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ON THE WEAK COMPACTNESS OF LIMITED COMPLETELY CONTINUOUS OPERATORS

ABDELMONAIM ELKADDOURI[1], JAWAD H'MICHANE[2,*]

[1]Department of Mathematical Sciences and Decision Support (SMAD), National School of Applied Sciences, Abdelmalek Essaadi University, BP: 2222 M'hannech, Tétouan, Morocco

[2]Moulay Ismail University, Faculty of Sciences, Department of Mathematics, B.P. 11201 Zitoune, Meknes, Morocco

*Corresponding author: hm1982jad@gmail.com

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Abstract. We characterize Banach lattices on which each limited completely continuous operator is weakly compact. As a consequence, we investigate some new characterizations of order continuous Banach lattices.

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1. Introduction

The class of limited completely continuous operators was introduced and studied by M. Salimi and S. M. Moshtaghioun in [6] and several interesting characterizations were given in [[7], [3]]. Also, the duality property for this class of operators is studied in [4].

Recall from [6] that an operator T from a Banach space X into another Banach space Y is said to be Limited completely continuous (abb. lcc) if it carries limited subsets of X to relatively compact subsets of Y. Note that every weakly compact operator is lcc (see Corollary 2.5 [6]), however the converse is not true in general. Indeed, the identity operator of the Banach lattice c_0 is lcc but it is not weakly compact.

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In this paper, we will focus to give characterizations of Banach lattices under which the converse of the previous fact stays true.

2. PRELIMINARIES

To state our results, we need to fix some notations and recall some definitions.

- A subset A of a Banach space X is called limited (resp., Dunford-Pettis (abb. DP)) if every weak* null (resp., weak null) sequence (f_n) in X' converges uniformly on A, that is, $\sup_{x\in A}|f_n(x)| \longrightarrow 0$. We note that every relatively compact subset of X is limited and clearly every limited set is DP, but the converse of these assertions, in general, are false.
- A Banach space X is said to have the Gelfand-Phillips property (abb. GP) if every limited subset of X is relatively compact.
- A subset A of the topological dual X' of a Banach space X is called L-limited if every limited weakly null sequence (x_n) of X converges uniformly in A, that is, $\sup_{f\in A}|f(x_n)| \longrightarrow 0$. Note that every relatively weakly compact subset of a dual Banach space X' is L-limited, but the converse is not true in general. In fact, the unit ball B_{ℓ^1} of the Banach space ℓ^1 is an L-limited set but it is not relatively weakly compact.
- Recall from [7] that a Banach space X is said to have the L-limited property if every L-limited set in X' is relatively weakly compact.
- A Banach space X has the Dunford-Pettis* property (abb. DP*) if every relatively weakly compact subset of X is limited, equivalently $f_n(x_n) \longrightarrow 0$ for every weakly null sequence (x_n) of X and every weak* null sequence (f_n) of X'.
- A norm bounded subset A of a Banach lattice E is called L-weakly compact if $\|y_n\| \longrightarrow 0$ for every disjoint sequence (y_n) contained in $Sol(A)$ [[5], Definition 3.6.1]. Every L-weakly compact set is relatively weakly compact, but the converse does not holds in general.
- We recall also from [5] that an operator T from a Banach lattice E into a Banach space X is called M-weakly compact if for each disjoint sequence (x_n) of B_E, we have $\|T(x_n)\| \longrightarrow 0$.

We denote by B_X the closed unit ball of X. The positive cone of E will be denoted by E^+. A Banach lattice is a Banach space $(E, \|.\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. A Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. Also, the solid hull of a set A is the smallest solid set including A and is exactly the set $Sol(A) := \{x \in E : \exists y \in A \text{ with } |x| \leq |y|\}$.

We will use the term operator $T : E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. The operator T is regular if $T = T_1 - T_2$, where T_1 and T_2 are positive operators from E to F. Note that each positive linear mapping on a Banach lattice is continuous. If an operator $T : E \longrightarrow F$ between two Banach lattices is positive, then its adjoint $T' : F' \longrightarrow E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$. For terminologies concerning Banach lattice theory and positive operators we refer the reader to the book of Aliprantis-Burkinshaw [1].

3. Main result

Proposition 3.1. *Let X be a Banach space, for a norm bounded subset A of X' the following statements are equivalent:*

(1) *A is L-limited.*
(2) *For each sequence (f_n) of A, we have $f_n(x_n) \longrightarrow 0$ for every sequence (x_n) of X which is weakly null and limited.*

Proof. (1) $\Rightarrow$ (2) Let (f_n) be a sequence of A and let (x_n) be a limited weakly null sequence of X. Since A is a limited set of X', then $\sup_{f\in A} |f(x_n)| \longrightarrow 0$, and by the inequality $|f_n(x_n)| \leq \sup_{f\in A} |f(x_n)|$ the proof is done.
(2) $\Rightarrow$ (1) Assume by way of contradiction that A is not an L-limited set of X. Then, there exists a limited weakly null sequence (x_n) of X such that $\sup_{f\in A} |f(x_n)| > \epsilon$ for some $\epsilon > 0$ and for all $n \in \mathbb{N}$. Hence, for every $n \in \mathbb{N}$ there exists some f_n in A such that $|f_n(x_n)| \geq \epsilon$, which is impossible. Therefore, A is an L-limited set of X'. □

As an immediate consequence, we obtain the following characterization of L-limited sets.

Corollary 3.2. *Let X be a Banach space, for a norm bounded sequence (f_n) of X' the following statements are equivalent:*

(1) *The subset $\{f_n; n \in \mathbb{N}\}$ is an L-limited set.*
(2) *$f_n(x_n) \longrightarrow 0$ for every limited weakly null sequence (x_n) of X.*

Now, we are in position to give our first major result.

Theorem 3.3. *For a Banach lattice E with the DP* property, the following statements are equivalent:*

(1) *Each lcc operator from E into an arbitrary Banach space X is weakly compact.*

(2) *E' is order continuous.*

(3) *Each L-limited set in E' is L-weakly compact.*

(4) *E has the L-limited property.*

Proof. $(1) \Rightarrow (2)$ Let T be an operator from E to the Banach lattice ℓ^1. Since ℓ^1 has the GP-property, then it follows from Theorem 2.2 [6] that T is an lcc operator, and so by our hypothesis T is weakly compact. By part (2) of Theorem 5.29 [1] the Banach lattice E' is order continuous.
$(2) \Rightarrow (3)$ Let A be an L-limited subset of E'; for each $x \in E$ we consider

$$\rho_A(x) = \sup\{|f|(|x|) : f \in E'\} = \sup\{f(y) : f \in E'; |y| \leq |x|\}.$$

Since A is norm bounded then $\rho_A(x) \in \mathbb{R}$, and it is clear that ρ_A is a semi-norm of the Banach lattice E. On the other hand, let x_n be a disjoint sequence of B_E and let $\epsilon > 0$, then for all n we can choose some $f_n \in A$ and $|y_n| \leq |x_n|$ with $\rho_A(x_n) < \epsilon + f_n(y_n)$. Since E' is order continuous and (y_n) is a norm bounded disjoint sequence (because $|y_n| \leq |x_n|$ and (x_n) is a disjoint sequence of E), it follows from Theorem 2.4.14 [5] and the DP* property of E that the sequence (x_n) is limited and weakly convergent to 0. Moreover, since A is an L-limited subset of E', then $f_n(y_n) \longrightarrow 0$, and hence $\limsup \rho_A(x_n) < \epsilon$ for each $\epsilon > 0$. So, $\lim \rho_A(x_n) \longrightarrow 0$, and it follows from Proposition 3.6.3 [5] that A is L-weakly compact.
$(3) \Rightarrow (4)$ Follows from Proposition 3.6.5 [5].
$(4) \Rightarrow (1)$ Let $T : E \longrightarrow X$ be a lcc operator, then it is clear that $T'(B_{X'})$ is an L-limited subset of E. So, by our hypothesis $T'(B_{X'})$ is relatively weakly compact, and hence T' is weakly compact. Finally, from the Gantmacher theorem we conclude that T is weakly compact. □

In the following theorem, we characterize Banach lattices under which each lcc operator is M-weakly compact.

Theorem 3.4. *Let E and F be two Banach lattices such that E has the DP* property. Then, the following assertions are equivalent:*

(1) *Each lcc operator $T : E \longrightarrow F$ is M-weakly compact.*

(2) *One of the following statements is holds:*

   (a) *E' is order continuous;*

(b) $F = \{0\}$.

Proof. (1) $\Rightarrow$ (2) Assume that the assertion (2) is false which means that E is not order continuous and $F \neq \{0\}$. By evoking Theorem 2.4.14 [5] and Proposition 2.3.11 [5] it follows that E contains a closed sublattice isomorphic to ℓ^1, and hence there exists a projection $P : E \longrightarrow \ell^1$. On the other hand, since $F \neq \{0\}$ there exists a non-null element $y \in F$. Now, we consider the operator $S : \ell^1 \longrightarrow F$ defined by:

$$S((\lambda_n)) = \left(\sum_{n=1}^{+\infty} \lambda_n\right) y \quad \text{for each} \quad (\lambda_n) \in \ell^1.$$

It is clear that S is well defined. Also S is lcc (because ℓ^1 has the GP-property), hence the operator $T = S \circ P : E \longrightarrow \ell^1 \longrightarrow F$ is lcc but it is not M-weakly compact. Indeed, if we design by (e_n) the canonical basis of $\ell^1 \subset E$, the sequence (e_n) is disjoint and bounded in E, moreover we have $T(e_n) = y$ for each $n \geq 1$; therefore the sequence $(T(e_n))$ is not converging to zero, and hence the operator T is not M-weakly compact.

$(2; a) \Rightarrow (1)$ Let $T : E \longrightarrow F$ be an $lcc-$operator and let (x_n) be a norm bounded disjoint sequence of E. Since E' is order continuous and E has DP* property, it follows from Corollary 2.9 [2] that (x_n) is limited and weakly null in E. As T is lcc, we have $\|T(x_n)\| \longrightarrow 0$, and hence the operator T is M-weakly compact.

$(2; b) \Rightarrow (1)$ In this case we have $T = 0$, and hence the operator T is M-weakly compact. □

As a consequence of the above theorem, we have the following characterization.

Corollary 3.5. *Let E be a Banach lattice with the DP* property and F be a non trivial Banach lattice, then the following assertions are equivalent:*

(1) *Each lcc operator $T : E \longrightarrow F$ is M-weakly compact.*

(2) *E' is order continuous.*

The following result present our second major result.

Theorem 3.6. *Let E and F be two Banach lattices such that E has the DP* property, then the following assertions are equivalent:*

(1) *Each lcc operator $T : E \longrightarrow F$ is weakly compact.*

(2) *One of the following statements is holds:*

  (a) *E' is order continuous;*

  (b) *F is reflexive.*

Proof. $(1) \Rightarrow (2)$ Assume that E' is not order continuous, by Theorem 2.4.14 [5] and Proposition 2.3.11 [5] it follows that E contains a closed sublattice isomorphic to ℓ^1, and hence there exists a projection $P : E \longrightarrow \ell^1$. To finish the proof, we have to show that F is reflexive. By the Eberlein-Smulian's Theorem, it suffices to show that every sequence (x_n) in the closed unit ball of F had a subsequence which converges weakly to an element of F.
We consider the operator $S : \ell^1 \longrightarrow F$ defined by

$$S((\lambda_i)) = \sum_{i=1}^{+\infty} \lambda_i x_i \quad \text{for each} \quad (\lambda_i) \in \ell^1,$$

the composed operator $T = S \circ P : E \longrightarrow \ell^1 \longrightarrow F$ is lcc, and hence by our hypothesis T is weakly compact. If we note by (e_n) the sequence with all term zero and the n'th equals 1, then the sequence $(x_n) = T(e_n)$ has a subsequence which converges weakly in F. This ends the proof.
$(2; a) \Rightarrow (1)$ Let $T : E \longrightarrow F$ be an lcc operator. Since E' is order continuous and E has the GP-property, it follows from the Theorem 3.4 that T is M-weakly compact, and so T is weakly compact.
$(2; b) \Rightarrow (1)$ In this case, each operator from E into F is weakly compact. □

As a consequence of the above theorem, we have the following characterization.

Corollary 3.7. *Let E be a Banach lattice such that E has the DP* property and let F be a non reflexive Banach lattice; then the following assertions are equivalent:*

(1) *Each lcc operator $T : E \longrightarrow F$ is weakly compact.*
(2) *E' is order continuous.*

An other consequence is given by:

Corollary 3.8. *Let T be an operator from a Banach lattice E into a Banach lattice F such that E has the DP* property and E' is order continuous; then the following assertion are equivalent:*

(1) *T is lcc.*
(2) *T is M-weakly compact.*
(3) *T is weakly compact.*

Proof. $(1) \Rightarrow (2)$ Follows from the Theorem 3.4.
$(2) \Rightarrow (3)$ Obvious.
$(3) \Rightarrow (1)$ Follows from Corollary 2.5 [6]. □

References

[1] C.D. Aliprantis, O. Burkinshaw, Positive operators. Reprint of the 1985 original. Springer, Dordrecht, 2006.

[2] P.G. Dodds, D.H. Fremlin, Compact operators in Banach lattices, Israel J. Math. 34 (1979), 287–320. https://doi.org/10.1007/BF02760610.

[3] A. El Kaddouri, J. H'michane, K. Bouras, M. Moussa, On the class of weak* Dunford–Pettis operators, Rend. Circ. Mat. Palermo. 62 (2013), 261–265. https://doi.org/10.1007/s12215-013-0122-x.

[4] J. H'Michane, A. El Kaddouri, K. Bouras, M. Moussa, Duality problem for the class of limited completely continuous operators, Oper. Matrices. 8 (2014) 593–599. https://doi.org/10.7153/oam-08-31.

[5] P. Meyer-Nieberg, Banach lattices. Universitext. Springer-Verlag, Berlin, 1991.

[6] M. Salimi, S.M. Moshtaghioun, The Gelfand–Phillips property in closed subspaces of some operator spaces, Banach J. Math. Anal. 5 (2011), 84–92. https://doi.org/10.15352/bjma/1313363004.

[7] M. Salimi, S.M. Moshtaghioun, A new class of Banach spaces and its relation with some geometric properties of Banach spaces, Abstr. Appl. Anal. 2012 (2012), 212957. https://doi.org/10.1155/2012/212957.