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ON SIMPLE POLYNOMIAL BOUNDS FOR THE EXPONENTIAL FUNCTION

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Abstract. In this article, we offer a new polynomial or polynomial-exponential bounds for the exponential function. Its main interest is to be both simple and sharp, under some clear conditions on the parameters involved. Applications are given for a probability function and the Kummer beta function.

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1. Introduction

The natural exponential functions are extremely important in many branches of science and mathematics. Sometimes we require the bounds of such a function on the interval $[0, 1]$ for a specific purpose. One obvious upper bound is given in the following inequality: For any $x \in [0, 1)$,

$$e^x \leq \frac{1}{1-x}. \tag{1}$$

The inequality (1) is coarser and its refinement is given in [3]. For some other sharp bounds, we refer to [1,3] and the references therein. The bounds in the present literature are somewhat complex in nature; there is still a need for tractable and sharp bounds in all branches of

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mathematics. The main goal of this paper is to achieve useful and simpler polynomial or polynomial-exponential bounds for exponential functions than those available in the literature.

2. Main Result

The following result presents the main finding of the study.

Proposition 1. *For any* $x \in [0,1]$ *and* $a \in \mathbb{R}$, *we have*

$$\operatorname{sign}(a)e^{ax} \leq \operatorname{sign}(a)\left[ax(1-x) + x^2(e^a - 1) + 1\right],$$

where $\operatorname{sign}(a) = -1$ *if* $a < 0$, $\operatorname{sign}(0) = 0$ *and* $\operatorname{sign}(a) = 1$ *for* $a > 0$.

Proof. We propose a proof based on the analysis of appropriate functions. To begin, let us consider the following function:

$$f(x;a) = e^{ax} - ax(1-x) - x^2(e^a - 1) - 1. \tag{2}$$

Then, upon differentiation, we have

$$f'(x;a) = ae^{ax} + 2(a - e^a + 1)x - a.$$

The equation $f'(x;a) = 0$ has two solutions only into $[0,1]$, which are $x_0 = 0$ and $x_1 = [y_a - W(y_a e^{y_a})]/a$, where $y_a = a^2/[2(a+1-e^a)]$ and $W(x)$ is the productlog function defined by $W(x)e^{W(x)} = x$. Clearly, the nature of x_0 and x_1 are informative on the possible sign of $f(x;a)$. Since $f(x;a)$ is continuous with $f(0;a) = 0$ and $f(1;a) = 0$, it is enough to study the nature of the extremum x_0; if x_0 a local minimum, then x_1 is a local maximum, and vice versa.

First, let us notice that

$$f''(x;a) = a^2 e^{ax} + 2(a - e^a + 1),$$

which implies that

$$f''(0;a) = a^2 + 2(a - e^a + 1) = 2\phi(a),$$

where

$$\phi(a) = 1 + a + \frac{a^2}{2} - e^a.$$

Since $e^a > 1 + a$ for any $a \in \mathbb{R}^*$, we have $\phi'(a) = 1 + a - e^a < 0$, which implies that $\phi(a)$ is decreasing for any $a \in \mathbb{R}$. Let us now study the nature of the extremum $x = 0$ according to $a > 0$ and $a < 0$ via the second derivative test.

- For $a > 0$, we have $\phi(a) < \phi(0) = 0$, implying that $f''(x_0; a) = f''(0; a) < 0$. Thus, x_0 is a local maximum point, implying that x_1 is necessary a local minimum point: For any $x \in [0, 1]$ and $a > 0$, we have

$$f(x; a) \leq \min(f(0; a), f(1; a)) = 0,$$

which implies the desired inequality.

- For $a < 0$, we have $\phi(a) > \phi(0) = 0$, implying that $f''(x_0; a) = f''(0; a) > 0$. Thus, x_0 is a local minimum point, implying that x_1 is necessary a local maximum point: For any $x \in [0, 1]$ and $a < 0$, we have

$$f(x; a) \geq \max(f(0; a), f(1; a)) = 0,$$

which implies the desired inequality.

This ends the proof. □

Remark 1. *For the case $a > 0$, some alternative proofs can be given. By the series expansion of the exponential function, since $x \in [0, 1]$, we have*

$$e^{ax} - 1 = \sum_{k=1}^{+\infty} \frac{(ax)^k}{k!} = ax + \sum_{k=2}^{+\infty} \frac{(ax)^k}{k!} \leq ax + x^2 \sum_{k=2}^{+\infty} \frac{a^k}{k!}$$
$$= ax + x^2(e^a - 1 - a) = ax(1 - x) + x^2(e^a - 1).$$

Remark 2. *For the case $a < 0$ and $x \in [0, 1]$, the proposed lower bound improves the famous inequality $e^{ax} \geq 1 + ax$. Indeed, by using $e^a \geq 1 + a$, we have*

$$e^{ax} \geq ax(1 - x) + x^2(e^a - 1) + 1 = 1 + ax + x^2(e^a - 1 - a) \geq 1 + ax.$$

Remark 3. *Proposition 1 can be extended to any bounded interval of the form $[0, c]$ for x, with $c > 0$. In this case, it is enough to replace x by x/c. That is, for any $x \in [0, c]$, since $x/c \in [0, 1]$, we have*

$$\text{sign}(a)e^{ax/c} \leq \text{sign}(a)\frac{1}{c^2}\left[ax(c - x) + x^2(e^a - 1) + c^2\right],$$

Remark 4. *For $x \in [0, 1]$ and $a = x$ or $a = -x$, Proposition 1 yields simple polynomial bounds for e^{x^2} and e^{-x^2}, respectively. More precisely, we have:*

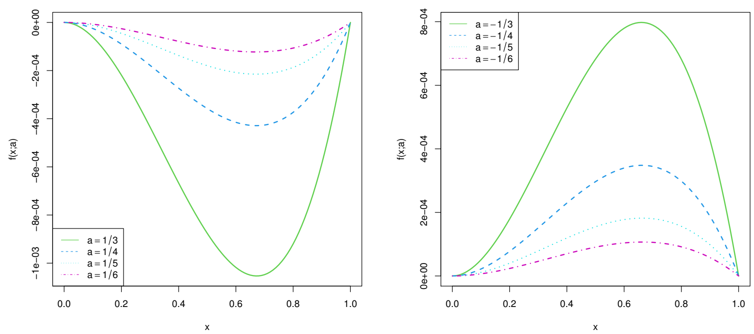
$$e^{x^2} \leq x^2(1 - x) + x^2(e^x - 1) + 1$$

and

$$-x^2(1 - x) + x^2(e^{-x} - 1) + 1 \leq e^{-x^2}.$$

The functions e^{x^2} and e^{-x^2} are involved in a plethora of mathematical and physical quantities, more or less complex. Our bounds can be of interest for direct bounds of these quantities. See, for instance, [2].

A graphical illustration of Proposition 1 is given in Figure 1. It shows the curve of the function $f(x;a)$ defined by (2) for some values of $a > 0$ and $a < 0$.



(a) (b)

FIGURE 1. Curves of $f(x;a)$ as defined by (2) for $x \in [0,1]$ and (a) some values of $a > 0$ and (b) some values of $a < 0$.

In Figure 1, we clearly identify the extrema x_0 and x_1 as described in the proof of Proposition 1. Also, the sharpness of the obtained bounds can be observed; the maximum of magnitude being between 10^{-4} and 10^{-3} for the considered values of a.

3. Applications

Two applications of Proposition 1 are examined in this section.

3.1. **Bound of a useful probability.** Let X be a random variable with the standard normal distribution, i.e., with probability density function $f(x) = 1/(2\pi)^{-1/2}e^{-x^2/2}$, $x \in \mathbb{R}$. Then, the following proposition gives an evaluation of the probability that the event $\{0 \leq X \leq t\}$ occurs with $t \in [0,1]$.

Proposition 2. *Let X be a random variable with the standard normal distribution. For any $t \in [0,1]$, we have*

$$P(0 \leq X \leq t) \geq \frac{1}{\sqrt{2\pi}}\left[-\frac{3}{2}t^3 + \frac{1}{8}t^4 + 8\gamma\left(3, \frac{t}{2}\right) + t\right],$$

where $\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt$ is the standard incomplete gamma function.

Proof. By applying Proposition 1 with $x \in [0,1]$ and $a = -x/2$, we have

$$e^{-x^2/2} \geq -\frac{1}{2}x^2(1-x) + x^2(e^{-x/2} - 1) + 1,$$

which implies that

$$\begin{aligned} P(0 \leq X \leq t) &\geq \frac{1}{\sqrt{2\pi}}\left(-\frac{1}{2}\int_0^t x^2 dx + \frac{1}{2}\int_0^t x^3 dx + \int_0^t x^2 e^{-x/2} dx - \int_0^t x^2 dx + t\right) \\ &= \frac{1}{\sqrt{2\pi}}\left[-\frac{3}{2}t^3 + \frac{1}{8}t^4 + 8\gamma\left(3, \frac{t}{2}\right) + t\right]. \end{aligned}$$

The proof of Proposition 2 ends. □

In Proposition 2, it is intriguing to see how polynomial and gamma functions appear to bound a probability function.

Figure 2 illustrates the lower bound in Proposition 2 by showing the curve of the function

$$F(x) = P(0 \leq X \leq x) - \frac{1}{\sqrt{2\pi}}\left[-\frac{3}{2}x^3 + \frac{1}{8}x^4 + 8\gamma\left(3, \frac{x}{2}\right) + x\right]$$

for $x \in [0,1]$.

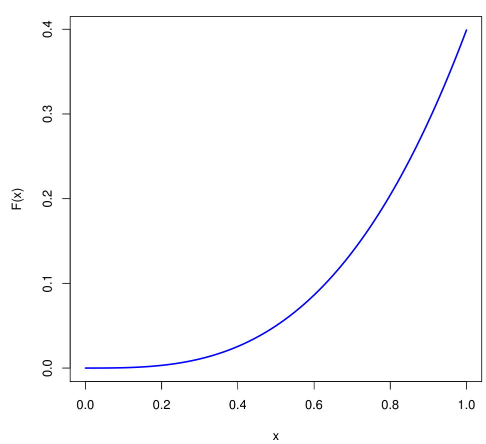


FIGURE 2. Curve for $F(x)$ for $x \in [0,1]$.

In Figure 2, we see that the bound is very sharp for $x \in [0, 0.6]$. However, we do not claim that it is the "sharpest lower bound ever" for $P(0 \leq X \leq t)$, but just an interesting application of our main result.

Remark 5. *Proposition 2 can be used to bound the cumulative distribution function of X; Since, for $t \in [0,1]$, $P(X \leq t) = 1/2 + P(0 \leq X \leq t)$, we have*

$$P(X \leq t) \geq \frac{1}{\sqrt{2\pi}}\left[-\frac{3}{2}t^3 + \frac{1}{8}t^4 + 8\gamma\left(3, \frac{t}{2}\right) + t\right] - \frac{1}{2},$$

3.2. **Bound of the Kummer beta function.** Proposition 1 can be used for approximation purposes. For instance, let us consider the Kummer beta function defined by

$$\mathcal{I}(a, \alpha, \beta) = \int_0^1 e^{ax} x^{\alpha-1}(1-x)^{\beta-1} dx, \tag{3}$$

with $a \in \mathbb{R}$, $\alpha > 0$ and $\beta > 0$. This function has found numerous applications in probability and statistics. See [4–7] in this regard, and it remained complicated to evaluate with simple functions. Thanks to Proposition 1, the following result can be proved.

Proposition 3. *Let $\mathcal{I}(a, \alpha, \beta)$ be the Kummer beta function as defined by (3). For any $a \in \mathbb{R}$, we have*

$$\text{sign}(a)\mathcal{I}(a, \alpha, \beta) \leq \text{sign}(a)\left[aB(\alpha+1, \beta+1) + (e^a - 1)B(\alpha+2, \beta) + B(\alpha, \beta)\right],$$

where $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx$ is the standard beta function.

Proof. The proof is a direct application of Proposition 1 and basic integral properties:

- For $a > 0$, by applying Proposition 1, we get

  $$e^{ax} \leq ax(1-x) + x^2(e^a - 1) + 1,$$

  which implies that, after some developments,

  $$\mathcal{I}(a, \alpha, \beta) \leq aB(\alpha+1, \beta+1) + (e^a - 1)B(\alpha+2, \beta) + B(\alpha, \beta).$$

- For $a < 0$, by also applying Proposition 1, we get the reverse inequality:

  $$\mathcal{I}(a, \alpha, \beta) \geq aB(\alpha+1, \beta+1) + (e^a - 1)B(\alpha+2, \beta) + B(\alpha, \beta),$$

The proof of Proposition 3 is complete. □

Thus, thanks to Proposition 3, the standard beta function can be used to evaluate quite precisely the Kummer beta function.

Other applications can be given; this study just opens a door for more in this direction.

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