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ASYMPTOTIC BEHAVIOR OF LINEAR ADVANCED DIFFERENCE EQUATIONS

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Abstract. In this article, we study the asymptotic behavior of a linear advanced difference equation. The idea used here is to construct appropriate mapping. Then we apply the Banach fixed point theorem to obtain sufficient conditions satisfying stability, asymptotic stability and exponential stability of the considered equation.

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1. Introduction

In mathematics, advanced differential equations (ADEs) are a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at future times. ADEs are of practical importance, which model many phenomenon in population dynamics, economics problems, or mechanical control engineering (see [11, 13] for details).

Many authors took this practical importance and studied the qualitative properties oh these equations. For examples, Chatzarakis and Stavroulakis in [6] studied the oscillations of first order linear difference equation with general advanced argument. Dannan and Elaydi in [7]

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are obtained necessary and sufficient conditions for the asymptotic stability of linear difference equations of advanced type. The reader can see more of this research here ([2,3,8–10]).

Despite the great importance of this type of equations, studies on this subject are very scarce. So, in this paper, we consider the linear advanced difference equation

$$\Delta x(n) + \sum_{k=1}^{N} a_k(n)\, x(n + h_k(n)) = 0,\ n \geq n_0, \tag{1}$$

where Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$ for any sequence $\{x(n),\ x(n_0) = x_0,\ n \in \mathbb{N}\}$, $\{a_k(n),\ n \in \mathbb{N},\ 1 \leq k \leq N\}$ are sequences of nonnegative real numbers such that $\sum_{k=1}^{N} a_k(n) \neq 1$, and $\{h_k(n),\ n \in \mathbb{N},\ 1 \leq k \leq N\}$ are sequences of nonnegative integers such that $n + h_k(n) \to \infty$ as $n \to \infty$.

Equation (1) can be viewed as a discrete analogue of the linear advanced differential equation

$$x'(t) + \sum_{k=1}^{N} a_k(t)\, x(t + h_k(t)) = 0,\ t \geq t_0 \geq 0, \tag{2}$$

where $a_k, h_k : [t_0, +\infty) \to \mathbb{R}$, $1 \leq k \leq N$ are continuous functions, with $h_k(t)$, $1 \leq k \leq N$ are positives. Dung in [12] studied the asymptotic behaviors of (2).

The fixed point theorem is considered one of the best and most effective methods in the study of stability, compared to the Lyapunov method for example, because the search for the Lyapunov function is a big problem for researchers (see [1,2,4,5,14–17]). Therefore, we will use in this paper the Banach fixed point theorem to study the stability of the equation (1).

2. Preliminaries

To facilitate the study of our equation (1) we take $N = 2$. So, we have the particular equation

$$\Delta x(n) + a(n)\, x(n + h(n)) + b(n)\, x(n + r(n)) = 0,\ n \geq n_0. \tag{3}$$

Using the relation

$$x(m) - x(n) = \sum_{s=n}^{m-1} \Delta x(s),$$

we can rewrite (3) as follows

$$\Delta x(n) = -(a(n) + b(n))\, x(n) - a(n) \sum_{s=n}^{n+h(n)-1} \Delta x(s) - b(n) \sum_{s=n}^{n+r(n)-1} \Delta x(s).$$

So,

$$\begin{aligned}\Delta x(n) &= -(a(n)+b(n))\,x(n) \\ &\quad + a(n)\sum_{s=n}^{n+h(n)-1}\left(a(s)\,x(s+h(s))+b(s)\,x(s+r(s))\right) \\ &\quad + b(n)\sum_{s=n}^{n+r(n)-1}\left(a(s)\,x(s+h(s))+b(s)\,x(s+r(s))\right).\end{aligned}$$

If we let $c(n)=1-(a(n)+b(n))$ and $z(s)=a(s)\,x(s+h(s))+b(s)\,x(s+r(s))$, we get

$$x(n+1)=c(n)\,x(n)+a(n)\sum_{s=n}^{n+h(n)-1} z(s)+b(n)\sum_{s=n}^{n+r(n)-1} z(s). \tag{4}$$

In the following lemma, we transform the equation (3) into a new equation that will facilitate us to apply our techniques.

Lemma 1. *Suppose that $c(n)\neq 0$ for all $n\in\mathbb{N}$. Then x is a solution of the equation (3) if and only if*

$$\begin{aligned}x(n) &= x_0\prod_{u=n_0}^{n-1} c(u)+\sum_{s=n_0}^{n-1}\prod_{u=s+1}^{n-1} c(u)\left(a(s)\sum_{m=s}^{s+h(s)-1} z_x(m)\right) \\ &\quad +\sum_{s=n_0}^{n-1}\prod_{u=s+1}^{n-1} c(u)\left(b(s)\sum_{m=s}^{s+r(s)-1} z_x(m)\right),\end{aligned} \tag{5}$$

where $c(n)=1-(a(n)+b(n))$ and $z_x(n)=a(n)\,x(n+h(n))+b(n)\,x(n+r(n))$.

Proof. By multiplying both sides of (4) by $\prod_{s=n_0}^{n-1} c^{-1}(s)$ and by summing from n_0 to $n-1$ we obtain

$$\begin{aligned}&\sum_{s=n_0}^{n-1}\Delta\left[\prod_{u=n_0}^{s-1} c^{-1}(u)\,x(s)\right] \\ &= \sum_{s=n_0}^{n-1}\prod_{u=n_0}^{s} c^{-1}(u)\left(a(s)\sum_{m=s}^{s+h(s)-1} z_x(m)\right) \\ &\quad + \sum_{s=n_0}^{n-1}\prod_{u=n_0}^{s} c^{-1}(u)\left(b(s)\sum_{m=s}^{s+r(s)-1} z_x(m)\right).\end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}&\prod_{u=n_0}^{n-1} c^{-1}(u)\, x(n) - \prod_{u=n_0}^{n_0-1} c^{-1}(u)\, x(n_0)\\&= \sum_{s=n_0}^{n-1} \prod_{u=n_0}^{s} c^{-1}(u) \left(a(s) \sum_{m=s}^{s+h(s)-1} z_x(m) \right)\\&+ \sum_{s=n_0}^{n-1} \prod_{u=n_0}^{s} c^{-1}(u) \left(b(s) \sum_{m=s}^{s+r(s)-1} z_x(m) \right).\end{aligned}$$

By dividing both sides of the above expression by $\prod_{u=n_0}^{n-1} c^{-1}(u)$ we get (5) □

3. Main results

Let $(\mathcal{X}, \|\cdot\|)$ be the Banach space of bounded sequences $x : n \in \mathbb{N} \to \mathbb{R}$ with the maximum norm $\|\cdot\|$.

Definition 1. *The zero solution of (3) is Lyapunov stable if the zero solution of (3) is asymptotically stable if for any $\epsilon > 0$ and any integer $n_0 \geq 0$ there exists a $\delta > 0$ such that $|x_0| \leq \delta$ implies $|x(n, n_0, x_0)| \leq \epsilon$ for $n \in \mathbb{N}$.*

Theorem 1. *Suppose there exist a positive constant M and a constant $\alpha \in (0,1)$ such that for $n \in n \in \mathbb{N}$*

$$\left| \prod_{u=n_0}^{n-1} c(u) \right| \leq M, \tag{6}$$

and

$$\begin{aligned}&\sum_{s=n_0}^{n-1} \left| \prod_{u=s+1}^{n-1} c(u) \right| \left(|a(s)| \sum_{m=s}^{s+h(s)-1} |a(m) + b(m)| \right)\\&+ \sum_{s=n_0}^{n-1} \left| \prod_{u=s+1}^{n-1} c(u) \right| \left(|b(s)| \sum_{m=s}^{s+r(s)-1} |a(m) + b(m)| \right)\\&\leq \alpha.\end{aligned} \tag{7}$$

Then the zero solution of (3) is stable.

Proof. Let $\epsilon > 0$ be given. Choose $\delta > 0$ such that for $|x(n_0)| \leq \delta$, we have

$$\delta M + \alpha\epsilon \leq \epsilon.$$

Define

$$\mathcal{S}_\epsilon = \{x \in \mathcal{X} : \|x\| \leq \epsilon\}.$$

Then $(\mathcal{S}_\epsilon, \|\cdot\|)$ is a complete metric space where $\|\cdot\|$ is the maximum norm.

By Lemma 1, we define the operator $\mathcal{H} : \mathcal{S}_\epsilon \to \mathcal{X}$ by

$$(\mathcal{H}x)(n) = x_0 \prod_{u=n_0}^{n-1} c(u) + \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} c(u) \left(a(s) \sum_{m=s}^{s+h(s)-1} z_x(m) \right)$$

$$+ \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} c(u) \left(b(s) \sum_{m=s}^{s+r(s)-1} z_x(m) \right), \tag{8}$$

for $n \in n \in \mathbb{N}$. We first show that $\mathcal{H}$ maps $\mathcal{S}_\epsilon$ into $\mathcal{S}_\epsilon$. So by (6) and (7)

$$\begin{aligned} |(\mathcal{H}x)(n)| &\leq M\delta + \sum_{s=n_0}^{n-1} \left| \prod_{u=s+1}^{n-1} c(u) \right| \left(|a(s)| \sum_{m=s}^{s+h(s)-1} |z_x(m)| \right) \\ &+ \sum_{s=n_0}^{n-1} \left| \prod_{u=s+1}^{n-1} c(u) \right| \left(|b(s)| \sum_{m=s}^{s+r(s)-1} |z_x(m)| \right) \\ &\leq M\delta + \alpha\epsilon \leq \epsilon. \end{aligned}$$

Thus $\mathcal{H}$ maps $\mathcal{S}_\epsilon$ into itself. We next show that $\mathcal{H}$ is a contraction.

Let $x, y \in \mathcal{S}_\epsilon$, then

$$\begin{aligned} &|(\mathcal{H}x)(n) - (\mathcal{H}y)(n)| \\ &\leq \sum_{s=n_0}^{n-1} \left| \prod_{u=s+1}^{n-1} c(u) \right| \left(|a(s)| \sum_{m=s}^{s+h(s)-1} |z_x(m) - z_y(m)| \right) \\ &+ \sum_{s=n_0}^{n-1} \left| \prod_{u=s+1}^{n-1} c(u) \right| \left(|b(s)| \sum_{m=s}^{s+r(s)-1} |z_x(m) - z_y(m)| \right) \\ &\leq \alpha \|x - y\|. \end{aligned}$$

Hence

$$\|\mathcal{H}x - \mathcal{H}y\| \leq \alpha \|x - y\|,$$

since $\alpha \in (0, 1)$, then $\mathcal{H}$ is a contraction.

Thus, by the contraction mapping principle, $\mathcal{H}$ has a unique fixed point x in $\mathcal{S}_\epsilon$ which is a solution of (3) with $x(n_0) = x_0$ and $|x(n)| = |x(n, n_0, x_0)| \leq \epsilon$ for $n \in \mathbb{N}$. This proves that the zero solution of (3) is stable. □

Definition 2. *The zero solution of (3) is asymptotically stable if it is Lyapunov stable and if for any integer $n_0 \geq 0$ there exists a $\delta > 0$ such that $|x_0| \leq \delta$ implies $|x(n, n_0, x_0)| \to 0$ as $n \to \infty$.*

Theorem 2. *Assume that the hypotheses of Theorem 1 hold. Also assume that*

$$\prod_{u=n_0}^{n-1} c(u) \to 0, \text{ as } n \to \infty, \tag{9}$$

hold. Then the zero solution of (3) is asymptotically stable.

Proof. We have already proved that the zero solution of (3) is stable. For a given $\epsilon > 0$ and $|x_0| \leq \delta$ define,

$$\mathcal{S}_\epsilon^0 = \{x \in \mathcal{S}_\epsilon \text{ such that } x(n) \to 0, \text{ as } n \to \infty\}.$$

Define $\mathcal{H} : \mathcal{S}_\epsilon^0 \to \mathcal{S}_\epsilon$ by (8). We must prove that for $x \in \mathcal{S}_\epsilon^0$, $(\mathcal{H}x)(n) \to 0$ when $n \to \infty$. By definition of $\mathcal{S}_\epsilon^0$, $x(n) \to 0$, as $n \to \infty$. Thus, we get

$$\begin{aligned} |(\mathcal{H}x)(t)| &\leq |x_0| \left|\prod_{u=n_0}^{n-1} c(u)\right| + \sum_{s=n_0}^{n-1} \left|\prod_{u=s+1}^{n-1} c(u)\right| \left(|a(s)| \sum_{m=s}^{s+h(s)-1} |z_x(m)|\right) \\ &\quad + \sum_{s=n_0}^{n-1} \left|\prod_{u=s+1}^{n-1} c(u)\right| \left(|b(s)| \sum_{m=s}^{s+r(s)-1} |z_x(m)|\right) \\ &:= I_1 + I_2 + I_3. \end{aligned} \tag{10}$$

By (9)

$$I_1 = |x_0| \left|\prod_{u=n_0}^{n-1} c(u)\right| \to 0 \text{ when } n \to \infty.$$

Moreover, let $x \in \mathcal{S}_\epsilon^0$ that for any $\epsilon_1 \in (0, \epsilon)$, there exists $N \geq n_0$ large enough such that $s \geq N$ implies $|x(s)| < \epsilon_1$. Hence, we get

$$I_2 \leq \epsilon_1 \sum_{s=n_0}^{n-1} \left|\prod_{u=s+1}^{n-1} c(u)\right| \left(|a(s)| \sum_{m=s}^{s+h(s)-1} |z_x(m)|\right) \leq \epsilon_1 \alpha < \epsilon_1.$$

Thus, $I_2 \to 0$ as $n \to \infty$. Also $I_3 \to 0$ as $n \to \infty$ by the same way of I_2.

Hence, $\mathcal{H}$ maps $\mathcal{S}_\epsilon^0$ into itself. By the contraction mapping principle, $\mathcal{H}$ has a unique fixed point $x \in \mathcal{S}_\epsilon^0$ which solves (3). Therefore, the zero solution of (3) is asymptotically stable. □

Example 1. *Consider the advanced difference equation (3) as the form*

$$\Delta x(n) + \frac{1}{3^{n+1}} (x(n+1) + x(n+2)) = 0, \; n \geq 0. \tag{11}$$

Then with $a(n) = b(n) = \frac{1}{3^{n+1}}$, $h(n) = 1$ *and* $r(n) = 2$, *we have*

$$\prod_{u=n_0}^{n-1} c(u) = \prod_{u=0}^{n-1} \left(1 - \frac{2}{3^{u+1}}\right) \to 0 \text{ when } n \to \infty.$$

And since $\left|\prod_{u=s+1}^{n-1} \left(1 - \frac{2}{3^{u+1}}\right)\right| < 1$, *we have also*

$$\begin{aligned}
&\sum_{s=0}^{n-1} \left|\prod_{u=s+1}^{n-1} \left(1 - \frac{2}{3^{u+1}}\right)\right| \left(\frac{1}{3^{s+1}} \sum_{m=s}^{s} \frac{2}{3^{m+1}}\right) \\
&+ \sum_{s=0}^{n-1} \left|\prod_{u=s+1}^{n-1} \left(1 - \frac{2}{3^{u+1}}\right)\right| \left(\frac{1}{3^{s+1}} \sum_{m=s}^{s+1} \frac{2}{3^{m+1}}\right) \\
&< \sum_{s=0}^{n-1} \frac{2}{3^{2s+2}} + \sum_{s=0}^{n-1} \frac{8}{3^{2s+3}} \\
&= \left(\frac{2}{9} + \frac{8}{27}\right) \sum_{s=0}^{n-1} \frac{1}{9^s} \\
&< \frac{7}{12}.
\end{aligned}$$

So, the conditions (6), (7) *and* (9) *are satisfied for the equation* (11). *Then by Theorem* 1 *the zero solution of* (11) *is stable and by Theorem* 2, *the zero solution is asymptotically stable.*

Definition 3. *The zero solution of* (3) *is exponentially stable, if there exist* $\sigma > 0$, $\delta > 0$ *and* $\lambda \in (0, 1)$ *such that*

$$|x(n, n_0, x_0)| < \sigma |x_0| \lambda^{n-n_0}, \ n \geq n_0, \tag{12}$$

whenever $|x_0| < \delta$.

Theorem 3. *Assume that there exist* $\sigma > 0$ *and* $\lambda \in (0, 1)$ *such that*

$$\left|\prod_{u=n_0}^{n-1} c(u)\right| \leq \frac{1}{2} \lambda^{n-n_0}, \ \forall n \geq n_0, \tag{13}$$

and

$$\sum_{s=n_0}^{n-1} \left(|a(s)| \sum_{m=s}^{s+h(s)-1} (|a(m)| + |b(m)|) + |b(s)| \sum_{m=s}^{s+r(s)-1} (|a(m)| + |b(m)|)\right) \leq \alpha, \tag{14}$$

hold, with $\alpha \in (0, 1)$. *Then the zero solution of* (3) *is exponentially stable.*

Proof. Since there exist $\lambda \in (0, 1)$ such that (13) holds. So, let us define the closed subspace $\mathcal{E}$ of $\mathcal{X}$ as

$$\mathcal{E} = \left\{x \in \mathcal{X} : |x(n)| \leq |x_0| \sigma \lambda^{n-n_0}, \ \forall n \geq n_0\right\}.$$

We will show that $\mathcal{H}(\mathcal{E}) \subset \mathcal{E}$. So, we use the same notation I_1 and I_2 in (10). Then by (13), we have

$$I_1 = |x_0| \left| \prod_{u=n_0}^{n-1} c(u) \right| \leq \frac{1}{2} |x_0| \sigma \lambda^{n-n_0}, \ n \geq n_0,$$

and

$$\begin{aligned} I_2 &= \sum_{s=n_0}^{n-1} \left| \prod_{u=s+1}^{n-1} c(u) \right| \left(|a(s)| \sum_{m=s}^{s+h(s)-1} |z_x(m)| + |b(s)| \sum_{m=s}^{s+r(s)-1} |z_x(m)| \right) \\ &\leq \frac{1}{2}\sigma |x_0| \sum_{s=n_0}^{n-1} \lambda^{n-s-1} \left(|a(s)| \sum_{m=s}^{s+h(s)-1} \left| a(m) \lambda^{m+h(m)-n_0} + b(m) \lambda^{m+r(m)-n_0} \right| \right) \\ &+ \frac{1}{2}\sigma |x_0| \sum_{s=n_0}^{n-1} \lambda^{n-s-1} \left(|a(s)| \sum_{m=s}^{s+r(s)-1} \left| a(m) \lambda^{m+h(m)-n_0} + b(m) \lambda^{m+r(m)-n_0} \right| \right) \\ &= \frac{1}{2}\sigma |x_0| \lambda^{n-n_0} \sum_{s=n_0}^{n-1} \lambda^{s-1} \left(|a(s)| \sum_{m=s}^{s+h(s)-1} \left(|a(m)| \lambda^{m+h(m)} + |b(m)| \lambda^{m+r(m)} \right) \right) \\ &+ \frac{1}{2}\sigma |x_0| \lambda^{n-n_0} \sum_{s=n_0}^{n-1} \lambda^{s-1} \left(|a(s)| \sum_{m=s}^{s+r(s)-1} \left(|a(m)| \lambda^{m+h(m)} + |b(m)| \lambda^{m+r(m)} \right) \right), \end{aligned}$$

since $\lambda \in (0,1)$, then $\lambda^{m+h(m)} \leq 1$, $\lambda^{m+r(m)} \leq 1$, and $\lambda^{s-1} \leq 1$. So, we have

$$I_2 \leq \frac{1}{2}\sigma |x_0| \lambda^{n-n_0} \alpha < \frac{1}{2}\sigma |x_0| \lambda^{n-n_0}.$$

We infer that

$$|(\mathcal{H}x)(t)| \leq I_1 + I_2 < \sigma |x_0| \lambda^{n-n_0}.$$

Then $\mathcal{H}(\mathcal{E}) \subset \mathcal{E}$.

By the contraction mapping principle, $\mathcal{H}$ has a unique fixed point $x \in \mathcal{E}$ which solves (3). Therefore, the zero solution of (3) is exponentially stable. □

Now, we state the theorems concerning the stability and asymptotic stability of the equation (1) without proof because the way is similarly as in Theorems 1 and 4. We begin by this lemma.

Lemma 2. *Suppose that $c(n) = 1 - \sum_{k=1}^{N} a_k(n) \neq 0$ for all $n \in \mathbb{N}$. Then x is a solution of the equation (3) if and only if*

$$x(n) = x_0 \prod_{u=n_0}^{n-1} c(u) + \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} c(u) \left(\sum_{k=1}^{N} \left(a_k(s) \sum_{m=s}^{s+h_k(s)-1} z_x(m) \right) \right), \tag{15}$$

where $z_x(n) = \sum_{k=1}^{N} a_k(n)\, x(n + h_k(n))$.

Proof. We use the same way in Lemma 1. □

Theorem 4. *Suppose there exist a positive constant* M_1 *and a constant* $\alpha \in (0,1)$ *such that for* $n \in n \in \mathbb{N}$

$$\left| \prod_{u=n_0}^{n-1} c(u) \right| \leq M_1, \tag{16}$$

where $c(u) = 1 - \sum_{k=1}^{N} a_k(u) \neq 0$, *and*

$$\sum_{s=n_0}^{n-1} \left| \prod_{u=s+1}^{n-1} c(u) \right| \left(\sum_{k=1}^{N} \left(|a_k(s)| \sum_{m=s}^{s+h_k(s)-1} |a_k(m)| \right) \right) \leq \alpha. \tag{17}$$

Then the zero solution of (1) is stable.

Theorem 5. *Assume that the hypotheses of Theorem 4 hold. Also assume that*

$$\prod_{u=n_0}^{n-1} c(u) \to 0, \text{ as } n \to \infty, \tag{18}$$

hold. Then the zero solution of (1) is asymptotically stable.

Theorem 6. *Assume that there exist* $\sigma > 0$ *and* $\lambda \in (0,1)$ *such that*

$$\left| \prod_{u=n_0}^{n-1} c(u) \right| \leq \frac{1}{2}\lambda^{n-n_0}, \ \forall n \geq n_0, \tag{19}$$

where $c(u) = 1 - \sum_{k=1}^{N} a_k(u) \neq 0$, *and*

$$\sum_{s=n_0}^{n-1} \left(\sum_{k=1}^{N} \left(|a_k(s)| \sum_{m=s}^{s+h_k(s)-1} |a_k(m)| \right) \right) \leq \alpha, \tag{20}$$

hold, with $\alpha \in (0,1)$. *Then the zero solution of (3) is exponentially stable.*

4. Conclusion

The linear advanced difference equation is considered. So, we have studied the stability, asymptotic stability, and exponential stability. The main tool used in the paper is Banach fixed point theorem. An example is given to illustrate the main results.

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