{A}1)$ *and* $(\mathcal{A}2)$ *holds. Then, the system* (1) *will be UH stable and consequently GUH stable.*

Proof. Let (x, y) be the solution of the system (2) and (v, k) be the solution of the proposed system (1). Using Cauchy–Schwarz inequality, Lemma 3.6 and It$\widehat{o}$ isometry, we get

$$\begin{aligned}\mathbb{E}\mathcal{D}^2_\infty\big(x(t), v(t)\big) &= \mathbb{E}\mathcal{D}^2_\infty\Big(x(t), M_{\gamma_1}(At^{\gamma_1})x_0 + \int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}} f_1\big(s, v(s), v(\lambda_1 s), k(s)\big)ds \\ &+ \int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}}\Big\langle \int_0^s g_1\big(u, v(u), v(\lambda_1 u), k(u)\big)dB(u)\Big\rangle ds\Big),\end{aligned}$$

$$\leq 2\mathbb{E}\mathcal{D}^2_\infty\Big(x(t), M_{\gamma_1}(At^{\gamma_1})x_0 + \int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}} f_1\big(s,x(s),x(\lambda_1 s),y(s)\big)ds$$
$$+ \int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}}\Big\langle \int_0^s g_1\big(u,x(u),x(\lambda_1 u),y(u)\big)dB(u)\Big\rangle ds\Big)$$
$$+ 2\mathbb{E}\mathcal{D}^2_\infty\Big(M_{\gamma_1}(At^{\gamma_1})x_0 + \int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}} f_1\big(s,x(s),x(\lambda_1 s),y(s)\big)ds$$
$$+ \int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}}\Big\langle \int_0^s g_1\big(u,x(u),x(\lambda_1 u),y(u)\big)dB(u)\Big\rangle ds, M_{\gamma_1}(At^{\gamma_1})x_0$$
$$+ \int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}} f_1\big(s,v(s),v(\lambda_1 s),k(s)\big)ds + \int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}}\Big\langle \int_0^s g_1\big(u,v(u),v(\lambda_1 u),k(u)\big)dB(u)\Big\rangle ds\Big),$$
$$\leq \frac{T^{2\gamma_1-1}\epsilon_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{2T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \mathbb{E}\mathcal{D}^2_\infty\Big(f_1\big(x(s),x(\lambda_1 s),y(s)\big), f_1\big(v(s),v(\lambda_1 s),k(s)\big)\Big)ds$$
$$+ \frac{2T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \Big(\int_0^s \mathbb{E}\Big\| g_1\big(x(s),x(\lambda_1 s),y(s)\big) - g_1\big(v(s),v(\lambda_1 s),k(s)\big)\Big\|^2 du\Big)ds,$$
$$\leq \frac{T^{2\gamma_1-1}\epsilon_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{2T^{2\gamma_1-1}L_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \Big(\mathbb{E}\mathcal{D}^2_\infty\big(x(s),v(s)\big) + \mathbb{E}\mathcal{D}^2_\infty\big(x(\lambda_1 s),v(\lambda_1 s)\big) + \mathbb{E}\mathcal{D}^2_\infty\big(y(s),k(s)\big)\Big)ds$$
$$+ \frac{2T^{2\gamma_1-1}N_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \Big[\int_0^s \Big(\mathbb{E}\mathcal{D}^2_\infty\big(x(u),v(u)\big) + \mathbb{E}\mathcal{D}^2_\infty\big(x(\lambda_1 u),v(\lambda_1 u)\big) + \mathbb{E}\mathcal{D}^2_\infty\big(y(u),k(u)\big)\Big)du\Big]ds,$$
$$\leq \frac{T^{2\gamma_1-1}\epsilon_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6T^{2\gamma_1}L_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\Big(\mathbb{E}\mathcal{D}^2_\infty(x,v) + \mathbb{E}\mathcal{D}^2_\infty(y,k)\Big) + \frac{6T^{2\gamma_1+1}N_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\Big(\mathbb{E}\mathcal{D}^2_\infty(x,v) + \mathbb{E}\mathcal{D}^2_\infty(y,k)\Big),$$
$$\leq \frac{T^{2\gamma_1-1}\epsilon_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \Big(\frac{6T^{2\gamma_1}L_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6T^{2\gamma_1+1}N_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\Big)\Big(\mathbb{E}\mathcal{D}^2_\infty(x,v) + \mathbb{E}\mathcal{D}^2_\infty(y,k)\Big),$$
$$\leq \frac{T^{2\gamma_1-1}\epsilon_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \xi_1\Big(\mathbb{E}\mathcal{D}^2_\infty(x,v) + \mathbb{E}\mathcal{D}^2_\infty(y,k)\Big), \text{where} \quad \xi_1 := \frac{6T^{2\gamma_1}L_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6T^{2\gamma_1+1}N_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}.$$

With a similar method, we get

$$\mathbb{E}\mathcal{D}^2_\infty\big(y(t),k(t)\big) \leq \frac{T^{2\gamma_2-1}\epsilon_2}{(2\gamma_2-1)(\Gamma(\gamma_2))^2} + \xi_2\Big(\mathbb{E}\mathcal{D}^2_\infty(x,v) + \mathbb{E}\mathcal{D}^2_\infty(y,k)\Big),$$

where $\xi_2 := \frac{6T^{2\gamma_2}L_2}{(2\gamma_2-1)(\Gamma(\gamma_2))^2} + \frac{6T^{2\gamma_2+1}N_2}{(2\gamma_2-1)(\Gamma(\gamma_2))^2}$.
Let $\epsilon = \max(\epsilon_1,\epsilon_2)$ and $\xi = \max(\xi_1,\xi_2)$, hence

$$\mathbb{E}\mathcal{D}^2_\infty\big((x,y)(t),(v,k)(t)\big) \leq R\epsilon + \xi\mathbb{E}\mathcal{D}^2_\infty\Big((x,y)(t),(v,k)(t)\Big),$$

where $R := \frac{T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{T^{2\gamma_2-1}}{(2\gamma_2-1)(\Gamma(\gamma_2))^2}$. Then

$$\mathbb{E}\mathcal{D}^2_\infty\big((x,y)(t),(v,k)(t)\big) \leq \frac{R}{1-\gamma}\epsilon := \omega\epsilon.$$

Hence the system (1) is UH stable. Therefore, if we put $\varphi(\epsilon) = \omega\epsilon$, we have $\varphi(0) = 0$ and $\mathbb{E}\mathcal{D}^2_\infty\big((x,y)(t),(v,k)(t)\big) \leq \varphi(\epsilon)$. Then, the system (1) is GUH stable. □

4. Conclusion

This research has examined a coupled system of fuzzy fractional pantograph stochastic differential equations. The fixed point technique is employed under Lipschitz conditions to demonstrate the existence and uniqueness of solution results. In addition, the Cauchy-Schwarz inequality, It$\widehat{o}$ isometry, and other methods are used to arrive at the stability result. Precisely, we have discussed two types of stability, called UH stability and GUH stability.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] F. Acharya, V. Kushawaha, J. Panchal, D. Chalishajar, Controllability of fuzzy solutions for neutral impulsive functional differential equations with nonlocal conditions, Axioms. 10 (2021), 84. https://doi.org/10.3390/axioms10020084.

[2] A.F. Jameel, A. Saaban, H. Ahadkulov, F.M. Alipiah, Approximate solution fuzzy pantograph equation by using homotopy perturbation method, J. Phys.: Conf. Ser. 890 (2017), 012023. https://doi.org/10.1088/1742-6596/890/1/012023.

[3] N. Mikaeilvand, L. Hossieni, The Taylor method for numerical solution of fuzzy generalized pantograph equations with linear functional argument, Int. J. Ind. Math. 2 (2010), 115-127.

[4] R. Hosseinzadeh, M. Zarebnia, Application and comparison of the two efficient algorithms to solve the pantograph Volterra fuzzy integro-differential equation, Soft Comput. 25 (2021), 6851–6863. https://doi.org/10.1007/s00500-021-05691-8.

[5] K. Agilan, V. Parthiban, (2020). Existence of solutions of fuzzy fractional panto-graph equations, Int. J. Math. Computer Sci. 15 (2020), 1117–1122.

[6] E. Arhrrabi, M. Elomari, S. Melliani, L.S. Chadli, Existence and stability of solutions of fuzzy fractional stochastic differential equations with fractional Brownian motions, Adv. Fuzzy Syst. 2021 (2021), 3948493. https://doi.org/10.1155/2021/3948493.

[7] S. Melliani, E. Arhrrabi, M. Elomari, L.S. Chadli, Ulam-Hyers-Rassias stability for fuzzy fractional integrodifferential equations under Caputo gH-differentiability, Int. J. Optim. Appl. 1 (2021), 51-55.

[8] L. Sajedi, N. Eghbali, H. Aydi, Impulsive coupled system of fractional differential equations with Caputo–Katugampola fuzzy fractional derivative, J. Math. 2021 (2021), 7275934. https://doi.org/10.1155/2021/7275934.

[9] T.M. Rassias, The problem of S. M. Ulam for approximately multiplicative mappings, J. Math. Anal. Appl. 246 (2000), 352–378. https://doi.org/10.1006/jmaa.2000.6788.

[10] T.M. Rassias, On the stability of functional equations in Banach spaces, J. Math. Anal. Appl. 251 (2000), 264–284. https://doi.org/10.1006/jmaa.2000.7046.

[11] T.M. Rassias, On the stability of functional equations and a problem of Ulam, Acta Appl. Math. 62 (2000), 23–130. https://doi.org/10.1023/a:1006499223572.

[12] T.M. Rassias, ed., Functional equations, inequalities and applications, Springer Dordrecht, 2003. https://doi.org/10.1007/978-94-017-0225-6.

[13] D. Chalishajar, A. Kumar, Existence, Uniqueness and Ulam's stability of solutions for a coupled system of fractional differential equations with integral boundary conditions, Mathematics. 6 (2018), 96. https://doi.org/10.3390/math6060096.

[14] T. Nabil, Ulam stabilities of nonlinear coupled system of fractional differential equations including generalized Caputo fractional derivative, AIMS Math. 6 (2021), 5088–5105. https://doi.org/10.3934/math.2021301.

[15] H. Khan, J.f. Gomez-Aguilar, T. Abdeljawad, A. Khan, Existence results and stability criteria for ABC-fuzzy-Volterra integro-differential equation, Fractals. 28 (2020), 2040048. https://doi.org/10.1142/s0218348x20400484.

[16] A. Ali, B. Samet, K. Shah, R.A. Khan, Existence and stability of solution to a toppled systems of differential equations of non-integer order, Bound Value Probl. 2017 (2017), 16. https://doi.org/10.1186/s13661-017-0749-1.

[17] H. Khan, Y. Li, W. Chen, D. Baleanu, A. Khan, Existence theorems and Hyers-Ulam stability for a coupled system of fractional differential equations with p-Laplacian operator, Bound Value Probl. 2017 (2017), 157. https://doi.org/10.1186/s13661-017-0878-6.

[18] T. Nabil, Existence results for nonlinear coupled system of integral equations of Urysohn Volterra-Chandrasekhar mixed type, Demonstr. Math. 53 (2020), 236–248. https://doi.org/10.1515/dema-2020-0017.

[19] Z. Ali, A. Zada, K. Shah, Ulam stability results for the solutions of nonlinear implicit fractional order differential equations, Hacettepe J. Math. Stat. 48 (2019), 1092-1109. https://doi.org/10.15672/hjms.2018.575.

[20] Y. Guo, X.B. Shu, Y. Li, F. Xu, The existence and Hyers–Ulam stability of solution for an impulsive Riemann–Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$. Bound. Value Probl. 2019 (2019), 59. https://doi.org/10.1186/s13661-019-1172-6.

[21] J. Wang, M. Feckan, Y. Zhou, Presentation of solutions of impulsive fractional Langevin equations and existence results, Eur. Phys. J. Spec. Top. 222 (2013), 1857–1874. https://doi.org/10.1140/epjst/e2013-01969-9.

[22] W. Hu, Q. Zhu, H.R. Karimi, Some improved Razumikhin stability criteria for impulsive stochastic delay differential systems, IEEE Trans. Automat. Contr. 64 (2019), 5207–5213. https://doi.org/10.1109/tac.2019.2911182.

[23] J. Priyadharsini, P. Balasubramaniam, Solvability of fuzzy fractional stochastic Pantograph differential system, Iran. J. Fuzzy Syst. 19 (2022), 47-60. https://doi.org/10.22111/ijfs.2022.6550.

[24] E. Arhrrabi, M. Elomari, S. Melliani, Averaging principle for fuzzy stochastic differential equations, Contrib. Math. 5 (2022), 25–29. https://doi.org/10.47443/cm.2022.003.

[25] E. Arhrrabi, A. Taqbibt, M. Elomari, S. Melliani, L. saadia Chadli, Fuzzy fractional boundary value problem, in: 2021 7th International Conference on Optimization and Applications (ICOA), IEEE, Wolfenbüttel, Germany, 2021: pp. 1–6. https://doi.org/10.1109/ICOA51614.2021.9442654.

[26] E. Arhrrabi, M. Elomari, S. Melliani, L.S. Chadli, Existence and uniqueness results of fuzzy fractional stochastic differential equations with impulsive, in: S. Melliani, O. Castillo (Eds.), Recent Advances in Fuzzy Sets Theory, Fractional Calculus, Dynamic Systems and Optimization, Springer International Publishing, Cham, 2023: pp. 147–163. https://doi.org/10.1007/978-3-031-12416-7_13.